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A characterisation of the minimal triangulations of permutation graphs

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Abstract A minimal triangulation of a graph is a chordal graph obtained from adding an inclusion-minimal set of edges to the graph. For permutation graphs, i.e., graphs that are both comparability and cocomparability graphs, it is known that minimal triangulations are interval graphs. We (negatively) answer the question whether every interval graph is a minimal triangulation of a permutation graph. We give a non-trivial characterisation of the class of interval graphs that are minimal triangulations of permutation graphs and obtain as a surprising result that only "a few" interval graphs are minimal triangulations of permutation graphs.

1 Introduction The study of minimal triangulations of arbitrary graphs and restricted graph classes has a long tradition. Given a graph, a triangulation is a chordal graph that is obtained by adding edges to the graph. Minimal triangulations are special triangulations. Using a characterisation of Rose, Tarjan, Lueker, we can say that a triangulation is called minimal, if the deletion of each single added edge yields a non-chordal graph [17]. Minimal triangulations are closely connected to the treewidth problem, since the treewidth of a graph is the smallest clique number minus 1 among its minimal triangulations [16]. Many NP-hard problems become tractable on graphs of bounded treewidth, so that computing the treewidth is of highly practical interest, which motivates the study of minimal triangulations. But also another research branch motivates the study of minimal triangulations: results show that graphs and their minimal triangulations share structural properties. An easy result is that every graph has a minimal triangulation with the same independent-set number. Another, stronger, result is that every minimal separator of a minimal triangulation is a minimal separator also of the base graph [12]. In this context also fits the fact that minimal triangulations can be used to characterise graphs and graph classes.

An early result shows that minimal triangulations of cographs are trivially perfect graphs [3]. Since trivially perfect graphs are interval graphs, this also shows that treewidth and pathwidth are equal for cographs. On the other hand, cographs are exactly the P_4-free graphs, and trivially perfect graphs are exactly the P_4-free chordal graphs, from which follows that every minimal triangulation of a cograph is a cograph. This result was extended by Parra and Scheffler to the following: for $k \leq 5$, a graph is P_k-free if and only if every of its minimal triangulations is P_k-free [15]. The cograph result was generalised also in another direction, in particular to permutation graphs:

Theorem 1 [2] *Minimal triangulations of permutation graphs are interval graphs.*

Later, it was shown that this result even holds for cocomparability graphs [11] and AT-free graphs [14]. By the following characterisation, the class of AT-free graphs is the largest class of graphs containing only graphs whose minimal triangulations are interval graphs: a graph is AT-free if and only if it has only minimal triangulations that are interval graphs [14], [15]. So, for some graph classes, minimal triangulations reflect structural properties of the base graphs: minimal triangulations of P_k-free graphs for $k \leq 5$ are P_k-free, minimal

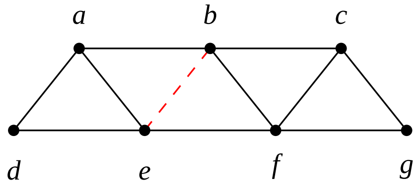

Figure 1 Depicted is a permutation graph G, and adding edge be gives an interval graph that is a minimal triangulation of G. This interval graph is not a permutation graph.

triangulations of cocomparability graphs and AT-free graphs are cocomparability graphs and AT-free graphs, respectively. In all these cases, the base graph class contains the class of minimal triangulations.

The case of permutation graphs is different. The class of interval graphs is not contained in the class of permutation graphs, and vice versa. A minimal triangulation of a permutation graph can be a graph that is not a permutation graph. An example is given in Figure 1: adding edge be gives a minimal triangulation, that is not a permutation graph. On the other hand, not every interval graph can be a minimal triangulation of a permutation graph. Such an example is depicted in Figure 5. Thus, the class of minimal triangulations of permutation graphs is a non-trivial subclass of the class of interval graphs, and the question arises how this class can be characterised. In this paper, we exactly address this problem. We give a characterisation of the class of minimal triangulations of permutation graphs, which even results in a recognition algorithm. This algorithm assumes the input graph to be given by an interval model and decides then in time linear in the number of vertices, independent of the number of edges.

The characterisation of the minimal triangulations of permutation graphs is based on minimal separators and potential maximal cliques. A potential maximal clique of a graph is a set of vertices that is a maximal clique in a minimal triangulation of the graph [4]. We define a graph over the set of potential maximal cliques of a graph in Section 3, which we call the *connectors graph*, and show a connection to minimal triangulations. In Sections 4 and 5, we investigate the structure of these graphs for chordal and permutation graphs. It turns out that every graph with two isolated vertices can be the connectors graph of a chordal, even an interval graph and that connectors graphs of permutation graphs have nice bipartite substructures. In Section 6, we give the final characterisation of the minimal triangulations of permutation graphs using connectors graphs, establishing a connection between minimal triangulations of permutation graphs and bipartite graphs. This result has two interesting implications. The one is that only a few interval graphs are minimal triangulations of permutation graphs. The other is the astounding observation that permutation graphs are cocomparability and comparability graphs, and minimal triangulations of permutation graphs are cocomparability graphs with a comparability (bipartite) structural property.

2 Graph preliminaries, chordal graphs and minimal triangulations We only consider simple finite undirected graphs. A *graph* is a pair $G = (V, E)$, where V denotes the set of *vertices* and E denotes the set of *edges*. Edges of G are 2-elementary subsets of V, and they are denoted as uv where u and v are vertices of G. In this case, u and v are called *adjacent*; otherwise, i.e., if uv is not an edge of G, u and v are *non-adjacent*. Let $H = (W, F)$ be a graph. We say that G is a *subgraph* of H, if $V \subseteq W$ and $E \subseteq F$. If G is not a proper subgraph of another subgraph of H on the vertex set of G, then G is called an *induced* subgraph of H, denoted as $G \sqsubseteq H$. For a set A of vertices of G, $G[A]$ denotes the subgraph of G induced by A, i.e., $G[A] = (A, X)$ for (A, X) an induced subgraph of G. Let $k \geq 0$, and let $x_0, \ldots, x_k$ be vertices of G. A *u, v-path* P of length k in G for u and v vertices of G is a sequence $(x_0, \ldots, x_k)$ of vertices of G where the vertices are mutually different and $x_i x_{i+1}$ is an edge of G for every $i \in \{0, \ldots, k-1\}$ and $u = x_0$ and $v = x_k$. If only consecutive path vertices are adjacent in G, P is an *induced path* in G. A *cycle* C of length k is a sequence $(x_1, \ldots, x_k)$ of mutually different vertices that is a path and where $x_1 x_k$ is an edge of G. If no non-consecutive vertices of C are adjacent in G, C is called an *induced cycle*.

