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# Bounding the Number of Minimal Dominating Sets: a Measure and Conquer Approach

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**Abstract** We show that the number of minimal dominating sets in a graph on $n$ vertices is at most $1.7697^n$, thus improving on the trivial $\mathcal{O}(2^n/\sqrt{n})$ bound. Our result makes use of the measure and conquer technique from exact algorithms, and can be easily turned into an $\mathcal{O}(1.7697^n)$ listing algorithm.
Based on this result, we derive an $\mathcal{O}(2.8805^n)$ algorithm for the domatic number problem, and an $\mathcal{O}(1.5780^n)$ algorithm for the minimum-weight dominating set problem. Both algorithms improve over the previous algorithms.

Keywords: *exact (exponential) algorithms, minimum dominating set, minimum set cover, domatic number, weighted dominating set*

## 1 Introduction

One of the typical questions in graph theory is: how many subgraphs satisfying a given property can a graph on $n$ vertices contain? For example, the number of perfect matchings in a simple $k$-regular bipartite graph on $2n$ vertices is always between $n!(k/n)^n$ and $(k!)^{n/k}$. (The first inequality was known as van der Waerden Conjecture [22] and was proved in 1980 by Egorychev [6] and the second is due to Bregman [2].) Another example is the famous Moon and Moser [18] theorem stating that every graph on $n$ vertices has at most $3^{n/3}$ maximal cliques (independent sets). Such combinatorial bounds are of interests not only on their own but also because they are used for algorithm design as well. Lawler [17] used Moon-Moser bound on the number of maximal independent sets to construct an $\mathcal{O}((1+\sqrt[3]{3})^n)$ time graph coloring algorithm which was the fastest coloring algorithm for 25 years. Recently Byskov and Eppstein [3] obtain an $\mathcal{O}(2.1020^n)$ time coloring algorithm which is, again, based on a $1.7724^n$ combinatorial upper bound on the number of maximal bipartite subgraphs in a graph.

The dominating set problem is a classic NP-complete graph optimization problem which fits into the broader class of domination and covering problems. Hundreds of papers have been written on them; see e. g. the survey [14] by Haynes et al. However, despite the importance of minimum dominating set problem, nothing better than the trivial $\mathcal{O}(2^n/\sqrt{n})$ bound was known on the number of minimal dominating sets in a graph.

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Our interest is motivated by the design of fast exponential-time algorithms for hard problems. The story of such algorithms dates back to the sixties and seventies. In 1962 Held and Karp presented an $\mathcal{O}(n^2\, 2^n)$ time algorithm for the travelling salesman problem which is still the fastest one known [15]. In 1977 Tarjan and Trojanowski [21] gave an $\mathcal{O}(2^{n/3})$ algorithm for maximum independent set problem. The last decade has seen a growing interest in fast exponential-time algorithms for NP-hard problems. Examples of recently developed fast exponential algorithms are algorithms for maximum independent set [1], satisfiability [4,16], coloring [7], treewidth [10], and many others. For a good overview of the field we refer to the recent survey written by Woeginger [23].

**Previous results.** Although minimum dominating set is a natural and very interesting problem concerning the design and analysis of exponential-time algorithms, no exact algorithm for it faster than $2^n \cdot n^{O(1)}$ had been known until very recently. In 2004 several different sets of authors obtained algorithms breaking the trivial "$2^n$-barrier". The algorithm of Fomin et al. [11] runs in time $\mathcal{O}(1.9379^n)$. The algorithm of Randerath and Schiermeyer [19] uses a very nice and cute idea (including matching techniques) to restrict the search space. The most time consuming part of their algorithm enumerates all subsets of nodes of cardinality at most $n/3$, thus the overall running time is $\mathcal{O}^*(1.8999^n)$. Grandoni [12,13], described a $\mathcal{O}(1.8019^n)$ algorithm and finally, Fomin et al. [9] reduced the running time to $\mathcal{O}(1.5137^n)$. All the mentioned results work only in the unweighted case, and cannot be used to list all the minimal dominating sets. The best algorithm for the weighted case prior to this paper is the trivial $\mathcal{O}(2^n n^{\mathcal{O}(1)})$ one.

