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Recurrence relation for the factors in the polar decomposition on Lie groups

A. Zanna*

11th May 2000

Abstract

In this paper we consider the polar decomposition on a Lie group G endowed with an involutive automorphism σ, as considered in (Munthe-Kaas, Quispel & Zanna 2000). In that paper, among other things, it was shown that $z = \exp(tZ) \in G$, $Z \in \mathfrak{g}$, can be decomposed as $z = xy$, where $x = \exp(X(t))$ and $y = \exp(Y(t))$, where $X(t) \in \mathfrak{p}_\sigma$ and $Y(t) \in \mathfrak{k}_\sigma$, $\mathfrak{g} = \mathfrak{p}_\sigma \oplus \mathfrak{k}_\sigma$ being a Cartan decomposition of the algebra $\mathfrak{g}$ induced by σ. The authors also derived a recurrence relation for the series expansion of $X(t)$. This paper is devoted to the derivation of the analogous formula for the function $Y(t)$. We obtain an explicit (but rather complicated) recurrence relation for the coefficients of the series expansion. We also point to some Matlab software for the computation of the series expansion of $X(t)$ and $Y(t)$.

1 Introduction

In the Linear Algebra community, the polar decomposition is a well known algorithm for decomposing a real matrix into the product of a positive semidefinite matrix and an orthogonal matrix (Horn & Johnson 1991). However, the matrix case is just a particular setting. In fact, structures like Lie groups, endowed with involutive automorphisms, possess all the necessary structure in order to define the analogue of matrix polar decomposition.

There exists a large literature describing the existence of such a polar decomposition in Lie groups. We refer the reader to the paper of Lawson (1994) and references therein, where, for instance, a proof of the existence of the polar decomposition is given.

This more general polar decomposition on Lie groups was also recently considered in Munthe-Kaas et al. (2000). In that paper, the authors derived an explicit recurrence relation for one of the factors. More specifically, if G is a Lie group and σ is an involutive automorphism, then one can consider the sets G^σ, of fixed points of σ, a subgroup of G, and G_σ the set of 'anti-fixed' points of G, i.e. of the type $\sigma(x) = x^{-1}$. The set G_σ doesn't have the structure of a group but that of a symmetric space (Helgason 1978), namely it is closed under a symmetric type of composition,

$$x \cdot y = xy^{-1}x \in G_\sigma, \qquad x, y \in G_\sigma,$$

where xy denotes the usual product in the Lie group G. Now, the polar decomposition of $z \in G$ is equivalent to writing

$$z = xy, \qquad x \in G_\sigma, y \in G^\sigma.$$

Such decomposition always exists when z is sufficiently close to the identity element $e \in G$, thus, if $z = \exp(tZ)$, then the factors x and y are differentiable functions and $x = \exp(X(t))$,

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$y = \exp(Y(t))$, for some function $X(t)$ and $Y(t)$ in $\mathfrak{g}$. Moreover, since σ induces an involutive automorphism $\mathrm{d}\sigma$ on the Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is split into the direct sum of two vector spaces,

$$\mathfrak{g} = \mathfrak{p}_\sigma \oplus \mathfrak{k}_\sigma,$$

$\mathfrak{p}_\sigma$ corresponding to the eigenvalue $\lambda = -1$ of $\mathrm{d}\sigma$ (i.e. $P \in \mathfrak{p}_\sigma$ implies that $\mathrm{d}\sigma(P) = -P$), $\mathfrak{k}_\sigma$ corresponding to the eigenvalue $\lambda = 1$ of $\mathrm{d}\sigma$ (i.e. $K \in \mathfrak{k}_\sigma$ implies that $\mathrm{d}\sigma(K) = K$). It is easily verified that $\mathfrak{k}_\sigma$ is a subalgebra of $\mathfrak{g}$, while $\mathfrak{p}_\sigma$ has the structure of a *Lie triple system* meaning that $\mathfrak{p}_\sigma$ is not closed under the commutator but under the double commutator. Thus, the function $X(t)$ and $Y(t)$ have the property that

$$X(t) \in \mathfrak{p}_\sigma, \qquad Y(t) \in \mathfrak{k}_\sigma.$$

As we have mentioned before, the functions $X(t)$ and $Y(t)$ are differentiable and can be expanded in a converging series at $t = 0$. Thus, $X(t) = \sum_{n=0}^{\infty} t^n X_n$ and $Y(t) = \sum_{n=0}^{\infty} t^n Y_n$. The recurrence relation for the X_n has been derived in (Munthe-Kaas et al. 2000). However, although in the same paper the authors showed that $Y(t)$ expands in odd powers of t only and obtained a differential equation for $Y(t)$, such an equation was not independent of the knowledge of $X(t)$, and this dependence rendered the derivation of a recurrence relation for $Y(t)$ a rather convoluted task.