Let $G = (V, E)$ be a graph. G is called *connected*, if for every pair u, v of vertices of G, there is a u, v-path in G. A *connected component* of G is a connected induced subgraph of G such that it is not properly contained in any other connected induced subgraph of G. For a set $S \subseteq V$, $G \setminus S$ denotes the subgraph of G induced by $V \setminus S$. Let a and b be vertices of G. A set S of vertices of G is an *a, b-separator* of G, if a and b are contained in different connected components of $G \setminus S$. If no proper subset of S is an a, b-separator of G, S is a *minimal* a, b-separator of G. A *minimal separator* of G is a minimal a, b-separator of G for some vertices a and b.

The class of chordal graphs has an interesting connection to minimal separators. A graph is called *chordal*, if it does not contain an induced cycle of length greater than 3. Chordal graphs are exactly the graphs whose minimal separators all are cliques [7]. Chordal graphs can also be characterised by a property of its maximal cliques. Let $G = (V, E)$ be a graph. A tree T is a *clique-tree* for G, if it has the following properties:

(vct) T has a vertex for every maximal clique of G

(ect) let C' and C'' be two maximal cliques of G, and let P be the path in T between the vertices corresponding to C' and C''. Then, every clique corresponding to a vertex on P contains the vertices in $C' \cap C''$.

Then, a graph is chordal if and only if it has a clique-tree [6], [8], [18]. A chordal graph may have more than one clique-tree. Clique-trees provide a nice characterisation of minimal separators of chordal graphs.

Theorem 2 **[1]** *Let $G = (V, E)$ be a chordal graph, and let T be a clique-tree for G. Then, a set S of vertices of G is a minimal separator of G if and only if there are maximal cliques C' and C'' of G that are neighbours with respect to T such that $S = C' \cap C''$.*

Minimal triangulations of graphs are chordal graphs. Let $G = (V, E)$ be a graph. A graph H on the vertex set of G is a *triangulation* of G, if H is chordal and contains G as a subgraph. If no proper subgraph of H is a triangulation of G then H is a *minimal triangulation* of G. An easy polynomial-time test for minimal triangulation is based on the following result: Let G and H be two graphs on the same vertex set. Then, H is a minimal triangulation of G if and only if G is a subgraph of H, H is chordal and $H - e$ is not chordal for every edge e of H not contained in G [17]. Minimal separators have an interesting characterisation.

Theorem 3 **[12]** *Let $G = (V, E)$ be a graph. A set S of vertices of G is a minimal separator of G if and only if S is a minimal separator of a minimal triangulation of G.*

A subclass of chordal graphs is the class of interval graphs. Interval graphs are defined as the intersection graphs of families of closed intervals of the real line in the following sense: given a family of closed intervals of the real line, the obtained interval graph has a vertex for every interval, and two vertices are adjacent if and only if the corresponding intervals have a non-empty intersection. Such a family of intervals is also called an *interval model* for an interval graph. Another characterisation uses clique-trees: a graph is an interval graph if and only if it has a clique-tree that is a path, i.e., a clique-tree with at most two leaves [9]. The alignment of the maximal cliques of an interval graph corresponding to a clique-tree that is a path is called *consecutive clique arrangement*.

3 An invariant for minimal triangulations In some sense, it is well understood how minimal triangulations are related to the corresponding base graphs. Several characterisation results are known. But only a few is known about structures of minimal triangulations that are "inherited" from the base graph. In this section, we identify a structural graph property that is inherited by its minimal triangulations. Let $G = (V, E)$ be a graph. A set C of vertices of G is called a *potential maximal clique* of G, if C is a maximal clique in a minimal triangulation of G. By definition, the potential maximal cliques of a chordal graph are the maximal cliques of the chordal graph. Potential maximal cliques were introduced by Bouchitté and Todinca [4] and turned out useful in connection with minimal triangulations problems such as treewidth and minimum fill-in (examples can be found in [4], [5], [13]). Potential maximal cliques contain minimal separators of the graph.

Definition 1 *Let $G = (V, E)$ be a graph. A potential maximal clique C of G is called a **connector** if and only if there is a **plug pair** for C, which is a pair $[a, b]$ of vertices in C such that there are two minimal separators S' and S'' of G contained in C such that $a \in S'$ and $b \in S''$ and there is no minimal separator of G contained in C that contains a and b.*

For a plug pair, we write $[a, b]$ instead of (a, b) to emphasise that there is no inherent order of the vertices. For a given graph, not every pair of vertices constitutes a plug pair, as the following lemma shows.

Lemma 4 *Let $G = (V, E)$ be a graph. Let $[a, b]$ be a plug pair for a potential maximal clique C of G. Then, a and b are adjacent in G.*

Proof: Let H be a minimal triangulation of G containing C as maximal clique. Let T be a clique-tree for H. Suppose there is a second maximal clique C' of H containing a and b. Let C'' be the maximal clique of H on the path between C and C' in T that is a neighbour of C with respect to T. Due to Theorem 2, $S =_{\mathrm{def}} C \cap C''$ is a minimal separator of H, and $\{a, b\} \subseteq S$. So, there is a minimal separator of H contained in C that contains a and b. Due to Theorem 3, S is a minimal separator also of G, which contradicts the definition of a plug pair. Hence, there is exactly one maximal clique of H containing a and b. Furthermore, the common neighbours of a and b in H are contained in C, i.e., $N_H(a) \cap N_H(b)$ is a clique in H. It follows that $H - ab$ is also a chordal graph, and if ab is not an edge of G, $H - ab$ is a triangulation of G, contradicting that H is a minimal triangulation of G. Thus, ab is an edge of G. ∎

It follows that the number of plug pairs of a graph is bounded above by the number of its edges, i.e., polynomially bounded in the number of vertices. The number of potential maximal cliques of a graph can be exponential in the number of vertices (see [4]). So, the question arises whether a graph can have only a very few connectors or whether a pair of vertices can be a plug pair for several connectors. The question is answered by an example. Figure 2 depicts a chordless cycle on six vertices. Every potential maximal clique of a chordless cycle contains three vertices, and every pair of non-adjacent vertices is a minimal separator. Then, it is not hard to see that the vertex pair $[a, b]$ is a plug pair in the two

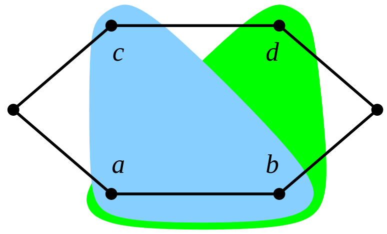

Figure 2 The chordless cycle on six vertices has a pair $[a, b]$ of vertices that is a plug pair for two potential maximal cliques.

potential maximal cliques $\{a, b, c\}$ and $\{a, b, d\}$. For minimal triangulations, however, the proof of Lemma 4 shows that there is a correspondence between the connectors and the plug pairs.

