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Broadcast-coloring of trees

Christian Sloper*

16th September 2002

Abstract

Broadcast-coloring is a new variation of coloring, where higher numbered colors can not be used as freely as lower numbered colors. In addition there is a correspondence between the eccentricity (max distance) of a vertex and the highest legal color for that vertex.

In this note we investigate broadcast-coloring of trees. We give the broadcast-chromatic number or a bound on the broadcast-chromatic number for several simple classes of trees. In particular we show the broadcast-chromatic number for paths ($\chi_b = 3$), spiders ($\chi_b = 3$) and caterpillars ($\chi_b \leq 7$).

Further, we discuss the broadcast-chromatic number of complete k-ary trees and show that the complete binary trees have broadcast-chromatic number $\chi_b \leq 7$. We also show that *large* binary trees are broadcast-colorable and have $\chi_b \leq 7$. We then conclude by showing that no complete k-ary tree, $k \geq 3$, is broadcast-colorable.

1 Introduction

We apologize for the lack of motivation in this report, but we will refer [6] and [4] for an introduction on broadcasts and broadcast-colorings in graphs.

Definition 1 *Distance*
The distance $d(u, v)$ *between two vertices* u, v *in a graph* G*, is the length (number of edges) of the shortest path between* u *and* v *in* G*.*

Definition 2 *Eccentricity*
The eccentricity *of a vertex* v *in a graph* $G = (V, E)$ *is* $e(v) = \max_{u \in V}\{d(v, u)\}$*.*

Definition 3 *Broadcast-coloring*
A broadcast-coloring *of a graph* $G = (V, E)$ *is a function* $color : V \to \mathbb{N}$ *s.t.*

(i) $\forall u, v \in V, \quad \Big(color(u) = color(v)\Big) \Rightarrow d(u, v) > color(u)$

(ii) $\forall v \in V, \quad color(v) \leq e(v)$

Definition 4 *Broadcast-chromatic number*
The broadcast-chromatic *number* $\chi_b \in \mathbb{N}$ *for a graph* G *is the lowest number of colors for which it is possible to broadcast-color* G *by color:* $V \to \{1, 2, ..., \chi_b\}$

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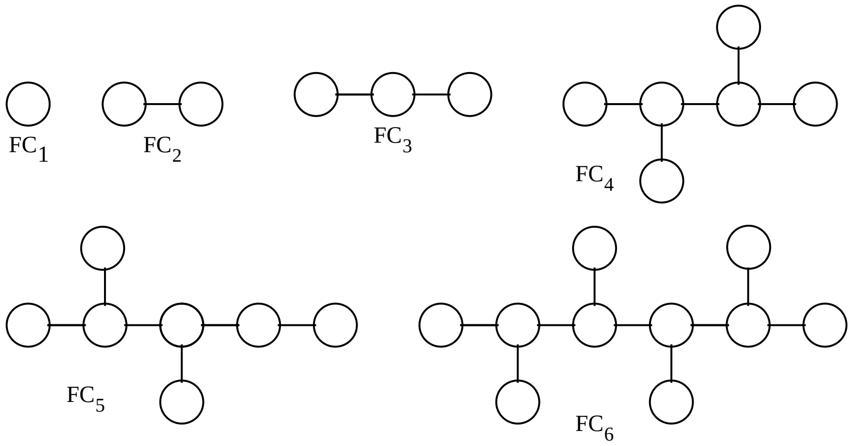

Figure 1: The forbidden subgraphs, up to symmetry, for broadcast-coloring of small caterpillars.

2 Paths, Spiders and Caterpillars

Graphs with few branchings can usually be broadcast-colored with a finite number of colors when they reach a certain size. This is due to sequences of colors that can be repeated indefinitely without violating the constraints on color-distance. Here we will give the broadcast-chromatic number for all paths, caterpillars and spiders.