In this paper we shall follow a slightly different approach and hence complete the work in (Munthe-Kaas et al. 2000) by deriving an explicit recurrence relation for the Y_ns. This will be accomplished first by deriving a differential equation obeyed by Y, this time independent of the direct knowledge of $X(t)$. Although the differential equation is implicit in Y', we will derive a recurrence relation for the Y_n expanding all the terms and operators in series and comparing powers of t. Finally, we point the reader to some Matlab software that can be used to compute the terms in the series expansion of X and Y, the software itself being based on the recurrence relations described in this paper.

1.1 Notation

We use the notation convention of (Helgason 1978) to denote group and algebra elements. For background theory on symmetric spaces we refer mainly to (Helgason 1978) and (Loos 1969). Group and algebra operators, like $\mathrm{dexp}, \mathrm{ad}, \mathrm{Ad}$ etc., are decribed at length in (Iserles, Munthe-Kaas, Nørsett & Zanna 2000).

Let G be a Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Denote by $\sigma : G \to G$ an involutive automorphism, and let $\mathrm{d}\sigma$ be the corresponding Lie-algebra automorphism, so that

$$\mathfrak{g} = \mathfrak{p}_\sigma \oplus \mathfrak{k}_\sigma$$

is a Cartan decomposition of $\mathfrak{g}$. The space $\mathfrak{k}_\sigma$ is a subalgebra of $\mathfrak{g}$ (corresponding to the eigenvalue $\lambda = 1$ of $\mathrm{d}\sigma$), while $\mathfrak{p}_\sigma$ is a Lie triple system (linear subspace of $\mathfrak{g}$ corresponding to the eigenvalue $\lambda = 1$).

Set $z = \exp(tZ)$ where $Z = P + K$ is the canonical projection of Z into $\mathfrak{p}_\sigma \oplus \mathfrak{k}_\sigma$, i.e. $\mathrm{d}\sigma(P) = -P$ and $\mathrm{d}\sigma(K) = K$. Set $w = (\sigma(z))^{-1}$. Then $w = \exp(tW)$, where $W = -\mathrm{d}\sigma(Z) = P - K$. We also set

$$X(t) = \sum_{n=0}^{\infty} X_n t^n, \qquad Y(t) = \sum_{n=0}^{\infty} Y_n t^n.$$

In (Munthe-Kaas et al. 2000) it was shown that

$$\begin{aligned}(n+1)X_{n+1} &= -[X_n, K] + \sum_{\substack{\ell \geq 1 \\ 2\ell \leq n}} c_{2\ell} \sum_{\substack{\ell_1,\ldots,\ell_{2\ell} > 0 \\ \ell_1 + \cdots + \ell_{2\ell} = n}} [X_{\ell_1}, [X_{\ell_2}, \cdots, [X_{\ell_{2\ell}}, P]]], \quad n = 1, 2, \ldots, \\ X_1 &= P,\end{aligned} \tag{1.1}$$

is the recurrence relation obeyed by the coefficients of $X(t)$. Here, $c_{2\ell} = \frac{2^{2\ell}\mathrm{B}_{2\ell}}{(2\ell)!}$, B_ℓ being the ℓ-th Bernoulli number (Abramowitz & Stegun 1965). Moreover, it was also shown that $Y(t)$ is an odd function of t, that is

$$Y_{2n} = O, \qquad n = 0, 1, 2, \ldots, \tag{1.2}$$

hence only the even coefficients in the expansion of $Y(t)$ need to be calculated.