Based on connectors and plug pairs, we are interested in the question how connectors are related to each other. For expressing such a relation, we define the so-called connectors graph using plug pairs. Let $G = (V, E)$ be a graph. The *connectors graph* of G, denoted as $\mathrm{con}(G)$, is defined as follows:

- (vc) $\mathrm{con}(G)$ contains a vertex for every potential maximal clique of G
- (ec) uv is an edge of $\mathrm{con}(G)$ if and only if the potential maximal cliques A_u and A_v are connectors and there are vertices a, b, c such that $[a, b]$ and $[b, c]$ are plug pairs for A_u and A_v, respectively.

The vertices of a connectors graph are associated with the corresponding potential maximal cliques, which can be considered as the *names* of the vertices. We are interested in the relationship between the connectors graphs of a graph and its minimal triangulations.

Bouchitté and Todinca showed a strong correspondence between the minimal separators of a graph and of its minimal triangulations [4]. In fact, this result is an extention of Theorem 3.

Theorem 5 [4] *Let $G = (V, E)$ be a graph and let H be a minimal triangulation of G. Let C be a maximal clique of H. Then, a set S of vertices is a minimal separator of H contained in C if and only if S is a minimal separator of G contained in C.*

Corollary 6 *Let $G = (V, E)$ be a graph, and let H be a minimal triangulation of G. Then, $\mathrm{con}(H) \sqsubseteq \mathrm{con}(G)$.*

Proof: Theorem 5 immediately implies the following: Let C be a maximal clique of H. By definition, C is a potential maximal clique of G. Let a and b be vertices of G. Then, $[a, b]$ is a plug pair for C in H if and only if $[a, b]$ is a plug pair for C in G. Hence, a maximal clique C of H is a connector if and only if the potential maximal clique C of G is a connector, and two connectors of H correspond to adjacent vertices in $\mathrm{con}(H)$ if and only if they correspond to adjacent vertices in $\mathrm{con}(G)$. ∎

4 Connectors graphs of chordal graphs Mainly, we are interested in the construction and structural properties of connectors graphs of chordal graphs. First, we show that the connectors graph of a chordal graph can be constructed from a clique-tree. Here, it is important to verify that the necessary properties are invariant with respect to clique-trees.

Lemma 7 *Let $G = (V, E)$ be a chordal graph. Let C be a maximal clique of G and let $[a, b]$ be a pair of vertices of G*

(1) If $[a,b]$ is a plug pair for C then for every clique-tree T for G, there are two maximal cliques C' and C'' of G such that C' and C'' are neighbours of C with respect to T and $a \in C \cap C'$ and $b \in C \cap C''$ and no maximal clique of G except for C contains a and b.

(2) Let T be a clique-tree for G. If there are two maximal cliques C' and C'' of G such that C' and C'' are neighbours of C with respect to T and $a \in C \cap C'$ and $b \in C \cap C''$ and no maximal clique of G except for C contains a and b, then $[a,b]$ is a plug pair for C.

Proof: We prove the two statements separately. We begin with the first statement. Let $[a,b]$ be a plug pair for C. By definition, there are minimal separators S_a and S_b of G contained in C such that $a \in S_a$ and $b \in S_b$ and there is no minimal separator of G contained in C that contains a and b. Let T be a clique-tree for G. Let C'_a and C''_a be maximal cliques of G that are neighbours with respect to T such that $C'_a \cap C''_a = S_a$; they exist due to Theorem 2. Let C_a be the maximal clique of G on the path between C and C''_a with respect to T that is a neighbour of C with respect to T. Then, $a \in C_a$ by the definition of clique-trees. Similarly, there is a maximal clique C_b of G that is a neighbour of C with respect to T such that $b \in C_b$. In the proof of Lemma 4, it is already shown that no maximal clique except for C contains a and b. Hence, we conclude this implication.

For the second statement, let C' and C'' be maximal cliques of G such that C' and C'' are neighbours of C with respect to T and $a \in C \cap C'$ and $b \in C \cap C''$ and there is no maximal clique of G except for C containing a and b. Due to Theorem 2, $C \cap C'$ and $C \cap C''$ are minimal separators of G. If there is a minimal separator of G containing a and b, there must be a second maximal clique of G containing a and b, which does not exist by assumption. Hence, there is no minimal separator of G contained in C that contains a and b, so that $[a,b]$ is a plug pair for C. ∎

A *blank graph* is a graph without edges.

Theorem 8 *Let $G = (V,E)$ be a chordal graph. If G is P_5-free, then the connectors graph of G is a blank graph.*

Proof: We show the contraposition of the statement. Let C_1 and C_2 be maximal cliques of G whose corresponding vertices in the connectors graph are adjacent. By definition, there are vertices a,b,c of G such that $[a,b]$ and $[b,c]$ are plug pairs for C_1 and C_2, respectively. Let T be a clique-tree for G. Applying Lemma 7, there is a maximal clique C_0 of G that is a neighbour of C_1 with respect to T and not on the path between C_1 and C_2 with respect to T and containing a. Since C_0 is a maximal clique, there is a vertex u in C_0 that is not contained in C_1. Hence, (u,a,b,c) is a chordless path on four vertices in G. Analogously, there is a vertex v in G such that (u,a,b,c,v) is a chordless path on five vertices in G. ∎

The converse of Theorem 8 is not true. Consider the chordal graph on the seven vertices a,b,c,d,e,f,g and with the following maximal cliques: ab, bcd, bde, df, fg. The path (a,b,d,f,g) is chordless, but the connectors graph is blank. However, when we delete vertex c, the connectors graph has an edge between bde and df. Note that bcd contains two minimal separators, that can be ordered by inclusion. Two graph classes whose connectors graphs are described completely by Theorem 8 are the classes of split graphs and of trivially perfect, or P_4-free chordal, graphs. Both graph classes contain only P_5-free graphs.