There are not so many known exact algorithms for the domatic number. Applying an algorithm similar to Lawler's dynamic programming algorithm [17] to the domatic number problem one obtains an $3^n \cdot n^{\mathcal{O}(1)}$ algorithm. Nothing better was known for this problem. For three domatic number problem, which is a special case of the domatic number problem, very recently Reige and Rothe succeed to break the $3^n$ barrier with an $\mathcal{O}(2.9416^n)$ algorithm [20].

**Our results.** In this paper we show that the number of minimal dominating sets in a graph on $n$ vertices is at most $1.7697^n$. Our result is inspired by the *measure and conquer* technique [9] from exact algorithms, which works as follows. The running time of exponential recursive algorithms is usually bounded by measuring the progress made by the algorithm at each branching step. Though these algorithms may be rather complicated, the measures used in their analysis are often trivial. For example in graph problems the progress is usually measured in terms of number of nodes removed. The idea behind measure and conquer is to chose the measure more carefully: a good choice can lead to a tremendous improvement of the running time bounds (for a fixed algorithm). One of the main contributions of this paper is showing that the same basic idea can be successfully applied to derive stronger combinatorial bounds. In particular, the inductive proof of Theorem 1 is based on the way we choose the measure of the problem.

Our combinatorial result is algorithmic in spirit, and can be easily turned into an algorithm listing all minimal dominating sets in time $\mathcal{O}(1.7697^n)$. Based on the listing algorithm, we derive an $\mathcal{O}(1.5780^n)$ algorithm for the minimum-weight dominating set problem, and an $\mathcal{O}(2.8805^n)$ algorithm for the domatic number. Both algorithms improve on previous best trivial bounds. Note that our algorithm for the domatic number is even faster than the (non-trivial) algorithm of Reige and Rothe [20] for the three domatic number problem, which is a special case.

## 2 Definitions and preliminaries

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a *dominating set* for $G$ if every vertex of $G$ is either in $D$, or adjacent to some node in $D$. A dominating set is *minimal* if all its proper subsets are not dominating. We define $\mathbf{DOM}(G)$ to be the number of minimal dominating sets in a graph $G$. The *domination number* $\gamma(G)$ of a graph $G$ is the cardinality of a smallest dominating set of $G$. The *Minimum Dominating Set* problem (`MDS`) asks to determine $\gamma(G)$. The *domatic number* $\mathbf{DN}(G)$ of a graph $G$ is the maximum $k$ such that the vertex set $V(G)$ can be split into $k$ pairwise nonintersecting dominating sets. Since any dominating set contains a minimal dominating set, the domatic number $\mathbf{DN}(G)$ is the maximum number of pairwise nonintersecting minimal dominating sets in $G$.

In the *Minimum Set Cover* problem (`MSC`) we are given a universe $\mathcal{U}$ of elements and a collection $\mathcal{S}$ of (non-empty) subsets of $\mathcal{U}$. The aim is to determine the minimum cardinality of a subset $\mathcal{S}^* \subseteq \mathcal{S}$ which *covers* $\mathcal{U}$, i. e. such that

$$\cup_{S \in \mathcal{S}^*} S = \mathcal{U}.$$

The *frequency* of $u \in \mathcal{U}$ is the number of subsets $S \in \mathcal{S}$ in which $u$ is contained.

A covering is *minimal* if it contains no smaller covering. We denote by $\mathbf{COV}(\mathcal{U}, \mathcal{S})$ the number of minimal coverings in $(\mathcal{U}, \mathcal{S})$.

The problem of finding $\mathbf{DOM}(G)$ can be naturally reduced to finding $\mathbf{COV}(\mathcal{U}, \mathcal{S})$ by imposing $\mathcal{U} = V$ and $\mathcal{S} = \{N[v] \mid v \in V\}$. Note that $N[v] = \{v\} \cup \{u \mid uv \in E\}$ is the set of nodes dominated by $v$. Thus $D$ is a dominating set of $G$ if and only if $\{N[v] | \ v \in D\}$ is a set cover of $(\mathcal{U}, \mathcal{S})$. So, each minimal set cover of $(\mathcal{U}, \mathcal{S})$ corresponds to a minimal dominating set of $G$.

The following properties of minimal coverings are easy to verify.