2 An implicit differential equation for $Y(t)$

Recall that $z = xy$ and that $\sigma(x) = x^{-1}$ and $\sigma(y) = y$, so that

$$z\sigma(z)^{-1} = xy\sigma(xy)^{-1} = xyy^{-1}x = x^2, \tag{2.1}$$

from which it is easily verified (taking logarithm on both sides) that

$$X(t) = \frac{1}{2}\mathrm{bch}(tZ, tW) = \frac{1}{2}\mathrm{bch}(t(P+K), t(P-K)),$$

where $\mathrm{bch}(\cdot,\cdot)$ is the familiar operator of the Baker–Campbell–Hausdorff formula (BCH), see (Varadarajan 1984).

Once $x = \exp(X(t))$ is known, one has

$$y = x^{-1}z,$$

hence, taking logarithm on both sides and using the bch formula, it is true that

$$Y(t) = \mathrm{bch}(-X(t), tZ).$$

Because of the fact that $X(t)$ is an infinite series whose coefficients are known recursion (see (1.1)), here we shall follow a different approach for determining $Y(t)$. As we have seen in (2.1), x can be expressed independently of y and as a function of z and $\sigma(z)$ only. We will derive a similar relation for y, only in terms of z and $\sigma(z)$. Then, differentiation of this relation, will allow us to obtain an explicit recurrence relation for Y only in terms of $Z = P + K$ and $W = P - K$.

To this goal, note that, from $z = xy$, we deduce that $\sigma(z) = x^{-1}y$, hence

$$x = y\sigma(z)^{-1} = yw.$$

Substitution in $z = xy$ yields

$$z = xy = ywy,$$

a relation invoking only y, z and $w = \sigma(z)^{-1}$, as desired. In order to differentiate, is however more convenient to multiply the above equality on both sides by y^{-1} on the right, to obtain

$$zy^{-1} = yw. \tag{2.2}$$

Differentiating (2.2) in the usual fashion, we obtain

$$\begin{aligned} Zzy^{-1} - z\mathrm{dexp}_{-Y}Y'y^{-1} &= \mathrm{dexp}_Y Y'w + yWw \\ &= \mathrm{dexp}_Y Y'zy^{-1} + zy^{-1}W, \end{aligned}$$

where we have made use of the fact that w and W commute, as well as (2.2). Recall that also z and Z commute. Hence, multiplication of both sides of the above equation by z^{-1} on the left and by y on the right yields

$$Z - \mathrm{dexp}_{-Y}Y' = z^{-1}\mathrm{dexp}_Y Y'z + y^{-1}Wy. \tag{2.3}$$

Note that
$$z^{-1}\mathrm{dexp}_Y Y' z = \mathrm{Ad}_{\exp(-tZ)}\mathrm{dexp}_Y Y' = \exp(-t\mathrm{ad}_Z)\mathrm{dexp}_Y Y',$$
and by a similar token,
$$y^{-1}Wy = \exp(-\mathrm{ad}_Y)W.$$
It is convenient to isolate the first term of the two exponential operators above, hence (2.3) becomes
$$Z - \mathrm{dexp}_{-Y}Y' = \mathrm{dexp}_Y Y' + \{\exp(-t\mathrm{ad}_Z) - 1\}\mathrm{dexp}_Y Y' + W + \{\exp(-\mathrm{ad}_Y) - 1\}W,$$
from which, collecting the free 'dexp' terms on the one side,
$$\{\mathrm{dexp}_{-Y} + \mathrm{dexp}_Y\}Y' = 2K - \{\exp(-t\mathrm{ad}_Z) - 1\}\mathrm{dexp}_Y Y' - \{\exp(-\mathrm{ad}_Y) - 1\}W. \qquad (2.4)$$
From
$$\mathrm{dexp}_{-u} = \frac{\mathrm{e}^{-u} - 1}{u}\Big|_{u=\mathrm{ad}_Y}, \qquad \mathrm{dexp}_u = \frac{\mathrm{e}^{u} - 1}{u}\Big|_{u=\mathrm{ad}_Y},$$
we observe that
$$\mathrm{dexp}_{-u} + \mathrm{dexp}_u = \frac{\mathrm{e}^{u} - \mathrm{e}^{-u}}{u} = 2\frac{\sinh u}{u},$$
hence (2.4) reduces to
$$2\frac{\sinh u}{u}\Big|_{u=\mathrm{ad}_Y} Y' = 2K - \sum_{k=1}^{\infty}\frac{(-t)^k}{k!}\mathrm{ad}_Z^k\mathrm{dexp}_Y Y' - \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\mathrm{ad}_Y^k W. \qquad (2.5)$$