Observation 5 *A path P is broadcast-colorable ($\chi_b = 3$) iff $|V(P)| \geq 4$*

Proof. That paths of length ≤ 3 are not broadcast-colorable is easily verifiable. Longer paths can be colored with the color-sequence $3121, 3121, ...$ which can be repeated indefinitely to color any path of length ≥ 4. □

Definition 6 *A* star *is a bipartite graph $K_{1,n}$. A* spider *is a star with subdivided edges.*

Observation 7 *A spider S can be broadcast-colored ($\chi_b = 3$) $\iff$ S is not a star.*

Proof. It is easy to verify that a star can not be broadcast-colored. If a spider S is not a star, then S is either a path of length ≥ 4 (colorable by Observation 5), or have one vertex v with degree ≥ 3 and a set of paths connected to v. Color v with color 2 and color the paths with the sequence $1312, ...,$. Shorter paths can safely be colored with parts of the sequence. □

Definition 8 *A* caterpillar *is a path, called the body, where each vertex except the end-vertices in the path may have any number of single vertices, called leaves, connected to it.*

Not all small caterpillars can be broadcast-colored, in Figure 1 we give the forbidden subgraphs for caterpillars of length 1 through 6. That is, if a caterpillar C of length i, $1 \leq i \leq 6$, has FC_i as a subgraph, then C can not be broadcast-colored.

Observation 9 *All* caterpillars *where the body is of length 7 or more have $\chi_b \leq 7$.*

Proof. Coloring the body, except the end-vertices, with the color-sequence $234256234257, ...$ and all the leaves and end-vertices with color 1, will color a caterpillar of any length ≥ 7 (use only first part of the sequence for lengths 7 through 13). Note that the sequence can not be used on caterpillars of length 6 or less due to eccentricity. □

A simple computer analysis shows that a color-sequence of length 34 (2342562342 5326423524 6235243265 2342) using only the colors 2 through 6 exists and can be used for caterpillars with body-length ≤ 36, though no such color-sequence using only 6 colors has length 35 or greater.

3 Binary Trees

In this section we investigate broadcast-coloring of binary trees and in particular complete binary trees.

Definition 10 *Binary tree*
A binary tree *is a tree where all vertices have degree* $1, 2,$ *or* 3.

Definition 11 *Complete binary tree*
We inductively define the complete binary tree B_i.

1. $B_1 \stackrel{def}{=}$ *1 vertex, the* root*. This vertex is* Level 1

2. $B_h \stackrel{def}{=}$ *Start with* B_{h-1} *and append* 2 *new leaves to each leaf of* B_{h-1}*. The new leaves are Level* h*.*

The height of a complete binary tree is $h = d(root, leaf\,) + 1$.

We show that all complete binary trees with height ≥ 3 can be broadcast-colored with less than 7 colors, and that the same number applies for *large* binary trees.

We will use induction to prove our result, and in fact we strengthen our induction hypothesis to carry through the proof. We introduce a set of extra rules and show that if a coloring adheres to these rules, then the coloring can be used as a basis for coloring a larger tree. We will call a coloring of a complete binary tree with these extra restrictions an *expandable* broadcast-coloring.

Definition 12 *Expandable broadcast-coloring*
An expandable broadcast-coloring *of a complete binary tree* $T = (V, E)$ *is a coloring s.t.*

(i) $\forall u, v \in V, \quad \Big(color(u) = color(v)\Big) \Rightarrow d(u, v) > color(u)$

(ii) $\forall v \in V, \quad color(v) \leq e(v)$

(iii) *The root(level 1) is colored* 1.

(iv) *All vertices on odd levels are colored* 1.

(v) *Every vertex colored* 1 *has at least one child colored* 2 *or* 3.

(vi) $color(v) = 6$ *and* $color(u) = 7 \Rightarrow d(u, v) \geq 5$

(vii) $color(p) \in \{4, 5, 6, 7\} \Rightarrow$ *p's children each have children colored* 2 *and* 3.

(viii) $\forall u \in V, \quad color(u) \leq 7$

Observation 13 Base case: *The graph in Figure 2 is an expandable broadcast-coloring of height* 4.

We now show that given an expandable coloring of a complete binary tree of height n, we can create an expandable coloring of a complete binary tree of height $(n + 1)$ by using the coloring for the tree of height n as a basis.

Lemma 14 *An expandable broadcast-coloring of a complete binary tree of height* n *can be extended to an expandable broadcast-coloring of a complete binary tree of height* $(n + 1)$.