**Proposition 1.** *Let $\mathcal{S}^*$ be a minimal covering of $(\mathcal{U}, \mathcal{S})$. Then the following statements hold.*

- *For every subset $S \in \mathcal{S}^*$ at least one of the elements $u \in S$ is covered only by $S$;*
- *If $\mathcal{S}^*$ contains two subsets $S_1$ and $S_2$ such that $S_1 \setminus S_2 = \{u_1\}$ and $S_2 \setminus S_1 = \{u_2\}$ then no other subset in $\mathcal{S}^*$ may contain $u_1$ or $u_2$.*

In the next section we prove an upper bound on $\mathbf{DOM}(G)$. Based on this result we show how to compute the domatic number of a graph in time $\mathcal{O}(2.8805^n)$.

## 3 Listing Minimal Dominating Sets

Here we prove the following

**Theorem 1.** *For any graph $G$ on $n$ vertices, $\mathbf{DOM}(G) < 1.7697^n$.*

*Proof.* Since every `MDS` problem can be reduced to `MSC` problem, we will prove an upper bound for $\mathbf{COV}(\mathcal{U}, \mathcal{S})$ first.

Consider an arbitrary instance of the `MSC` problem with a universe $\mathcal{U}$ of elements and a collection $\mathcal{S}$ of (non-empty) subsets of $\mathcal{U}$. Denote by $s_i$ the number of subsets of cardinality $i$ for $i = 1, 2, 3$ and by $s_4$ the number of the subsets of cardinality at least 4 in $S$. We use the following measure $k(\mathcal{U}, \mathcal{S})$ of $(\mathcal{U}, \mathcal{S})$:

$$k(\mathcal{U}, \mathcal{S}) = |\mathcal{U}| + \sum_{i=1}^{4} \varepsilon_i s_i,$$

where the values of $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4$ will be defined later. We refer to the value $k = k(\mathcal{U}, \mathcal{S})$ as to the *size* of the `MSC` problem $(\mathcal{U}, \mathcal{S})$.

Let $\mathbf{COV}(k)$ be the maximum value of $\mathbf{COV}(\mathcal{U}, \mathcal{S})$ among all `MSC` problems of size at most $k$. Let $d_2 = \min\{\varepsilon_1, \varepsilon_2 - \varepsilon_1\}, d_3 = \min\{d_2, \varepsilon_3 - \varepsilon_2\}$, and $d_4 = \min\{d_3, \varepsilon_4 - \varepsilon_3\}$. We need the following

**Lemma 1.** *$\mathbf{COV}(k) \le \alpha^k$, where $\alpha$ satisfies the following inequalities:*

$$\alpha^k \ge \max \begin{cases} r\alpha^{k-r\varepsilon_1-1}, \ r \ge 2 \\ \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4} \\ \alpha^{k-\varepsilon_4} + \alpha^{k-4-\varepsilon_4-4d_4} \\ \alpha^{k-1-\varepsilon_1-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-d_3} \\ 2\alpha^{k-2-2\varepsilon_2} \\ 2\alpha^{k-2-2\varepsilon_2-d_3} + \alpha^{k-3-2\varepsilon_2-2d_3} \\ \alpha^{k-\varepsilon_3} + \alpha^{k-3-\varepsilon_3-6d_3} \\ \alpha^{k-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-3d_2} \\ \alpha^{k-\varepsilon_2} + \alpha^{k-2-2\varepsilon_2-2d_2} \\ 3\alpha^{k-2-3\varepsilon_2-2d_2} + 3\alpha^{k-3-6\varepsilon_2} + \alpha^{k-4-6\varepsilon_2} \\ \alpha^{k-\varepsilon_2} + 2\alpha^{k-2-4\varepsilon_2-3d_2} \end{cases} \tag{1}$$

*Proof.* We use induction on $k$. Clearly, $\mathbf{COV}(0) = 1$. Suppose that $\mathbf{COV}(l) \le \alpha^l$ for every $l < k$. Let $\mathcal{S}$ be a set of subsets of $\mathcal{U}$ such that the `MSC` problem $(\mathcal{U}, \mathcal{S})$ is of size $k$. We consider different cases.