3 The series expansion and recurrence relation

Instead of inverting the operator $\frac{\sinh u}{u}$, $u = \mathrm{ad}_Y$, on the left-hand-side of (2.5), we observe that (2.5) can already be used to derive, after some algebra, an explicit recurrence relation for the Y_ns. Let us begin with expanding in series the term on the left hand side of (2.5). Since $Y(t) = \sum_{n=0}^{\infty} t^{2n+1}Y_{2n+1}$, it is easily seen that
$$Y'(t) = \sum_{n=0}^{\infty}(2n+1)t^{2n}Y_{2n+1}.$$
Moreover,
$$\frac{\sinh u}{u}\Big|_{u=\mathrm{ad}_Y} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\mathrm{ad}_Y^{2k},$$
(note that it is an even function of t, as it is Y'), hence, changing order of summation and isolating the first term,
$$\begin{aligned}\frac{\sinh u}{u}\Big|_{u=\mathrm{ad}_Y} Y' &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\mathrm{ad}_Y^{2k}\sum_{n=0}^{\infty}(2n+1)t^{2n}Y_{2n+1}\\ &= \sum_{n=0}^{\infty}(2n+1)t^{2n}Y_{2n+1} + \sum_{n=0}^{\infty}(2n+1)t^{2n}\sum_{k=1}^{\infty}\frac{1}{(2k+1)!}\mathrm{ad}_Y^{2k}(Y_{2n+1}).\end{aligned}$$

[1] We perform next a series expansion of the second term on the above right hand side, which, after matching powers of t, results into
$$\sum_{n=1}^{\infty}t^{2n}\sum_{q=1}^{n}\sum_{\substack{k\geq 1\\ k\leq q}}\frac{1}{(2k+1)!}\sum_{\substack{k_1,\ldots,k_{2k}>0\\ k_1+\cdots+k_{2k}=2q}}[Y_{k_1},\ldots,[Y_{k_{2k}},Y_{2(n-q)+1}],\ldots].$$

[1] Note the lower limit of the index k ($k=1$) and the fact that $Y(t) = \mathcal{O}(t)$.

It turns out that the coefficient of the t^{2n} term of the left hand side of (2.5) is given by:

$$2(2n+1)Y_{2n+1}+2\sum_{q=1}^{n}\sum_{\substack{k\geq 1\\ k\leq q}}\frac{1}{(2k+1)!}\sum_{\substack{k_1,\ldots,k_{2k}>0\\ k_1+\cdots+k_{2k}=2q}}[Y_{k_1},\ldots,[Y_{k_{2k}},Y_{2(n-q)+1}],\ldots] \tag{3.1}$$

Next, we turn our attention to the right hand side of (2.5). Let us start from the term

$$\sum_{k=1}^{\infty}\frac{(-t)^k}{k!}\mathrm{ad}_Z^k\mathrm{dexp}_Y Y'. \tag{3.2}$$

Since

$$\begin{aligned}\mathrm{dexp}_Y Y' &= \sum_{n=0}^{\infty}(2n+1)t^{2n}\mathrm{dexp}_Y Y_{2n+1}\\ &= \sum_{n=0}^{\infty}(2n+1)t^{2n}Y_{2n+1}+\sum_{n=0}^{\infty}(2n+1)t^{2n}\{\mathrm{dexp}_Y-1\}Y_{2n+1},\end{aligned}$$

the term (3.2) can be split into two parts, say T_1+T_2, where

$$T_1=\sum_{k=1}^{\infty}\frac{(-t)^k}{k!}\mathrm{ad}_Z^k\sum_{n=0}^{\infty}(2n+1)t^{2n}Y_{2n+1},$$

and

$$T_2=\sum_{k=1}^{\infty}\frac{(-t)^k}{k!}\mathrm{ad}_Z^k\sum_{n=0}^{\infty}(2n+1)t^{2n}\left(\sum_{m=1}^{\infty}\frac{1}{(m+1)!}\mathrm{ad}_Y^m Y_{2n+1}\right).$$

