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Abstract

We show that minimum feedback vertex set in a bipartite graph on n vertices can be found in $1.8621^n \cdot n^{\mathcal{O}(1)}$ time.

Keywords: *minimum feedback vertex set, bipartite graph, exact algorithm*

1 Introduction

Let $G = (V, E)$ be an undirected graph on n vertices. The set $X \subset V$ is called a *feedback vertex set* or an *FVS* if $G \setminus X$ is a forest. The problem of finding minimum FVS has many applications and its history can be traced back to the early 60s (see [10] for the survey). The problem is also one of the classical NP complete problem from Karp's list [12]. Thus not surprisingly, for several decades almost every new algorithmic paradigm was tried on this problem including approximation algorithms [1, 2, 9, 13], linear programming [7], local search [4], polyhedral combinatorics [6, 11], probabilistic algorithms [14], and parameterized complexity [8, 15].

In recent years the topic of exact (exponential-time) algorithms for NP hard problems has led to much research [17]. However, despite much progress on exponential-time solutions to other graph problems such as chromatic number [3, 5] or maximum independent set [16], the only worst-case bound known for finding minimum FVS is that of $\mathcal{O}^*(2^n)$ obtained by trying all possible vertex subsets. Throughout this paper we use a modified big-Oh notation that suppresses all polynomially bounded factors. For functions f and g we write $f(n) = \mathcal{O}^*(g(n))$ if $f(n) = g(n) \cdot n^{\mathcal{O}(1)}$.

In this note we present an algorithm finding a minimum FVS in a bipartite graph in $\mathcal{O}^*(1.8621^n)$ time. Note that an easy reduction (subdividing all edges) shows that the problem remains NP-hard even for bipartite graphs.

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2 Algorithm

We need the following statement, which can be easily proved by Stirling's formula.

Proposition 2.1. *Let V be a set of cardinality n and let $0 < c \leq 1/2$. Then the number of all subsets $S \subset V$ of cardinality at most cn is bounded by $\mathcal{O}^*(1/(c^c(1-c)^{1-c})^n)$.*

First we present an algorithm finding the minimum FVS in graphs having large independent sets, and then improve it slightly for bipartite graphs.

Lemma 2.2. *Let A be an independent set in a graph G and let X be a minimum FVS in G having the largest intersection with $B = V \setminus A$. Also let $C \subseteq A$ be the set of all vertices from A having at least 3 neighbors in $B \setminus X$. Then*

a) $X \cap A \subseteq C$;

b) $|C| \leq \frac{3 \cdot |B \setminus X|}{2}$ and $|C \setminus X| \leq \frac{|B \setminus X|}{2}$.

Proof. Let $S = B \setminus X$ and $X_A = X \cap A$.

a) Let us note that X_A cannot contain a vertex v having at most one neighbor in S since then $X \setminus \{v\}$ would be an FVS of smaller cardinality. If a vertex $u \in X_A$ has two neighbors in S (say, v and w) then $X \cup \{v\} \setminus \{u\}$ is a minimum FVS having larger intersection with B, contradicting the choice of X. Thus $X_A \subset C$.

b) Since B is an FVS and X is a minimum FVS, we have that $|C \cap X| \leq |S|$. Let $S' = C \setminus X$. The set $S \cup S'$ induces a forest in G with at least $3 \cdot |S'|$ edges and therefore $3|S'| < |S| + |S'|$. Thus $|S'| \leq |S|/2$ and $|C| \leq 3 \cdot |S|/2$. □

Lemma 2.3. *Let G be a graph with a given independent set of size $(1-\varepsilon)n$. Then a minimum FVS in G can be found in $\mathcal{O}^*((1 + 3\sqrt{3}/2)^{\varepsilon n})$ time.*

Proof. Let A be an independent set of size $(1-\varepsilon)n$ and let $B = V \setminus A$.

Consider the following procedure of finding a minimum FVS X having maximum number of vertices from B. We perform in three steps.

Step 1. First we try to guess how the set $B \setminus X$ looks like. We try all subsets $S \subseteq B$. If $G[S]$ contains a cycle then we reject this subset. Otherwise let $X_B = B \setminus S$. For every vertex $u \in A$ having exactly two neighbors (say, v and w) in S add an edge vw to $G[S]$. If the resulting graph contains a cycle we again can reject S by property a) of Lemma 2.2. If this is not the case, consider subset $C_S \subset A$ of all vertices having at least 3 neighbors in S. If $|C_S| > 3|S|/2$ we again can reject S by property b) of Lemma 2.2.

Step 2. For each set S that was not rejected at the previous step, we try all possible subsets $S' \subset C_S$ of size at most $|S|/2$. We may restrict the search to subsets of such size due to property b) of Lemma 2.2. If $S \cup S'$ induces a forest then we put $X = X_B \cup (C_S \setminus S')$ in the list of potential candidates for minimum FVS.

Step 3. Among all potential candidates we choose the one of minimum cardinality. It is a desired minimum FVS.

The correctness of the procedure follows from the fact that the list of potential candidates does not contain only the sets X which do not satisfy the conditions of Lemma 2.2.