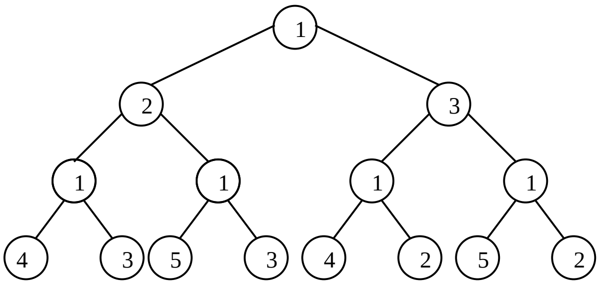

Figure 2: An expandable broadcast-coloring of a complete binary tree of height 4.

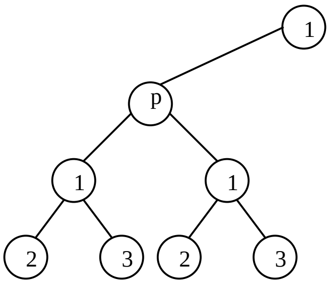

Figure 3: If $color(p) \in \{4, 5, 6, 7\}$ then the leaves must, due to rule (vii), be colored $2, 3$ and $2, 3$ as shown. Note that this coloring of the leaves is not in violation with the coloring of other vertices.

Proof.

We will construct the broadcast-coloring for the $(n + 1)$-height tree by coloring the n-first levels as the n-height tree and show that the new level always can be colored in such a way that it adheres to the expandable coloring-rules.

If n is even then, as a consequence of rule (iv), all vertices at level $(n - 1)$ are colored 1, and hence all the leaves of level $(n + 1)$ can be colored 1.

If n is odd, we will for each leaf consider the value of its grandparent p. If $color(p) \in \{4, 5, 6, 7\}$ then according to rule (vii) we must color its grandchildren $2, 3$ and $2, 3$. Observe that this is always a legal coloring (Figure 3).

The situation is more complicated if $color(p) = 2$ (color 3 will not be argued for but is analogous). The four grandchildren of p can have any color with the exception of 2. We will examine the part of the graph which can affect the coloring of p's grandchildren, consisting of vertices at distance ≤ 7 on even levels from these grandchildren. This subgraph can be seen in Figure 4, note that the following discussion uses the labeling seen in this figure. Due to rule (v), we can assume w.l.o.g. that w_1, w_2 and v's respective siblings are colored 2 or 3. Vertex l_1's and vertex l_2's siblings are colored 3.

We now have, due to rule (i), that $l_1 \in \{4, 5, 6, 7\}$ and $l_2 \in \{4, 5, 6, 7\}$. It remains to show that we can always color l_1 and l_2 with these colors. We examine y, p's grandparent, and show by a case analysis on y's color how to color l_1 and l_2.

We say that vertex v *blocks* color c from vertex u iff $color(v) = c$ and $d(v, u) \leq c$, i.e. coloring vertex u with color c would violate rule (i).

1. $color(y) \in \{1, 2\}$

   This is impossible due to rule (i).

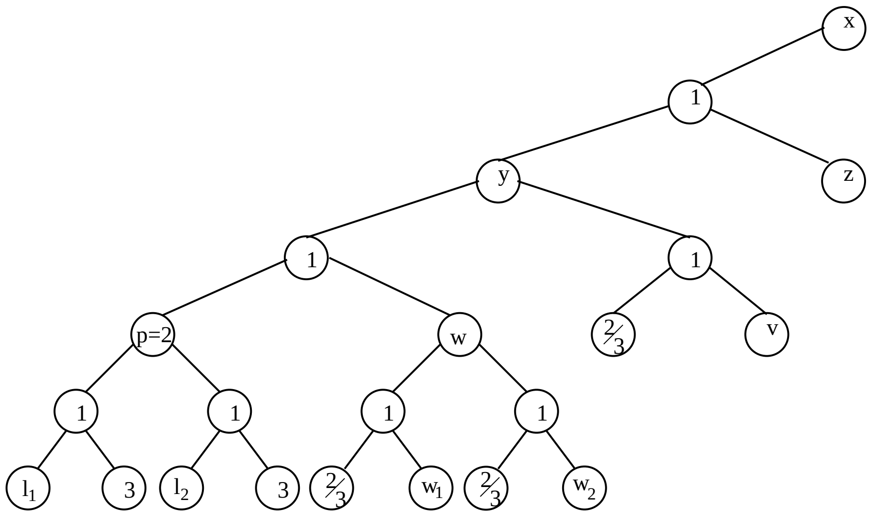

Figure 4: If $color(p) = 2$ we examine the subgraph consisting of vertices at distance ≤ 7 from l_1 and l_2.