**Case 0.** *There is an element $u \in \mathcal{U}$ of frequency one.* Since $u$ must be covered by the only set $S$ containing it, we have that every minimal cover contains $S$. So,

$$\mathbf{COV}(\mathcal{U}, \mathcal{S}) \le \mathbf{COV}(\mathcal{U} \setminus \{u\}, \mathcal{S} \setminus S) \le \alpha^{k-1-\varepsilon_1}.$$

**Case 1.** *$\mathcal{U}$ has an element $u$ belonging only to subsets of cardinality one.* Let $S_1 = S_2 = \cdots = S_r = \{u\}$, where $r \ge 2$ be all the subsets containing $u$. Then by Proposition 1, every minimal covering should contain exactly one subset from $S_1, \ldots, S_r$. Thus

$$\mathbf{COV}(\mathcal{U}, \mathcal{S}) \le r \cdot \mathbf{COV}(k - r\varepsilon_1 - 1) \le r\alpha^{k-r\varepsilon_1-1}.$$

**Case 2.** *$\mathcal{S}$ has a subset with $r \ge 5$ elements.* Let $S = \{u_1, \ u_2, \ldots, u_r\}$ be such a subset. Every minimal set cover either contains $S$, or does not. The number of minimal set covers that do not contain $S$ is at most $\mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S)$. Clearly, the number of minimal set covers containing $S$ is at most $\mathbf{COV}(\mathcal{U} \setminus \{u_1, \ u_2, \ldots, u_r\}, \mathcal{S}')$. Here $\mathcal{S}'$ consists of all nonempty subsets $S' \setminus \{u_1, \ u_2, \ldots, u_r\}$ where $S' \in \mathcal{S}$. Note that $S \notin \mathcal{S}'$. Thus

$$\begin{aligned} \mathbf{COV}(\mathcal{U}, \mathcal{S}) &\le \mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S) + \mathbf{COV}(\mathcal{U} \setminus \{u_1, \ u_2, \ldots, u_r\}, \mathcal{S}') \\ &\le \mathbf{COV}(k - \varepsilon_4) + \mathbf{COV}(k - 5 - \varepsilon_4) \le \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4}. \end{aligned}$$

So, we may suppose that all subsets contain at most four elements and the minimum frequency of the elements is two.

**Case 3.** *$\mathcal{S}$ has a subset of cardinality four.* Let $S = \{u_1, \ u_2, \ u_3, \ u_4\}$ be such a subset. Again, the number of minimal set covers that do not contain $S$ is at most $\mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S)$ and the number of minimal set covers containing $S$ is at most $\mathbf{COV}(\mathcal{U} \setminus \{u_1, \ u_2, \ u_3, \ u_4\}, \mathcal{S}')$. Since there are no elements of frequency one and all subsets have cardinality at most four, removal of every element $u_1, u_2, u_3, u_4$ from $\mathcal{U}$ reduces the size of the problem by at least $d_4 + 1$. Thus

$$\begin{aligned} &\mathbf{COV}(\mathcal{U}, \mathcal{S}) \le \mathbf{COV}(\mathcal{U}, \mathcal{S} \setminus S) + \mathbf{COV}(\mathcal{U} \setminus \{u_1, \ u_2, \ u_3, u_4\}, \mathcal{S}') \\ &\le \mathbf{COV}(k - \varepsilon_4) + \mathbf{COV}(k - \varepsilon_4 - 4(d_4 + 1)) \le \alpha^{k-\varepsilon_4} + \alpha^{k-4-\varepsilon_4-4d_4}. \end{aligned}$$

Now we may suppose that all subsets contain at most three elements.

**Case 4.** *There is $u \in \mathcal{U}$ of frequency two.* Denote by $S_1$ and $S_2$ the subsets containing $u$. Let $|S_2| \geq |S_1|$. Since the condition of Case 1 does not hold, we have that $S_2$ is of cardinality at least two. There are three subcases.

**Subcase 4A.** $|S_1| = 1$. Then every minimal cover contains exactly one of these subsets. If it contains $S_1$, then we remove $u, S_1$, and $S_2$. Otherwise, we remove also the other element of the subset $S_2$ from $\mathcal{U}$ and thus, in addition, reduce the size of the problem by at least $d_3 + 1$. So we have

$$\begin{aligned}\mathbf{COV}(\mathcal{U}, \mathcal{S}) &\leq \mathbf{COV}(k - 1 - \varepsilon_1 - \varepsilon_2) + \mathbf{COV}(k - 2 - \varepsilon_1 - \varepsilon_2 - d_3) \\ &\leq \alpha^{k-1-\varepsilon_1-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-d_3}.\end{aligned}$$

**Subcase 4B.** $|S_1| \geq 2$ and $S_1 \subseteq S_2$. Again, every minimal cover contains exactly one of these subsets and we clearly have

$$\mathbf{COV}(\mathcal{U}, \mathcal{S}) \leq 2\mathbf{COV}(k - 2 - 2\varepsilon_2) \leq 2\alpha^{k-2-2\varepsilon_2}.$$