To finish this section, we show that connectors graphs of interval graphs can have any structure. We prove this statement by giving a concrete construction. A vertex of a graph is called *isolated*, if does not have any neighbour.

Theorem 9 *Let $G = (V, E)$ be a graph with two isolated vertices. Then, there is an interval graph whose connectors graph is (isomorphic to) G.*

Proof: For simplicity, we assume that G has the vertices $0, 1, \ldots, n + 1$, where 0 and $n + 1$ are isolated vertices. Let G have m edges. We construct an interval graph H with $n + 2$ maximal cliques and $2n + 2 + m$ vertices as follows. Let $a_1, b_1, a_2, \ldots, b_n$ and c' and c'' be vertices. For every $i \in \{1, \ldots, n\}$, let $C_i =_{\text{def}} \{a_1, \ldots, a_i, b_i, \ldots, b_n\}$, and let $C_0 =_{\text{def}} \{c', a_1, \ldots, a_n\}$ and $C_{n+1} =_{\text{def}} \{c'', b_1, \ldots, b_n\}$. Let H be the graph whose maximal cliques are exactly the sets $C_0, \ldots, C_{n+1}$. Note that $\langle C_0, C_1, \ldots, C_{n+1}\rangle$ is a consecutive clique arrangement for H, so that H is an interval graph. Furthermore, $[a_i, b_i]$ is a plug pair for C_i, $i \in \{1, \ldots, n\}$, so that C_i is a connector. C_0 and C_{n+1} are not connectors. Note that the connectors graph of H is blank. To add edges to the connectors graph, we add further vertices to H, but no edges between already contained vertices. Hence, $C_1, \ldots, C_n$ will remain connectors in H. Let xy be an edge of G, $x < y$. Let u_{xy} be a new vertex. Add u_{xy} to the cliques $C_x, \ldots, C_y$. Then, $\langle C_0, \ldots, C_{n+1}\rangle$ is still a consecutive clique arrangement for the new graph H, and $[a_x, u_{xy}]$ and $[u_{xy}, b_y]$ are plug pairs for C_x and C_y, respectively. Hence, the connectors graph of H contains an edge between the vertices corresponding to C_x and C_y. After adding a vertex for every edge of G, the connectors graph of the finally constructed interval graph is G. ■

We see that connectors graphs of well-known chordal graph classes have no special property. This is an important property of connectors graphs, since they are made for analysing minimal triangulations of non-chordal graphs. So, any structural property of the connectors graphs of a considered graph class may yield a non-trivial result about the structure of the minimal triangulations. To complete Theorem 9, note that every chordal graph that is not complete has two maximal cliques that are not connectors.

5 Connectors graphs of permutation graphs Let M be a finite set of size n, and let σ_1 and σ_2 be two orderings over M. The graph on M defined by σ_1 and σ_2 has a vertex for every element of M, and two vertices of M are adjacent if and only if the corresponding elements are ordered differently by σ_1 and σ_2. Considering σ_1 and σ_2 as binary relations, the edge set of the defined graph corresponds to the complement of the intersection, $\sigma_1 \cap \sigma_2$. Graphs defined in this way are called *permutation graphs*, and we denote the graph defined by the pair (σ_1, σ_2) as $G(\sigma_1, \sigma_2)$. With a permutation graph, we associate a so-called *permutation diagram*, which is defined as follows: on two horizontal lines, mark points for each vertex and label the points with the names of the vertices in order defined by σ_1 for the upper line and σ_2 for the lower line. Connect the two points with the same label by a line segment. It is an easy property that two vertices of the permutation graph are adjacent if and only if the corresponding line segments of the permutation diagram intersect. For further information on permutation graphs and diagrams, we refer to the book by Golumbic [10]. To give a sketch of this section, we will characterise the connectors of a permutation graph, identify a structural property of adjacent connectors in the connectors graph and combine these results to the main theorem about connectors graphs of permutation graphs. Our results are best understood using permutation diagrams as representation models for permutation graphs.

For studying potential maximal cliques of permutation graphs, minimal separation lines are the best means. Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph on n vertices. A *separation line* s of G is a pair (a, e) where $a, e \in \{\frac{1}{2}, 1\frac{1}{2}, \ldots, n\frac{1}{2}\}$. Separation lines for permutation graphs were introduced by Bodlaender, Kloks, Kratsch [2] (under the name *scanlines*). Let $s_1 = (a_1, e_1)$ and $s_2 = (a_2, e_2)$ be two separation lines of G. If $a_1 \leq a_2$ and $e_1 \leq e_2$, we also write $s_1 \leq s_2$; if $s_1 \leq s_2$ and $s_1 \neq s_2$, we write $s_1 < s_2$. Let $\sigma_1 = \langle x_1, \ldots, x_n \rangle$ and $\sigma_2 = \langle x_{\pi(1)}, \ldots, x_{\pi(n)} \rangle$ for some permutation π over $\{1, \ldots, n\}$. Separation line (a, e) *crosses* vertex x_i, if either $i < a$ or $\pi^{-1}(i) < e$. The set of vertices crossed by s is denoted as $\text{int}(s)$. If $i < a$ and x_i is not crossed by s, then x_i is *to the left of* s; if $i > a$ and x_i is not crossed by s, then x_i is *to the right of* s. For s_1 and s_2 and vertex x, if $s_1 < s_2$ and x is to the left of s_2 and to the right of s_1, we say that x is *between* s_1 *and* s_2. If $\{x_i : i \in \{a - \frac{1}{2}, a\frac{1}{2}, \pi(e - \frac{1}{2}), \pi(e\frac{1}{2})\}\} \cap \text{int}(s) = \emptyset$, s is called a *minimal separation line* of G. Note that the first set may contain between one and four vertices. If s_1 and s_2 are minimal separation lines of G and $s_1 < s_2$ and there is no minimal separation line t of G such that $s_1 < t < s_2$, then s_2 is called a *successor* of s_1.

Theorem 10 **[13]** *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let C be a set of vertices of G. Then, C is a potential maximal clique of G if and only if there is a pair of minimal separation lines of G, s_1 and s_2, such that s_2 is a successor of s_1 and C is equal to the union of $\text{int}(s_1) \cup \text{int}(s_2)$ and the set of vertices between s_1 and s_2. Separation lines s_1 and s_2 are unique for C.*

Minimal separators of permutation graphs have a representation that is similar to the one for potential maximal cliques.