2. $color(y) = 3$

$$color(y) = 3 \text{ and } color(p) = 2 \overset{(i)}{\Rightarrow} color(w) \notin \{2,3\} \overset{(v)}{\Rightarrow} \\ color(w_1) \in \{2,3\}, color(w_2) \in \{2,3\}$$

The distance from x, z, and v to l_1 and l_2 is 6, hence x, z, and v can only block colors 6 and 7 from l_1 and l_2. Due to rule (vi) only one of x, z, v, or w can be colored 6 or 7.

This implies that from the set of colors $\{4,5,6,7\}$, vertex w will block one color and x, z, and v will block at most one other color. This leaves at least two colors for l_1 and l_2.

3. $color(y) \in \{4,5\}$

Note that w, x, z, and v do not block $4, 5, 6$, or 7 from l_1 and l_2 since:

$$color(y) \in \{4,5\} \overset{(vii)}{\Rightarrow} color(w) \in \{2,3\}$$
$$color(y) \in \{4,5\} \overset{(vii)}{\Rightarrow} color(v) \in \{2,3\}$$
$$color(y) \in \{4,5\} \overset{(v)}{\Rightarrow} color(z) \in \{2,3\}$$

Also note that since $color(y) \in \{4,5\}$, y's grandparent x can not be in $\{4,5,6,7\}$ as this would violate rule (vii). Hence, $color(x) \in \{2,3\}$.

We can now see that w_1 and w_2 can not block both color 6 and color 7 from l_1 and l_2 because of rule (vi), and y blocks either color 4 or color 5. This leaves at least two colors for l_1 and l_2.

4. $color(y) \in \{6,7\}$

From rule (vii) we have:

$$color(y) \in \{6,7\} \overset{(vii)}{\Rightarrow} color(w) \in \{2,3\}$$

Since x, z, v, w_1, and w_2 can not block colors 4 and 5 from l_1 and l_2 and $color(w) \in \{2, 3\}$, we can always use color 4 and color 5 as a valid coloring for l_1 and l_2.

It is easy to verify that the new coloring is expandable. □

Theorem 15 *Any complete binary tree of height ≥ 3 is broadcast-colorable with ≤ 7 colors.*

Proof. For height 3, use Observation 13 without leaves. For larger heights the proof is by induction on the height. For height 4 use Observation 13 as a base and for the inductive step use Lemma 14. □

Corollary 16 *Any tree T with degrees $1, 2$ and 3 and diameter ≥ 14 can be colored using ≤ 7 colors.*

Proof.
Let C be a complete binary tree s.t. T is a subtree of C. By Theorem 15 we can color C with 7 colors. Remove vertices from C to obtain T, T is now legally colored as the diameter of 14 ensures that any vertex in T can be colored with 7 without violating the eccentricity. □

4 Ternary trees

In section 3 we proved that all complete binary trees can be broadcast-colored using a constant number of colors. Now we show the surprising result that complete *ternary* trees can not be broadcast-colored. We even prove the stronger result that ternary trees can not be *eccentrically radio-colored.*

Definition 17 *Ternary tree*
A ternary tree *is a tree where all vertices have degree* $1, 2, 3,$ *or* 4.

Definition 18 *Complete ternary tree*
We inductively define the complete ternary tree T_i.
1. $T_1 \overset{def}{=}$ *1 vertex, the* root. *This vertex is* Level 1
2. $T_h \overset{def}{=}$ *Start with* T_{h-1} *and append* 3 *new leaves to each leaf of* T_{h-1}. *The new leaves are Level h.*
The height of a complete ternary tree is $h = d(root, leaf) + 1$. *We define* T_h^- *as a* T_h *with one leaf missing.*

Definition 19 *Eccentrical Radio-coloring*
An eccentrical radio-coloring *of a graph* $G = (V, E)$ *is a function* $color : V \to \mathbb{N}$ *s.t.*

(i) $\forall u, v \in V, \quad \Big(color(u) = color(v)\Big) \Rightarrow d(u, v) > color(u)$

(ii) $\forall v \in V, \quad color(v) \leq \max_{u \in V}\{e(u)\}$

A coloring that adheres to rule (i) is known as a radio-coloring.