**Subcase 4C.** $|S_1| \geq 2$ and $S_1 \not\subseteq S_2$. Now every minimal cover contains either exactly one of these subsets, or both of them. If the first alternative happens (there are two possibilities for that), then we remove two subsets, two elements and reduce the cardinality of at least one other subset. Otherwise, we remove two subsets, three elements and decrease either the cardinalities of at least two other subsets by one or the cardinality of one subset by two (anyway, reducing the size of the instance by at least $2d_3$). Hence,

$$\begin{aligned}\mathbf{COV}(\mathcal{U}, \mathcal{S}) &\leq 2\mathbf{COV}(k - 2 - 2\varepsilon_2 - d_3) + \mathbf{COV}(k - 3 - 2\varepsilon_2 - 2d_3) \\ &\leq 2\alpha^{k-2-2\varepsilon_2-d_3} + \alpha^{k-3-2\varepsilon_2-2d_3}.\end{aligned}$$

Now we assume that the minimum frequency of the elements is three.

**Case 5.** *$\mathcal{S}$ has a subset of cardinality three.* This case is analyzed similar to Case 3, but since now all elements have frequency at least three and all subsets have cardinality at most three, removing each of the elements decrease the cardinalities of at least two other subsets, reducing the size of the instance by $1 + 2d_3$. Therefore

$$\begin{aligned}\mathbf{COV}(\mathcal{U}, \mathcal{S}) &\leq \mathbf{COV}(k - \varepsilon_3) + \mathbf{COV}(k - \varepsilon_3 - 3(1 + 2d_3)) \\ &\leq \alpha^{k-\varepsilon_3} + \alpha^{k-3-\varepsilon_3-6d_3}.\end{aligned}$$

Now we may suppose that all subsets contain either one or two elements.

**Case 6.** *There are $S, S' \in \mathcal{S}$ such that $S' \subset S$.* Since we are not in Case 1, we have that $|S| \geq 2$. By Proposition 1, every minimal covering containing $S$ does not contain $S'$. Thus for $|S'| = 1$ we have

$$\begin{aligned}\mathbf{COV}(\mathcal{U}, \mathcal{S}) &\leq \mathbf{COV}(k - \varepsilon_2) + \mathbf{COV}(k - 2 - \varepsilon_1 - \varepsilon_2 - 3d_2) \\ &\leq \alpha^{k-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-3d_2}\end{aligned}$$

and for $|S'| > 1$, we have

$$\begin{aligned}\mathbf{COV}(\mathcal{U}, \mathcal{S}) &\leq \mathbf{COV}(k - \varepsilon_2) + \mathbf{COV}(k - 2 - 2\varepsilon_2 - 2d_2) \\ &\leq \alpha^{k-\varepsilon_2} + \alpha^{k-2-2\varepsilon_2-2d_2}.\end{aligned}$$

(Here we use the fact that by Case 4 the minimum frequency of the elements is three.)

Now we assume that all subsets are of cardinality two.

**Case 7.** *There is $u \in \mathcal{U}$ of frequency three.* Let $S_i = \{u, u_i\}$, $i = 1, 2, 3$ be the subsets containing $u$. Then

— There are at most $3 \cdot \mathbf{COV}(k - 2 - 3\varepsilon_2 - 2d_2)$ minimal covers of $\mathcal{S}$ containing exactly one of these subsets. In each of the three cases we remove three subsets $S_1, S_2, S_3$, two elements ($u$ and one of $u_i$), and reduce the cardinalities of at least two other subsets.
— There are at most $3 \cdot \mathbf{COV}(k - 3 - 6\varepsilon_2)$ minimal covers containing exactly two of these subsets. Indeed, if $S_1$ and $S_2$ are in a minimal cover then by Proposition 1 no other subset containing $u_1$ or $u_2$ may lie in this cover. Since the frequencies of $u_1$ and $u_2$ are at least three (Case 4) and at most one subset may contain $u_1$ and $u_2$ together (Case 6) we can remove at least three other subsets containing $u_1$ or $u_2$.
— There are at most $\mathbf{COV}(k - 4 - 6\varepsilon_2)$ minimal covers containing all three of these subsets.