Theorem 11 **[13]** *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph on n vertices. A set S of vertices of G is a minimal separator of G if and only if there is a minimal separation line s of G where $(\frac{1}{2}, \frac{1}{2}) < s < (n\frac{1}{2}, n\frac{1}{2})$ such that $S = \text{int}(s)$.*

Using the two characterisations of potential maximal cliques and of minimal separators, we can characterise the connectors and plug pairs of permutation graphs. For the rest of this section, every potential maximal clique of a permutation graph is associated with a pair of minimal separation lines that define the potential maximal clique in the sense of Theorem 10. Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let s_1 and s_2 be minimal separation lines of G where s_2 is a successor of s_1. Then, we define $L(s_1, s_2) =_{\text{def}} \text{int}(s_1) \setminus \text{int}(s_2)$ and $R(s_1, s_2) =_{\text{def}} \text{int}(s_2) \setminus \text{int}(s_1)$.

Lemma 12 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let C be a potential maximal clique of G defined by the pair (s_1, s_2) of minimal separation lines of G where s_2 is a successor of s_1. Then, $[a, b]$ is a plug pair for C if and only if $a \in L(s_1, s_2)$ and $b \in R(s_1, s_2)$ or $b \in L(s_1, s_2)$ and $a \in R(s_1, s_2)$.*

Proof: Let G have n vertices. By definition of C, $\text{int}(s_1)$ and $\text{int}(s_2)$ are contained in C. First, let $a \in L(s_1, s_2)$ and $b \in R(s_1, s_2)$. In particular, $a \in \text{int}(s_1)$ and $b \in \text{int}(s_2)$, and $\text{int}(s_1)$ and $\text{int}(s_2)$ are non-empty. So, $(\frac{1}{2}, \frac{1}{2}) < s_1 < s_2 < (n\frac{1}{2}, n\frac{1}{2})$. Then, $\text{int}(s_1)$ and $\text{int}(s_2)$ are minimal separators of G due to Theorem 11. Observe that C does not contain another minimal separator: there is no minimal separation line s such that $s_1 < s < s_2$ by the choice of s_2, and every minimal separation line different from s_1 and s_2 that crosses a vertex crosses a vertex that is not contained in C. Hence, there is no minimal separator of G contained in C that contains a and b, so that $[a, b]$ is a plug pair for C according to the definition.

For the converse, let $[a, b]$ be a plug pair for C, and let S_1 and S_2 be the minimal separators of G contained in C that contain a and b, respectively. Since S_1 and S_2 must be different minimal separators and since C contains at most two minimal separators, we

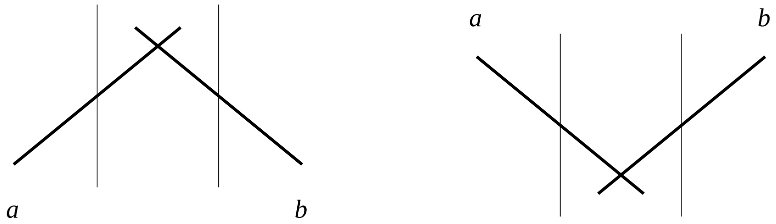

Figure 3 The figure shows the two possible situations for the vertices of a plug pair $[a, b]$ in a connector of a permutation graph. The vertical lines represent the two minimal separation lines defining the connector. The left hand figure illustrates the case when vertices a and b meet at the top whereas a and b meet at the bottom in the right hand figure.

can assume $S_1 = \text{int}(s_1)$ and $S_2 = \text{int}(s_2)$. Hence, $a \in L(s_1, s_2)$ and $b \in R(s_1, s_2)$. ■

Using Lemma 12, the connectors graph of a permutation graph can be generated easily. The same lemma and the definition of permutation diagrams justify the following definition.

Definition 2 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let C be a potential maximal clique of G defined by the pair (s_1, s_2) of minimal separation lines of G where s_2 is a successor of s_1. Let C be a connector, and let $[a, b]$ be a plug pair for C where we assume $a \in L(s_1, s_2)$ and $b \in R(s_1, s_2)$. We say that a and b* ***meet at the top*** *if and only if $b \prec_{\sigma_1} a$. Otherwise, we say that a and b* ***meet at the bottom****.*

From Lemma 4, we know that the vertices of a plug pair are adjacent. Taking into account this fact, the definition of "meeting at the top/bottom" has a clear geometric representation. It is illustrated in Figure 3. We show now that "meeting at the top/bottom" is a property of the connector, not only of a single plug pair.

Lemma 13 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let C be a connector of G. Then, there is a plug pair $[a, b]$ for C such that a and b meet at the top if and only if for every plug pair $[c, d]$ for C, c and d meet at the top.*

Proof: Let C be defined by the pair (s_1, s_2) of minimal separation lines of G where s_2 is a successor of s_1. According to Lemma 12, every pair $[c, d]$ of vertices where $c \in L(s_1, s_2)$ and $d \in R(s_1, s_2)$ is a plug pair for C. It follows from Lemma 4 that c and d are adjacent in G for every $c \in L(s_1, s_2)$ and $d \in R(s_1, s_2)$. Now, let $[a, b]$ be a plug pair for C such that a and b meet at the top. Without loss of generality, $a \in L(s_1, s_2)$ and $b \in R(s_1, s_2)$. Then, a and d for every $d \in R(s_1, s_2)$ meet at the top; similarly, c and b for every $c \in L(s_1, s_2)$ meet at the top. If there is a pair $[c, d]$, $c \in L(s_1, s_2)$ and $d \in R(s_1, s_2)$, such that c and d meet at the bottom, it follows that $b \prec_{\sigma_1} c \prec_{\sigma_1} d \prec_{\sigma_1} a$, which means that c and d cannot be adjacent. This, however, is a contradiction. ■

Lemma 13 motivates the following definition for connectors.