Note that any broadcast-coloring is trivially an eccentrical radio-coloring. We seek to prove the following theorem.

Theorem 20 *The trees $T_h, h \geq 4$, can not be eccentrically radio-colored.*

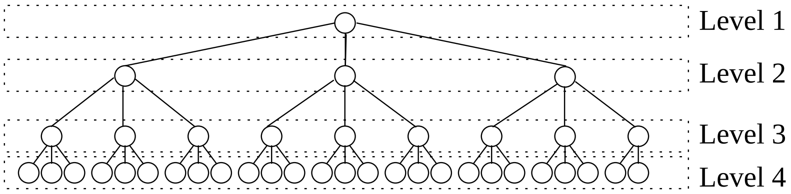

Figure 5: An example of T_4^-.

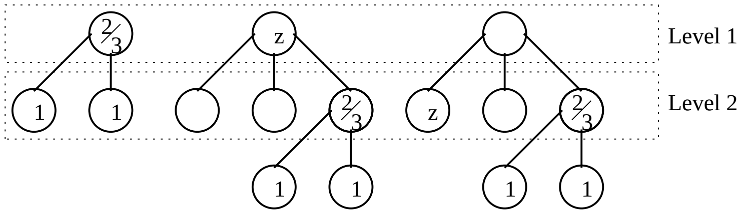

Figure 6: Illegal colorings by Claim 3 if $z \in \{4, 5, 6\}$.

To prove this, we will use induction on the height of the tree. However, first we must establish a suitable base case.

Lemma 21 *T_4^- can not be eccentrically radio-colored (See Figure 5).*

Proof. Maximum eccentricity in T_4^- is 6, but as we will see six colors is insufficient. To prove this we will use a case analysis on the placement of vertices colored 1 on level 1 and 2. First we will establish some facts:

Claim 1: Only one vertex on level 1 and level 2 can be colored 4 or greater.

> *Proof.* Assume in contradiction to the claim that there exists two vertices x and y on level 1 and level 2 s.t. $color(x) > color(y) \geq 4$. Let $c \geq 4, c \notin \{color(x), color(y)\}$. Color c can be used at most three times on level 3 and level 4, no more than once on a non-leaf. This implies that there exists a subtree T_3 where every vertex but the root is colored $1, 2,$ and 3 and a single leaf l is colored c. Then, there exists at least one T_2 in this T_3 where 3 is used as a root, but this is impossible as we have to use a 3 to color a neighbor of l. □

Claim 2: There are at least four vertices colored 1 on levels $1, 2,$ and 3.

> *Proof.* We try to color as many vertices on level 1 through level 3 without using color 1 as possible. We can then use one 6, one 5, one 4, three 3's and three 2's, a total of 9 vertices. Level $1, 2,$ and 3 have 13 vertices in total, thus at least four vertices must be colored 1.□

Claim 3: If a vertex v is colored 2 or 3 and v is on level 1 *or* there exists a vertex z on level 1 or level 2, where $color(z) \in \{4, 5, 6\}$, and v is on level 2 then v can not be the parent of two vertices x and y colored 1. (See Figure 6)

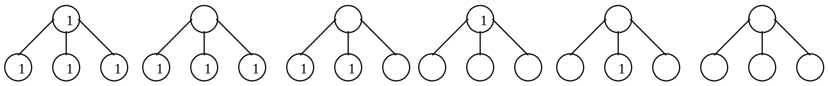

Figure 7: The six different cases we investigate in Lemma 21. Because of the nature of our case-analysis we can assume that T_4^- is symmetrical on levels 1 and 2.

Proof. If v is colored 2 (color 3 is analogous) and x and y are colored 1 we can not color the children of x and y. If v is on level 1 then x and y has 6 children which must be colored using $\{3,4,5,6\}$ which is impossible. Otherwise, we can assume w.l.o.g that $color(z) = 6$, we now have only the colors $\{3,4,5\}$ to color the 5 or 6 children of x and y, which can not be done either. □

We will now begin our case analysis on which vertices on level 1 and level 2 which are colored 1. We have six different cases to investigate (see Figure 7).