Therefore,

$$\mathbf{COV}(\mathcal{U}, \mathcal{S}) \leq 3\alpha^{k-2-3\varepsilon_2-2d_2} + 3\alpha^{k-3-6\varepsilon_2} + \alpha^{k-4-6\varepsilon_2}.$$

**Case 8.** *$\mathcal{S}$ does not satisfy any of the conditions from Cases* 1–7. Let $S = \{u, v\}$ be a subset of $\mathcal{S}$. Denote by $\mathcal{S}_u$ and $\mathcal{S}_v$ all other subsets containing $u$ and $v$ respectively. Since the minimum frequency of the elements is four (Case 7), $|\mathcal{S}_u| \geq 3$ and $|\mathcal{S}_v| \geq 3$. By Proposition 1, if $\mathcal{S}^*$ is a minimal cover containing $S$, then $\mathcal{S}^* \cap \mathcal{S}_u = \emptyset$ or $\mathcal{S}^* \cap \mathcal{S}_v = \emptyset$. Thus we have at most $\mathbf{COV}(k - \varepsilon_2)$ minimal covers that do not contain $S$ and at most $2 \cdot \mathbf{COV}(k - 2 - 4\varepsilon_2 - 3d_2)$ covers containing $S$. Then

$$\mathbf{COV}(\mathcal{U}, \mathcal{S}) \leq \alpha^{k-\varepsilon_2} + 2\alpha^{k-2-4\varepsilon_2-3d_2}.$$

Summarizing Cases 1–7 (recurrence of Case 0 is trivial), we obtain the inequalities (1). This completes the proof of Lemma 1. □

For any graph $G$ on $n$ vertices the size of the corresponding instance of `MSC` is at most $|\mathcal{U}| + \varepsilon_4|\mathcal{S}| = (1 + \varepsilon_4)n$. Thus the estimation of $\mathbf{COV}(k)$ boils up to choosing the weights $\varepsilon_i$, $i = 1, \ldots, 4$ and $\alpha$, minimizing $\alpha^{1+\varepsilon_4}$. This optimization problem is interesting in its own and we refer to Eppstein's work [8] on quasi-convex programming for general treatment of such problems. We numerically obtained the following values of the variables: $\varepsilon_1 = 2.9645, \varepsilon_2 = 3.5218, \varepsilon_3 = 3.9279, \varepsilon_4 = 4.1401$, and $\alpha < 1.117446$. Therefore, $\mathbf{DOM}(G) \leq \mathbf{COV}((1 + \varepsilon_4)n) < 1.117446^{5.1401n} < 1.7697^n$. This completes the proof of Theorem 1. □

Using standard methods one can easily transform the proof of the Theorem 1 into an algorithm listing all minimal dominating sets.

**Corollary 1.** *There is an algorithm for listing all minimal dominating sets in an $n$ vertex graph $G$ in time $\mathcal{O}(1.7697^n)$.*

In the *Minimum Weighted Dominating Set* problem each vertex $v$ of the graph has weight $w(v)$ and we search for the dominating set $D$ of minimum weight $w(D) = \sum_{v \in D} w(v)$. Clearly, the Corollary 1 allows us to solve this problem in time $\mathcal{O}(1.7697^n)$. Note, however, that the running time can be greatly reduced by exploiting the fact that, if all the subsets have cardinality at most two, Minimum Weighted Set Cover can be solved in polynomial time via reduction to the minimum-weight perfect matching problem (see [5]). This way we obtain the following theorem (we omit the proof details in this extended abstract)

**Theorem 2.** *There is an algorithm computing a minimum-weight dominating set in time $O(1.5780^n)$.*

## 4 Algorithm for the domatic number

The results of the previous section can be used to compute the domatic number of a graph $G = (V, E)$. Our algorithm has similarities with the classical algorithm computing the chromatic number due to Lawler [17] (see also [7]), but the analysis of our algorithm is based on Lemma 1.