Definition 3 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let C be a connector of G. We say that C is* ***oriented to the top*** *if and only if there is a plug pair $[a, b]$ for C such that a and b meet at the top. Otherwise, we say that C is* ***oriented to the bottom****.*

Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. For two connectors A and B of G, we write $A \parallel B$, if both A and B are oriented to the top or to the bottom; otherwise, we write $A \perp B$. Remember that the definition of a connector and Lemma 13 imply that a connector of a permutation graph is oriented either to the top or to the bottom. The following lemma presents the crucial property from which we will derive our main result in this section. Let

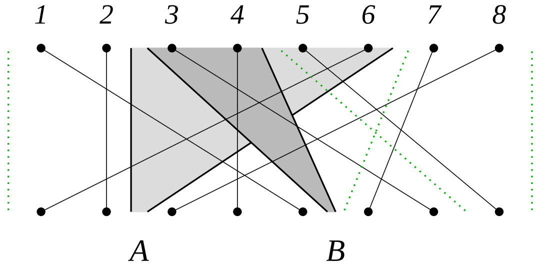

Figure 4 Depicted is a permutation diagram of a permutation graph on eight vertices. The dotted and the thick line segments represent the minimal separation lines.

A and B be potential maximal cliques of G defined by pairs (s_1, s_2) and (t_1, t_2) of minimal separation lines of G, respectively. Let s_2 and t_2 be successors of s_1 and t_1, respectively. We say that A and B are *parallel*, if $s_1 < s_2 \leq t_1 < t_2$ or $t_1 < t_2 \leq s_1 < s_2$.

Lemma 14 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let A and B be parallel connectors of G. Let $[a, b]$ be a plug pair for A and let $[b, c]$ be a plug pair for B. Then, $A \perp B$.*

Proof: Let A and B be defined by the pairs (s_1, s_2) and (t_1, t_2) of minimal separation lines of G, respectively, where s_2 and t_2 are successors of s_1 and t_1, respectively. Without loss of generality, we assume $s_2 \leq t_1$. Then, $b \in R(s_1, s_2) \cap L(t_1, t_2)$. Let A be oriented to the top. Then, a and b meet at the top (in A) due to Lemma 13, i.e., $b \prec_{\sigma_1} a$. Since $c \in R(t_1, t_2)$, $b \prec_{\sigma_1} c$, and b and c meet at the bottom, i.e., B is oriented to the bottom. Similarly, if A is oriented to the bottom, B is oriented to the top. ∎

It would be nice, if the prerequisite of parallel connectors in Lemma 14 was not necessary. But a simple example shows that the statement does not hold for non-parallel connectors in general. Consider the permutation graph G represented by the permutation diagram in Figure 4. The two pairs of strong lines (minimal separation lines) define two potential maximal cliques A and B of G. Obviously, A and B are not parallel. Both A and B are connectors: A has plug pair $[3, 6]$ and B has plug pair $[3, 4]$. And both connectors are oriented to the top.

Now, we can prove our main result. It is an extension of the result of Lemma 14.

Theorem 15 *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph. Let S be a set of vertices of the connectors graph $\mathrm{con}(G)$ of G that correspond to pairwise parallel potential maximal cliques. Then, S induces a bipartite subgraph of $\mathrm{con}(G)$.*

Proof: Let $A_1, \ldots, A_k$ be the potential maximal cliques of G corresponding to the vertices in S. Let $S = \{x_1, \ldots, x_k\}$, where x_i corresponds to A_i, $i \in \{1, \ldots, k\}$. Due to Theorem 10, there is a unique pair (s'_i, s''_i) of minimal separation lines of G such that s''_i is a successor of s'_i and (s'_i, s''_i) represents A_i in the sense of Theorem 10, $i \in \{1, \ldots, k\}$. Since $A_1, \ldots, A_k$ are pairwise parallel, we can assume without loss of generality that $s'_1 < s''_1 \leq s'_2 < \cdots < s''_k$. Now, assume that S does not induce a bipartite subgraph of $\mathrm{con}(G)$. Then, there is a cycle $C = (x_{i_1}, \ldots, x_{i_l})$ of odd length in G. Due to Lemma 14, $A_{i_1} \perp A_{i_2}, A_{i_2} \perp A_{i_3}, \ldots, A_{i_{l-1}} \perp A_{i_l}, A_{i_l} \perp A_{i_1}$. Remember that it is important to require the potential maximal cliques to be mutually parallel. Since $l - 1$ is an even number, we obtain $A_{i_1} \parallel A_{i_l}$, from which follows the contradiction $A_{i_1} \perp A_{i_1}$. Hence, C cannot exist, and S induces a bipartite subgraph of $\mathrm{con}(G)$. ∎

Note that Theorem 15 does not make a statement about the structure of the whole

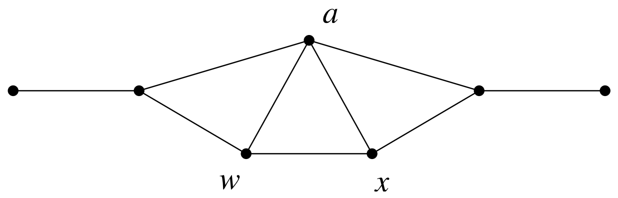

Figure 5 The depicted graph H is an interval graph and minimal triangulation of the graphs H, $H-aw$, $H-ax$, $H-aw,ax$. None of the four graphs is a permutation graph, which can be verified easily, since a chordless path on six vertices has an almost unique representation in the permutation diagram.

connectors graph.

6 Characterising minimal triangulations of permutation graphs In this section, we present our main result. We will give the first non-trivial characterisation of the class of those chordal graphs that are minimal triangulations of permutation graphs. It is known that minimal triangulations of permutation graphs are interval graphs (Theorem 1). However, it is not known whether every interval graph can be a minimal triangulation of a permutation graph. And this question must be answered negatively. Figure 5 depicts an interval graph H (easy to see: a path on six vertices and a vertex adjacent with four consecutive path vertices). This interval graph is a minimal triangulation of H itself and the three subgraphs $H-aw$, $H-ax$ and $H-aw,ax$, and none of these four graphs is a permutation graph.

Our characterisation proof consists of two parts: combining all the results of the previous sections, we can exclude a large number of interval graphs from the class of minimal triangulations of permutation graphs, and for showing that the remaining interval graphs are minimal triangulations of permutation graphs, we construct a witness (a permutation graph). The proof also relies on the following theorem, that characterises the set of minimal triangulations of a single permutation graph.

Theorem 16 [13] *Let $G = G(\sigma_1, \sigma_2)$ be a permutation graph, and let H be a chordal graph on the vertex set of G. Then, H is a minimal triangulation of G if and only if the set of maximal cliques of G is a maximal set of pairwise parallel potential maximal cliques of G.*

Theorem 16 does not directly say that minimal triangulations of permutation graphs are interval graphs. This, however, is an immediate corollary from the characterisation of the potential maximal cliques of a permutation graph applying Theorem 10 and the definition of parallel potential maximal cliques.