1. Four vertices colored 1.
2. Three vertices colored 1.
3. Two vertices colored 1.
4. The root colored 1.
5. One vertex on level 2 colored 1.
6. No vertices colored 1.

1. If all four vertices of level 1 and level 2 are colored 1 we contradict the requirements on distance.

2. If three vertices on level 1 and level 2 are colored 1, we have only one possibility, all three vertices on level 2 are colored 1. This is not viable as we must then color level 3 without 1s. The nine vertices can then only be colored $((2,3,4),(2,3,5),(2,3,6))$, parentheses group siblings, which leaves no free color for the root.

3. If two vertices on level 1 and level 2 are colored 1, we again have only one possibility, two level 2 vertices are colored 1. By Claim 3 we can not have color 2 or 3 as root, thus the root has color 4 or greater. This implies, by Claim 1, that 2 or 3 is used on vertex v on level 2. Claim 2 implies that v must have at least two children colored 1, but this contradicts Claim 3.

4. If the root is colored 1 then the three level 2 vertices must be colored $(2,3,c), c \geq 4$. We can w.l.o.g assume that $c = 6$. Let v be the vertex colored 2 and w be the vertex colored 3. Vertex v's children can choose from the colors $\{1,4,5\}$, but because of Claim 3 we can not select more than one 1. This implies that v's children is colored $(1,4,5)$. Unfortunately, this leaves only $\{1,2\}$ for w's children, and again due to Claim 3 we can not select more than one 1. Thus we can not color w's children.

5. If we have one vertex v colored 1 on level 2, we must, by Claim 1, use the colors $\{2,3,c\}, c \geq 4$, to color the other vertices on level 1 and level 2. We can w.l.o.g assume that $c = 6$. We now have several sub-cases.

   If c is used on level 1 then level 2 is colored $(2,1,3)$ and due to Claim 2, either the vertex colored 2 or the vertex colored 3 must have two children colored 1, violating Claim 3.

   If c is used on level 2, we have either color 2 or color 3 on the root. If the root is colored 2 then it is impossible to color the children of v (vertex colored

1). If the root is colored 3 then v's children must be colored $\{4, 5, 2\}$, but this leaves only the color 1 for the children of the vertex w colored 2. Due to Claim 3, we can not color more than one child of w with 1.

6. If no vertices on level 1 and level 2 is colored 1 then we must use four colors ≥ 2 contradicting Claim 1.

As we have demonstrated, it is not possible to color T_4^- with six colors, and since the maximum eccentricity of T_4^- is 6 we have that T_4^- can not be eccentrically radio-colored. This completes the proof of Lemma 21. □

Lemma 22 *T_5 can not be radio-colored with less than 10 colors.*

Proof.
We first prove the following general facts about T_5.

Claim 1: In any radio-coloring of T_5 there exists at least one vertex colored ≥ 7 in each of the roots three T_4 subtrees.

Proof. Assume in contradiction to the claim that there exists a radio-coloring of T_5 with a T_4-subtree T where T has no vertex colored ≥ 7. Then we can obtain an eccentricial radio-coloring of T_4^- by removing an arbitrary leaf. This contradicts Lemma 21. □

Claim 2: In any radiocoloring of T_5 at least one T_4-subtrees have either no vertex colored 6 or one leaf colored 6.

Proof. The claim is trivially true since 6 is the maximum distance between two non-leaf vertices in T_4.□

Assume in contradiction to the stated lemma that T_5 can be radio-colored with 9 colors. We will do a case analysis where we will split the problem into two cases.
1. At most one vertex colored 7 in T_5.
2. More than one vertex colored 7 in T_5.

1. By Claim 1 each T_4-subtree have at least one vertex colored ≥ 7 and since only one vertex is colored 7, all three subtrees must have not more than one vertex v colored ≥ 7. By Claim 2 at least one of these subtrees contain either no vertex colored 6 or a leaf colored 6. If no vertex is colored 6, then recolor v with 6 and remove an arbitrary leaf. If a leaf is colored 6, then remove this leaf and recolor v with 6. In both cases we obtain an eccentric radio-coloring of T_4^-. This contradicts Lemma 21.