We analyze T_1 first. We observe that T_1 consists in even and odd powers of t, however, given that the left hand side of (2.5) possesses even powers of t only, we disregard the odd powers in T_1 and consider the even part only, $T_{1,\mathrm{even}}$, given by

$$T_{1,\mathrm{even}}=\sum_{n=1}^{\infty}t^{2n}\sum_{m=1}^{n}\frac{2(n-m)+1}{(2m)!}\mathrm{ad}_Z^{2m}Y_{2(n-m)+1},$$

hence, the coefficient of t^{2n} is

$$\sum_{m=1}^{n}\frac{2(n-m)+1}{(2m)!}\mathrm{ad}_Z^{2m}Y_{2(n-m)+1}. \tag{3.3}$$

The analysis of the T_2 term is more complicated. Firstly, merging the first two series, we obtain

$$T_2=-t\,\mathrm{ad}_Z\sum_{\ell=0}^{\infty}t^{\ell}\sum_{j=0}^{\ell}\frac{(-1)^{\ell-j}(j+1)}{(\ell-j+1)!}\mathrm{ad}_Z^{\ell-j}.$$

Secondly, observe that

$$\sum_{m=1}^{\infty}\frac{1}{(m+1)!}\mathrm{ad}_Y^m Y_{j+1}=\sum_{q=0}^{\infty}t^{q+1}\sum_{\substack{k\geq 1\\ k\leq q+1}}\frac{1}{(k+1)!}\sum_{\substack{j_1,\ldots,j_k>0\\ j_1+\cdots+j_k=q+1}}[Y_{j_1},\ldots,[Y_{j_k},Y_{j+1}]\ldots],$$

from which we obtain

$$
\begin{aligned}
T_2 &= \sum_{m=0}^{\infty} t^{m+2} \sum_{q=0}^{m} \sum_{j=0}^{m-q} \frac{(-1)^{m-q-j+1}(j+1)}{(m-q-j+1)!} \mathrm{ad}_Z^{m-j-q+1} \\
&\qquad \sum_{\substack{k\geq 1\\ k\leq q+1}} \frac{1}{(k+1)!} \sum_{\substack{j_1,\ldots,j_k>0\\ j_1+\cdots+j_k=q+1}} [Y_{j_1},\ldots,[Y_{j_k},Y_{j+1}]\ldots].
\end{aligned}
$$

To obtain the coefficient of t^{2n} in the above expression, it is sufficient to set $m+2=2n$, from which we deduce that $m=2(n-1)$. Substituting above, the required coefficient is

$$
\begin{aligned}
&\sum_{q=0}^{2(n-1)} \sum_{j=0}^{2(n-1)-q} \frac{(-1)^{2n-q-j-1}(j+1)}{(2n-q-j-1)!} \mathrm{ad}_Z^{2n-j-q-1} \\
&\sum_{\substack{k\geq 1\\ k\leq q+1}} \frac{1}{(k+1)!} \sum_{\substack{j_1,\ldots,j_k>0\\ j_1+\cdots+j_k=q+1}} [Y_{j_1},\ldots,[Y_{j_k},Y_{j+1}]\ldots].
\end{aligned}
$$

It remains to analyze the last term of (2.5),

$$
\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathrm{ad}_Y^k W,
$$

a task that does not require particular difficulties. Let us recall, however, that Y is an odd function of t, therefore terms of the form ad_Y^{2k+1} only contribute with odd powers of t to the expansion, while we are interested in even powers of t. Therefore, we disregard the odd powers altogether, to consider the expansion of

$$
\sum_{k=1}^{\infty} \frac{1}{(2k)!} \mathrm{ad}_Y^{2k} W
$$

instead. This reduces to

$$
\sum_{n=1}^{\infty} t^{2n} \sum_{\substack{\ell\geq 1\\ \ell\leq n}} \frac{1}{(2\ell)!} \sum_{\substack{\ell_1,\ldots,\ell_{2\ell}>0\\ \ell_1+\cdots+\ell_{2l}=2n}} [Y_{\ell_1},\ldots,[Y_{\ell_{2\ell}},W],\ldots],
$$

and the coefficient of t^{2n} is

$$
\sum_{\substack{\ell\geq 1\\ \ell\leq n}} \frac{1}{(2\ell)!} \sum_{\substack{\ell_1,\ldots,\ell_{2\ell}>0\\ \ell_1+\cdots+\ell_{2l}=2n}} [Y_{\ell_1},\ldots,[Y_{\ell_{2\ell}},W],\ldots]. \tag{3.4}
$$

Now we have all the elements to complete the proof of the main result of this paper.