For every set $X \subseteq V$ denote by $\mathbf{DN}(G|X)$ the maximum number of pairwise nonintersecting subsets of $X$ such that each of these subsets is a minimal dominating set in $G$. Clearly, $\mathbf{DN}(G|V) = \mathbf{DN}(G)$. Note that if $X$ is not dominating, then $\mathbf{DN}(G|X) = 0$

We use an array $A$, indexed by the $2^n$ subsets of $V$, for which we compute the numbers $\mathbf{DN}(G|X)$ for all subsets $X \subseteq V$. We initialize this array by assigning 0 to all $A[X]$. Then we run through the subsets $X$ of $V$ in an order such that all proper subsets of each set $X$ are visited before $X$. To compute $A[X]$, we run through all minimal dominating sets $D \subseteq X$ of $G$, and put

$$A[X] = \max\{A[X \setminus D] + 1 \mid D \subseteq X \text{ and } D \text{ is a minimal dominating set in } G\}.$$

Finally, after running through all subsets, we return the value in $A[V]$ as the domatic number of $G$.

**Theorem 3.** *The domatic number of a graph $G$ on $n$ vertices can be computed in time $\mathcal{O}(2.8805^n)$.*

*Proof.* The correctness of the algorithm `DN` can be shown by an easy induction. Let $X$ be a subset of $V$. Suppose that after running the algorithm, for every proper subset $S$ of $X$ the value $A[S]$ is equal to $\mathbf{DN}(G|S)$. Note that $A[\emptyset] = 0$. If $X$ contains no dominating subsets (i. e. $X$ is not dominating), then we have that $A[X] = \mathbf{DN}(G|X) = 0$. Otherwise, $\mathbf{DN}(G|X)$ is equal to $\max\{\mathbf{DN}(G|(X \setminus D))+1\}$, where maximum is taken over all minimal dominating sets $D \subset X$, and thus the value $A[X]$ computed by the algorithm is equal to $\mathbf{DN}(G|X)$.

For a set $X \subseteq V$, let $\mathbf{DOM}(G|X)$ be the number of minimal dominating sets of $G$ which are subsets of $X$. To estimate the running time of the algorithm, let us bound first $\mathbf{DOM}(G|X)$. We use the following reduction to the `MSC` problem. Let $\mathcal{U} = V$ and $\mathcal{S} = \{N[v] \mid v \in X\}$. Then, $\mathbf{DOM}(G|X) = \mathbf{COV}(\mathcal{U}, \mathcal{S})$. Note that the size of this problem is at most $|\mathcal{U}| + \varepsilon_4 \cdot |\mathcal{S}| = n + \varepsilon_4 \cdot |X|$. By Lemma 1, $\mathbf{COV}(\mathcal{U}, \mathcal{S}) \leq \mathbf{COV}(n + \varepsilon_4 \cdot |X|) \leq \alpha^{n+\varepsilon_4 \cdot |X|}$, where $\alpha$ and $\varepsilon_i, i = 1, \ldots, 4$ must satisfy (1). As in Theorem 1, one also can list in time $\mathcal{O}(\alpha^{n+\varepsilon_4 \cdot |X|})$ (and polynomial space) all minimal dominating sets contained in $X$. The main loop of the algorithm generating all minimal dominating sets contained in $X$ can be performed in time $\mathcal{O}(\alpha^{n+\varepsilon_4 \cdot |X|})$ and the running time of the algorithm can be bounded by

$$\mathcal{O}(\sum_{i=0}^{n} \binom{n}{i} \alpha^{n+\varepsilon_4 \cdot i}) = \mathcal{O}(\alpha^n (1 + \alpha^{\varepsilon_4})^n).$$

We numerically found the following values : $\varepsilon_1 = 3.3512, \varepsilon_2 = 4.0202, \varepsilon_3 = 4.4664, \varepsilon_4 = 4.7164$, and $\alpha < 1.105579$. Then $\mathcal{O}(\alpha^n (1 + \alpha^{\varepsilon_4})^n) = \mathcal{O}(2.8805^n)$ □

## 5 Conclusions and open problems

We have shown that the number of minimal dominating sets in a graph on $n$ vertices is at most $1.7697^n$. Dieter Kratsch (private communication) found graphs ($n/6$ disjoint copies of the octahedron) containing $15^{n/6} \approx 1.5704^n$ minimal dominating sets. We conjecture that Kratsch's graphs are the graphs with the maximal number of minimal dominating sets. This suggests the possibility that minimal dominating sets can be listed even faster.

As an algorithmic application of our combinatorial bound based on measure and conquer technique, we obtained a faster exponential algorithm to compute the domatic number of a graph. We also obtained an algorithm computing a minimum-weight dominating set in time $O(1.5780^n)$. An interesting open question is if the analysis of our algorithms can be refined, possibly via a further refined measure of the size of the set cover.

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