2. Now we examine the case where more than one vertex is colored 7. In this case the vertices colored 7 are leaves and we must have at least one subtree T_4 T with no vertex colored 8 or 9. Remove the leaf colored 7 in T to obtain an eccentric radio-coloring of T_4^-. This contradicts Lemma 21.

We reached a contradiction in each case, thus the assumption must be false, completing the proof of Lemma 22. □

Proof of Theorem 20 An easy consequence of Lemma 21 is that T_4 is not eccentrically radio-colorable. To prove the result for higher h we will use induction on the height of the trees. In fact we will prove the stronger result, that any radio-coloring of $T_h, h \geq 5$, will have at least two vertices u, v s.t. $color(u) > color(v) > \max_{w \in V(T_h)}\{e(w)\}$.
Proof by induction on the height of the tree.

Base: Eccentricity of T_5 is 8, by Lemma 22 we have that no radio-coloring of T_5 have less than 10 colors. That is, $\exists u, v \in T_5$, $color(u) > color(v) > \max_{w \in V(T_5)}\{e(w)\} = 8$.

Inductive Hypothesis: We assume that any radio-coloring of a $T_{h-1}, h-1 \geq 5$, has at least two vertices u, v s.t. $color(u) > color(v) > \max_{w \in V(T_{h-1})}\{e(w)\} = (2h-4)$.

Inductive Step: T_h is formed by a root vertex and three T_{h-1}. By IH each of the T_{h-1} subtrees have at least two vertices u, v colored greater than $\max_{w \in V(T_{h-1})}\{e(w)\} = (2h-4)$. Each subtree potentially has one vertex colored $(2h-3)$ that is not in violation with other vertices, but this leaves three vertices with color $\geq (2h-2)$. Max eccentricity in T_h is $(2h-2)$, thus only one vertex in T_h can have the color $(2h-2)$, leaving two vertices with color greater than $(2h-2)$, the maximum eccentricity in T_h.□

Corollary 23 *No complete ternary tree is broadcast-colorable.*

Proof. For complete ternary trees of height ≤ 3 we can easily verify by hand that no broadcast-coloring exists. For higher trees we prove the result by contradiction.

Assume in contradiction to the stated corollary that there exists a complete ternary tree T_h, $h \geq 4$, with a valid broadcast-coloring. The coloring is a valid eccentricial coloring of T_h, this contradicts Theorem 20. □

5 Complete k-ary trees

We now extend our results from Section 4 to complete k-ary trees. We then wrap things up by showing that no complete trees other than the binary trees and paths (unary complete trees) are broadcast-colorable.

Definition 24 *k-ary tree*
A k-ary tree *is a tree T where $\forall v \in V(T), deg(v) \leq k+1$.*

Definition 25 *Complete k-ary tree*
We inductively define the complete k-ary tree T_i.

1. $T_1 \stackrel{def}{=}$ *1 vertex, the* root.
2. $T_i \stackrel{def}{=}$ *Start with T_{i-1} and append k new leaves to each leaf of T_{i-1}.*

The height of a complete k-ary tree is $h = d(root, leaf) + 1$.

Theorem 26 *No complete k-ary tree, $k \geq 3$, of height h, $h \geq 4$ is eccentrically radio-colorable.*

Proof. Assume in contradiction to the stated theorem that there exists a complete k-ary tree C with $k \geq 3$ of height h, $h \geq 4$, with a valid eccentrical radio-coloring. Then the complete 3-ary tree T of height h will be a subtree of C. Color C and remove vertices from C to obtain T, the remaining vertices are properly eccentrically radio-colored since the maximum eccentricity has not changed. This contradicts Theorem 20.□

Theorem 27 *No complete k-ary tree, $k \geq 3$, is broadcast-colorable.*

Proof. Assume in contradiction to the stated theorem that there exists a complete k-ary tree C, $k \geq 3$, of some height h that can be broadcast-colored. The complete 3-ary tree T of height h will be a subtree of C. Broadcast-color C and remove vertices from C to obtain T. The remaining vertices are now properly broadcast-colored since the eccentricity of the vertices have not changed. This contradicts Corollary 23. $\square$

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