Theorem 3.1 *Let G be a Lie group and σ an involutive automorphism so that the differentiable polar decomposition $z=\exp(tZ)=xy=\exp(X(t))\exp(Y(t))$ exists in the sense of (Munthe-Kaas et al. 2000). Then, the function $Y(t)$ is analytic at the origin and it admits a series expansion $Y(t)=\sum_{n=0}^{\infty} t^n Y_n$ whose coefficient obey the explicit recurrence relation*

$$
\begin{aligned}
Y_1 &= K, \qquad\qquad (3.5)\\
Y_{2n} &= O, \qquad n=0,1,2,\ldots,
\end{aligned}
$$

$$
\begin{aligned}
2(2n+1)Y_{2n+1} \quad = \quad & -2\sum_{q=1}^{n}\sum_{\substack{k\geq 1\\ k\leq q}}\frac{1}{(2k+1)!}\sum_{\substack{k_1,\ldots,k_{2k}>0\\ k_1+\cdots+k_{2k}=2q}}[Y_{k_1},\ldots,[Y_{k_{2k}},Y_{2(n-q)+1}],\ldots] \qquad (3.6)\\
& -\sum_{m=1}^{n}\frac{2(n-m)+1}{(2m)!}\mathrm{ad}_Z^{2m}Y_{2(n-m)+1}\\
& -\sum_{q=0}^{2(n-1)}\sum_{j=0}^{2(n-1)-q}\frac{(-1)^{2n-q-j-1}(j+1)}{(2n-q-j-1)!}\mathrm{ad}_Z^{2n-j-q-1}\\
& \qquad \sum_{\substack{k\geq 1\\ k\leq q+1}}\frac{1}{(k+1)!}\sum_{\substack{j_1,\ldots,j_k>0\\ j_1+\cdots+j_k=q+1}}[Y_{j_1},\ldots,[Y_{j_k},Y_{j+1}]\ldots]\\
& -\sum_{\substack{\ell\geq 1\\ \ell\leq n}}\frac{1}{(2\ell)!}\sum_{\substack{\ell_1,\ldots,\ell_{2\ell}>0\\ \ell_1+\cdots+\ell_{2l}=2n}}[Y_{\ell_1},\ldots,[Y_{\ell_{2\ell}},W],\ldots].
\end{aligned}
$$

Here, $Z = P + K$ and $W = P - K$, whereby $\mathrm{d}\sigma(P) = -P$ and $\mathrm{d}\sigma(K) = K$ (i.e. $P \in \mathfrak{p}_\sigma$ and $K \in \mathfrak{k}_\sigma$).

Proof. Having derived the series expansion of the differential equation (2.5) obeyed by $Y(t)$, we need only to match the coefficients of the t^{2n} term in (2.5), for $n = 0, 1, \ldots$, since we already know that the function $Y(t)$ is odd in t (Munthe-Kaas et al. 2000). Thus, (3.1)–(3.4), in tandem with (2.5), result into the relations (3.5) and (3.6), thus proving the assertion. □

It is immediate to observe that the recurrence relation obeyed by the coefficients of the series expansion of Y is by far much more complicated and involved than that obeyed by the coefficients of X, although, being $Y(t)$ and odd function of t, one might expect to have a simpler recurrence. However, one might notice that the relations obeyed by $x = \exp(X(t))$ and $y = \exp(Y(t))$ are of a different nature, as (2.1) and (2.2) reveal. Furthermore, in view of the relation

$$
z = ywy, \qquad tZ = \mathrm{symmbch}(Y(t), tW),
$$

one also realizes tZ is the result of the symmetric BCH formula $\mathrm{symmbch}(Y(t), tW)$ (Yoshida 1990). In other words, the function $Y(t)$ solves the following inverse problem: Given $z = \exp(tZ)$ and $w = \exp(tW)$, what is the function $Y(t)$ so that $y = \exp(Y(t))$ and the symmetric composition formula $z = ywy$ is obeyed? Because of this connection, we expect our recurrence relation to be of interest in the context of time-reversible numerical integrators for ordinary differential equations, where constructs *a la* Yoshida are of large application.