Now let us evaluate the running time of the algorithm. At step 1 we consider all $2^{\varepsilon n}$ subsets. In the worst case, for each of them at step 2 we have to consider at most $\binom{3|S|/2}{|S|/2}$ subsets which is $\mathcal{O}^*((3/2^{2/3})^{3|S|/2})$ by Proposition 2.1. So, the overall running time of the algorithm is

$$
\begin{aligned}
\sum_{s=0}^{\varepsilon n} \binom{\varepsilon n}{s} \sum_{i=0}^{s/2} \binom{3s/2}{i} &= \mathcal{O}^*(\sum_{s=0}^{\varepsilon n} \binom{\varepsilon n}{s} (3/2^{2/3})^{3s/2}) \\
&= \mathcal{O}^*(\sum_{s=0}^{\varepsilon n} \binom{\varepsilon n}{s} (3^{3/2}/2)^s) = \mathcal{O}^*((1+3\sqrt{3}/2)^{\varepsilon n}).
\end{aligned}
$$

□

Note that $(1+3\sqrt{3}/2)^\varepsilon < 2$ for $\varepsilon \leq 0.54135$. Since a bipartite graph has an independent set of cardinality at least $n/2$, Lemma 2.3 yields that a minimum FVS in bipartite graphs can be found in $\mathcal{O}^*((1+3\sqrt{3}/2)^{n/2}) = \mathcal{O}^*(1.8968^n)$ time. In Lemma 2.3 we used only the fact that a graph contains independent set A of size $(1-\varepsilon)n$. To prove the next theorem we exploit the fact that the graph is bipartite.

Theorem 2.4. *A minimum FVS of a bipartite graph can be found in time* $\mathcal{O}^*(1.8621^n)$.

Proof. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, $|A| = (1-\varepsilon)n$, $|B| = \varepsilon n$. We assume that $|B| \leq |A|$, i. e. $\varepsilon = |B|/n \leq 1/2$.

Consider the following procedure. Steps 1 and 2 of this procedure are similar to the steps from Lemma 2.3 and Steps 3 and 4 can be seen as applying the procedure from Lemma 2.3 to the set B.

Step 1. We try all subsets $S \subseteq B$ of cardinality at most $\varepsilon n/2$. Let $X_B = B \setminus S$. For every vertex $u \in A$ having exactly two neighbors (say, v and w) in S add an edge vw to S. If after that $G[S]$ contains a cycle, reject S. Otherwise, compute the subset $C_S \subset A$ of all vertices having at least 3 neighbors in S. If $|C_S| > 3|S|/2$ reject S.

Step 2. For each set $S \subseteq B$ that was not rejected at the first step, consider all subsets $S' \subset C_S$ of cardinality at most $|S|/2$. If $S \cup S'$ induces a forest then mark $X = X_B \cup (C_S \setminus S')$ as a candidate for a minimum FVS.

Step 3. Try all subsets $P' \subset A$. Let $Y_A = A \setminus P'$. For every vertex $u \in B$ having exactly two neighbors (say, v and w) in P' add an edge vw to P'. If after that $G[P']$ contains a cycle, reject it. Otherwise try all subsets $C'_{P'} \subseteq B$ of all vertices having at least 3 neighbors in P'. If $|C'_{P'}| > |P'|/2 + \varepsilon n/2$, reject P'.

Step 4. For each set P' that was not rejected at the third step, consider all subsets $P \subset C'_{P'}$ of cardinality at most $|P'|/2$. If $P \cup P'$ induces a forest then $Y = Y_A \cup (C'_{P'} \setminus P)$ is a candidate to be a minimum FVS.

Step 5. Choose among all the candidates X and Y found the one having minimum cardinality. It is a desired minimum FVS.

Let us argue first that the described procedure finds a minimum FVS. The argumentation is similar to those used in Lemma 2.3. Let X and Y be minimum FVSs having the largest intersections with B and A respectively. If $|X \cap B| \geq \varepsilon n/2$ then X is found at steps 1–2. Otherwise, $|Y \cap B| \leq |X \cap B| \leq \varepsilon n/2$, and Y will be found at steps 3–4.

We claim that the running time of the described procedure is

$$
\mathcal{O}^*(\max\{(108^{\varepsilon n/4}), 2^{\varepsilon n/2}(1+\sqrt{2})^{(1-\varepsilon)n}\}). \tag{1}
$$

In fact, steps 1–2 require

$$
\begin{aligned}
\sum_{s=0}^{\varepsilon n/2} \binom{\varepsilon n}{s} \sum_{i=0}^{s/2} \binom{3s/2}{i} = \mathcal{O}^*(\sum_{s=0}^{\varepsilon n/2} \binom{\varepsilon n}{s} (3\sqrt{3}/2)^s) &= \mathcal{O}^*(2^{\varepsilon n}(3\sqrt{3}/2)^{\varepsilon n/2}) \\
&= \mathcal{O}^*(108^{\varepsilon n/4}),
\end{aligned}
$$

time and steps 3–4 require

$$\sum_{s=0}^{(1-\varepsilon)n} \binom{(1-\varepsilon)n}{s} \sum_{i=0}^{s/2} \binom{(\varepsilon n+s)/2}{i} = \mathcal{O}^*\Big(\sum_{s=0}^{(1-\varepsilon)n} \binom{(1-\varepsilon)n}{s} 2^{(\varepsilon n+s)/2}\Big) = \mathcal{O}^*(2^{\varepsilon n/2}(1+\sqrt{2})^{(1-\varepsilon)n})$$

time.

Now for $\varepsilon \geq 0.4856$ the theorem follows from (1) and for $\varepsilon \leq 0.4856$ the theorem follows from Lemma 2.3. □

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