Theorem 17 *Let $H = (V, E)$ be an interval graph. Then, H is a minimal triangulation of a permutation graph if and only if* $\mathrm{con}(H)$ *is bipartite.*

Proof: Let H be a minimal triangulation of a permutation graph G. Let $\mathcal{C}$ be the set of maximal cliques of H. Due to Theorem 16, $\mathcal{C}$ is a set of pairwise parallel potential maximal cliques of G. Due to Theorem 15, the set of vertices of $\mathrm{con}(G)$ corresponding to the sets in $\mathcal{C}$ induces a bipartite subgraph of $\mathrm{con}(G)$, which is equal to $\mathrm{con}(H)$ according to Corollary 6. Hence, $\mathrm{con}(H)$ is bipartite.

For showing the converse, we construct a permutation graph G and show that it has H as a minimal triangulation. So, let $\mathrm{con}(H)$ be bipartite. Then, there exists a 2-colouring f for $\mathrm{con}(H)$, using the colours *yellow* and *blue*. Let $\mathfrak{A} = \langle A_1, \ldots, A_k \rangle$ be a consecutive clique arrangement for H, and let $f(A_i)$ be the colour of the vertex in $\mathrm{con}(H)$ A_i corresponds to. A maximal clique of H can have the following kinds of vertices (we assume $A_0 =_{\mathrm{def}} A_{k+1} =_{\mathrm{def}} \emptyset$):

(K1) vertices in $A_i \setminus (A_{i-1} \cup A_{i+1})$; these vertices are called *inner* vertices of A_i

(K2) vertices in $(A_{i-1} \cap A_i) \setminus A_{i+1}$; these vertices are called *left end* vertices of A_i

(K3) vertices in $(A_i \cap A_{i+1}) \setminus A_{i-1}$; these vertices are called *right end* vertices of A_i.

It is clear that (K1), (K2) and (K3) define pairwise disjoint sets of vertices. Furthermore, vertices in $A_{i-1} \cap A_{i+1}$ do not belong to any of these vertex sets. However, every vertex of H is either inner vertex or left end vertex of a maximal clique of H. Note that by Lemma 7, a maximal clique of H is a connector if and only if it has left end and right end vertices.

The definition of G is done by giving an appropriate permutation diagram. The two defining vertex orderings can immediately be read from the permutation diagram. The permutation diagram is defined in several steps. The description of each step is accompanied by an illustrating figure (see Figure 6). These illustrations depict the situation during the generation of a permutation diagram after each step of the algorithm: we generate a permutation graph with the interval graph represented by the interval model in Figure 6(a) as a minimal triangulation. The algorithm first defines a rough structure and then refines it. We begin with the two horizontal lines for the permutation diagram and define k rectangles between the two lines. Every rectangle corresponds to a maximal clique of H, ordered according to $\mathfrak{A}$. Colour every rectangle with the colour of the corresponding maximal clique with respect to f. Assign orientations to those rectangles that correspond to connectors: *yellow* means 'oriented to the top', *blue* means 'oriented to the bottom' (Figure 6(b)). This step was a preparing step. Now, add the line segments, that represent the vertices of the permutation graph, to the permutation diagram:

- an inner vertex is contained only in the box corresponding to its maximal clique
- the line segment of a left end vertex has its left end in the corresponding box
- the line segment of a right end vertex has its right end in the corresponding box.

If possible, orient the line segments according to the orientations of the boxes. The result of this step is illustrated in Figure 6(c). Note that this step heavily relies on the fact that the connectors graph of H is bipartite, so that the line segment of every vertex that is end vertex in two connectors can have its two endpoints on the two horizontal lines without violating the orientation rule (for example, vertices d and c in Figure 6(c)). Now, every line segment whose corresponding vertex is end vertex in at least one connector is properly oriented. For the remaining line segments (the line segments of vertices b, e, f in the example), we choose an arbitrary orientation. A possible result for our example is given in Figure 6(d). The last step arranges the line segment endpoints in each rectangle such that they are ordered from left to right obeying the following rule on each horizontal line: left end vertices, inner vertices, right end vertices. Additionally, line segments of inner vertices pairwise intersect. The result is presented in Figure 6(e). This defines a permutation graph G. The final permutation diagram for our example is depicted in Figure 6(f).

It remains to show that H is indeed a minimal triangulation of G. Applying Theorem 16, it suffices to show that every rectangle defines a potential maximal clique of G. First observe that the common border of two rectangles defines a minimal separation line (thin lines in Figure 6(f)): every rectangle corresponds to a maximal clique of H, and every maximal clique contains an inner vertex or left and right end vertices. Second, there is no minimal separation line that is properly contained in a rectangle: a minimal separation line s lies between two non-adjacent vertices, i.e., non-intersecting line segments in the permutation diagram. Two non-adjacent vertices of G that are contained in one rectangle are either both left end or both right end vertices or one vertex is left or right end vertex and the other is unclassified. In all these cases, s intersects with the border of a rectangle. Hence, the borders of the rectangles define a maximal set of pairwise non-intersecting minimal separation lines and therefore, every rectangle defines a potential maximal clique of G in the sense of Theorem 10. This shows that H is a minimal triangulation of G. ∎

The characterisation of minimal triangulations of permutation graphs of Theorem 17 can be used to design a fast recognition algorithm for this class of interval graphs. Furthermore, we can use the algorithm of the proof of the characterisation to construct a permutation graph witnessing that an input interval graph is indeed a minimal triangulation of a permutation graph. Our algorithms require the input graph to be given by an interval model. Then, the algorithms take time only proportional in the number of vertices. We call such a running time *vertex-linear*.