4 Matlab software for the computation of the series expansion of $X(t)$ and $Y(t)$

In this section we describe some software to compute the series expansion for the functions $X(t)$ and $Y(t)$ described in this paper and in (Munthe-Kaas et al. 2000). The Matlab codes described here require the Matlab package *DiffMan* written by K. Engø, A. Marthinsen and H. Munthe-Kaas and freely available from the home page

`http://www.ii.uib.no/diffman/`

We refer to the user manual (Engø, Marthinsen & Munthe-Kaas 1999) for explanations on how to install the software and how to initiate it.

The software presented here can instead be found at

```
http://www.ii.uib.no/~anto/SOFTWARE/
```

We expect the software to be included in the next release of *DiffMan*. The software includes the files:

```
lafpolar.m
nbch.m
xpolar.m
ypolar.m
```

together with the auxiliary files

```
bernoulli.m
firspartition.m
nextpartition.m
```

The above files are archived in the file `polar.tar` by means of the unix `tar` command, and

```
tar xvf polar.tar
```

will create a directory called `polar/` containing the abovementioned files. (This directory might be for instance created under the `diffman/` directory.)

After having initiated the *DiffMan* package by the `dminit` command,

```
  DiffMan Version 1.6 is initialized - 1999.03.10

  Please report any problems and/or bugs to:
       diffman@math.ntnu.no

  For more information and how to get started, try:
      >> dmtutorial
      >> dmhelp
      >> demo

>>
```

change directory to `polar/`. Then, the command

```
>> format rat
>> Y = ypolar(n);
```

calculates the n-dimensional vector `Y` whose entries $\texttt{Y(i)} = Y_i$, for $i = 1, 2, \ldots, n$, according to the recurrence relation presented in Section 3. Thus, for instance,

```
>> Y = ypolar(5);
>> Y(3)
```

returns

```
    ans =
       LieAlgebra class: lafree
       Data:
       -1/12*[1,[1,2]]
>>
```

which is the component Y_3 of $Y(t)$. Also, all the component up to Y_5 are calculated and stored in `Y`. Note that, according to the notation of this paper,

$$\begin{array}{lll} [1] & \text{corresponds to} & P, \\ [2] & \text{corresponds to} & K. \end{array}$$

The function `xpolar.m` works in an analogous manner, except that now the output `X = xpolar(n)` returns the component `X(i)` $= X_i$, for $i = 1, 2, \ldots, n$ computed according to (1.1).

The function `lafpolar.m` can be used to calculate directly the functions X and Y,

```
>> [X,Y] = lafpolar(n),
```

to order n. This program may be slower than `xpolar.m` and `ypolar.m` for large n because it requires the solution of a differential equation with a numerical method of order (at least) n. The function `nbch.m` computes the BCH formula according to the recurrence relation in (Varadarajan 1984) instead of solving a differential equation as the 'built-in' `bch.m` in *DiffMan*, therefore `nbch.m` results faster than `bch.m` when the computation of the formula is required to high order.

Finally, a brief description of the auxiliary functions `bernoulli.m, firstpartition.m` and `nexpartition.m`. The first file is used to compute Bernoulli numbers, the remaining two are fundamental in finding the interval partitions that occurr time and again in the recursion formulas (1.1) and (3.6).

5 Conclusions

In this paper we have completed the description of the differentiable factors of the polar decomposition on Lie groups (Munthe-Kaas et al. 2000) for elements of the form $z = \exp(tZ) = x(t)y(t)$. We require $y(t) \in G^\sigma$, the set of fixed points of the Lie-group involutive automorphism σ, and $x(t) \in G_\sigma$, the set of anti-fixed point of σ.

In particular, explicit recurrence relations are given not only for the logarithm $X(t)$ of symmetric space factor $x(t) = \exp(X(t))$ but also for the logarithm $Y(t)$ of the subgroup factor $y(t) = \exp(Y(t))$. These expansions are valid nearby the identity e of the Lie group, so that the polar decomposition above exists.

The recurrence relation for $Y(t)$ is particularly relevant since the subgroup factor $y(t)$ possesses some optimality features, being a the best approximant to z in G^σ, under opportune circumstances (see (Munthe-Kaas et al. 2000)).

The recurrence relations have been implemented in a Matlab software package, freely available on the internet, that can be used under the *DiffMan* Matlab package.

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