Corollary 18 *There is a vertex-linear-time algorithm deciding the following question: given an interval graph H represented by an interval model, is H a minimal triangulation of a permutation graph. In the positive case, the algorithm outputs a permutation graph G represented by a permutation diagram and a set of pairwise non-intersecting minimal separation lines proving that H is a minimal triangulation of G.*

Proof: Let $\mathfrak{I}$ be the given interval model for H. To decide whether H is a minimal triangulation of a permutation graph, it suffices to generate the connectors graph of H and decide whether it is bipartite (Theorem 17). Using the definitions of the proof of Theorem 17 and the result of Lemma 7, a maximal clique C of H is a connector if and only if it has a left end and a right end vertex with respect to any consecutive clique arrangement of H. Let $\mathfrak{A} = \langle A_1, \ldots, A_k \rangle$ be a consecutive clique arrangement for H. Remember that $\mathfrak{A}$ corresponds to a clique-tree that is a path, and note that it can be obtained directly from $\mathfrak{I}$. It is clear that it is not always possible to determine all maximal cliques of an interval graph in vertex-linear time. However, for every vertex x of H, the smallest and largest numbers i, denoted as $L(x)$ and $R(x)$, such that $x \in A_i$ can be determined in total vertex-linear time. Then, a maximal clique is a connector if and only if it contains two vertices a and b such that $L(a) < R(a) = L(b) < R(b)$. Note that $[a, b]$ then is a plug pair for the maximal clique. The connectors graph of H can be generated as follows:

(1) determine the number of maximal cliques of H and generate a vertex for every maximal clique

(2) two vertices u and v are adjacent where u and v correspond to the maximal cliques A_i and A_j, respectively, $i < j$, if and only if A_i and A_j are connectors and there is a vertex w such that $L(w) = i$ and $R(w) = j$.

Having computed the numbers $L(x)$ and $R(x)$ for every vertex x, the connectors of H can be determined by sweeping through the vertex list. This takes vertex-linear time. Then, the edges of $\mathrm{con}(H)$ are listed again by a sweep through the vertex list. Thus, in vertex-linear time, $\mathrm{con}(H)$ is generated. Note that the number of vertices and edges of $\mathrm{con}(H)$ is bounded by the number of vertices of H. Then, in vertex-linear time, it can be decided whether $\mathrm{con}(H)$ is bipartite, and if so, a 2-colouring can be computed. Checking every step of the proof of Theorem 17, it is clear that a permutation graph G represented by a permutation diagram can be generated in vertex-linear time. It also shows how to define

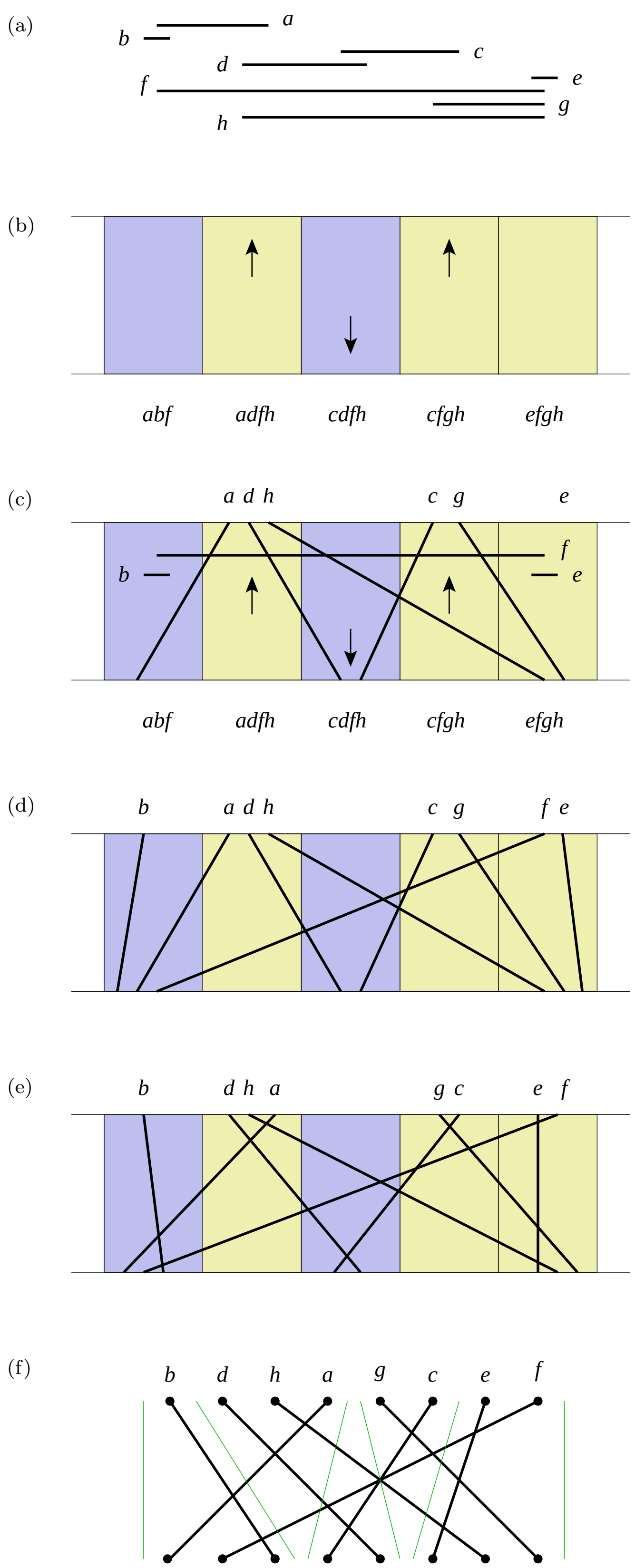

Figure 6 Illustrating the steps of an algorithm for generating a permutation graph having a given interval graph with bipartite connectors graph as minimal triangulation. The algorithm is developed in the proof of Theorem 17.

the set of minimal separation lines. So, we conclude the proof. ∎

7 Final remarks We gave a characterisation of the minimal triangulations of permutation graphs. This characterisation is based on connectors graphs, that were introduced here, and shows a connection between interval graphs that are minimal triangulations of permutation graphs and bipartite graphs. This result also implies a statement about the number of interval graphs that are minimal triangulations of permutation graphs. Even though there is not much known about connectors graphs of interval graphs, it seems quite natural to assume that interval graphs are equally distributed among the connectors graphs. And if we accept that the class of bipartite graphs is only a small, nevertheless important, graph class, we can conclude that only “a few” interval graphs can be minimal triangulations of permutation graphs.

Finding a characterisation of the class of minimal triangulations of permutation graphs was a non-trivial and interesting task, since the class of permutation graphs does not contain the class of minimal triangulations of permutation graphs. Other graph classes with this property are the class of P_6-free graphs [15] and the class of gem-free graphs. The gem is a minimal triangulation of the C_5. The most natural such graph class, however, is the class of comparability graphs. The question is whether our techniques can be applied to these or similar graph classes to obtain results about the classes of their minimal triangulations. One step could be to identify structural properties that are preserved during the filling process.

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