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Abstract

The weight hierarchy of a linear $[n, k; q]$ code C over $GF(q)$ is the sequence $(d_1, d_2, \ldots, d_k)$ where d_r is the size of the smallest support of an r–dimensional subcode of C. An $[n, k; q]$ code satisfies the chain condition if there exists subcodes $D_1 \subset D_2 \subset \cdots \subset D_k = C$ of C such that D_r has dimension r and support of size d_r for all r. Further, C satisfies the almost chain condition if it does not satisfy the chain condition, but there exists subcodes D_r of dimension r and support of size d_r for all r such that $D_2 \subset D_3 \subset \cdots \subset D_k = C$ and $D_1 \subset D_3$. A simple necessary condition for a sequence to be the weight hierarchy of a code satisfying the almost chain condition is given. Further, explisit constructions of such codes are given, showing that in almost all cases, the necessary conditions are also sufficient.

1 Introduction

The weight hierarchy of linear codes has been much studied. For a code of dimension k, it is a sequence of parameters $(d_1, d_2, \ldots, d_k)$. In particular, d_1 is the minimum distance of the code. The parameters were first introduced in [12]. In [19] it was shown that these parameters are important in the analysis of an application of linear codes to the wiretap channel of type II. Later, the weight hierarchy has been shown to be important in the analysis of the trellis complexity of linear codes, see e.g. [11], [14], [18]; and analysis of linear codes for error detection on the local binomial channel, see [16]. The possible weight hierarchies of binary linear codes of dimension up to 4 were determined in [15]. In [1]–[9] we studied the possible weight hierarchies of linear codes, both for dimension 4 or less in detail and for arbitrary dimensions for codes satisfying the chain condition. Further results on codes satisfying the chain condition are given in [17].

2 Notations and problem formulation

Throughout this paper, unless otherwise stated, C will be an $[n,k]$ code, that is, a binary linear code of length n and dimension k.

For any subcode D of C, the *support* of D is the set of positions where not all the codewords of D are zero, and it is denoted by $\chi(D)$. Further, the *support weight* of D is the size of $\chi(D)$, and it is denoted by $w_S(D)$.

For $1 \le r \le k$, the *the r-th minimum support weight* (or Generalized Hamming weight) of C is defined by

$$d_r(C) = \min\{w_S(D) \mid D \text{ is an } [n,r] \text{ subcode of } C\}.$$

The sequence $(d_1, d_2, \ldots, d_k)$ is the *weight hierarchy* of C. The chain condition was introduced in [20].

Definition 1 *A linear $[n,k;q]$ code C satisfies the* chain condition *if there exist subcodes D_r of C for $1 \le r \le k$ such that $dim(D_r) = r$, $w_s(D_r) = d_r$, and*

$$D_1 \subset D_2 \subset \cdots \subset D_k.$$

The weight hierarchies of codes satisfying the chain condition have been studied in [5], [10], [17], and [20]. In this paper we will study the weight hierarchies of a class of codes which almost satisfies the chain condition.

Definition 2 *A linear $[n,k;q]$ code C satisfies the* almost chain condition *if C does not satisfy the chain condition, but there exist subcodes D_r of C for $1 \le r \le k$ such that $dim(D_r) = r$, $w_s(D_r) = d_r$,*

$$D_2 \subset D_3 \subset \cdots \subset D_k,$$

and $D_1 \subset D_3$.

We note that if we add a zero-position to an $[n,k]$ code C we get an $[n+1,k]$ code

$$C' = \big\{(\mathbf{c}|0) \bigm| \mathbf{c} \in C\big\}.$$

The codes C and C' have the same weight hierarchy. Therefore, without loss of generality, we can restrict ourselves to codes without zero-positions, that is, we will assume that $n = d_k$. Our problems can then be reformulated in terms of projective geometry and we do this next.

The difference sequence (DS) $(i_0, i_1, \cdots, i_{k-1})$ of C is defined by

$$i_r = d_{k-r} - d_{k-r-1} \text{ for } 0 \leq r \leq k-1$$

where $d_0 = 0$.

The difference sequence can easily be computed from the weight hierarchy and vice versa. It was shown in [12] that $i_r \geq 1$ for all r.

Let G be a generator matrix for C. For any $\mathbf{x} \in GF(q)^k$, $m(\mathbf{x})$, the *value* of $\mathbf{x}$, will denote the number of occurrences of $\mathbf{x}$ as a column in G. In [13] it was shown that there is a one-one correspondence between the subspaces of C of dimension r and the subspaces of $GF(q)^k$ of dimension $(k-r)$ such that if D corresponds to U, then

$$w_S(D) + \sum_{\mathbf{x} \in U} m(\mathbf{x}) = d_k. \tag{1}$$

We may view the vectors as points in the projective space $PG(\kappa, q)$, where $\kappa = k-1$. To U of dimension $k-r$ there is a corresponding projective subspace of $PG(\kappa, q)$ of projective dimension $\kappa - r$. From now on, we will only consider projective spaces (subspaces of $PG(\kappa, q)$) and by dimension we will always mean the projective dimension.

A *value assignment* is a function

$$m : PG(\kappa, q) \rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}.$$

For $p \in PG(\kappa, q)$, we call $m(p)$ the *value* of p. A value assignment defines a generator matrix and a code (up to equivalence). We define the value of a subset S of $PG(\kappa, q)$ as follows:

$$m(S) = \sum_{p \in S} m(p).$$

By (1), if m corresponds to the code C, then for $0 \leq r \leq \kappa$ we have

$$\max\{m(S) \mid S \text{ subspace of } PG(\kappa, q) \text{ of dim. } r\} = \sum_{j=0}^{r} i_j.$$

For $0 \leq r \leq \kappa$, let

$$M_r = \left\{ P_r \;\middle|\; P_r \text{ subspace of } PG(\kappa, q) \text{ of dim. } r,\ m(P_r) = \sum_{j=0}^{r} i_j \right\}.$$

In particular $M_\kappa = \{PG(\kappa, q)\}$. Note that in general, M_r depends on the sequence $(i_0, i_1, \cdots, i_\kappa)$. The definition of chain condition and almost chain condition can be reformulated as follows.

Definition 3 *If there exist subspaces P_r of $PG(\kappa, q)$ for $0 \leq r \leq \kappa - 1$ such that $P_r \in M_r$ and $P_0 \subset P_1 \subset \cdots \subset P_{\kappa-1}$, then $(i_0, i_1, \cdots, i_\kappa)$, is called a chain difference sequence (CDS).*

Definition 4 *If $(i_0, i_1, \cdots, i_\kappa)$ is not a CDS, but there exist subspaces P_r of $PG(\kappa, q)$ for $0 \leq r \leq \kappa - 1$ such that $P_r \in M_r$,*

$$P_0 \subset P_1 \subset \cdots \subset P_{\kappa-3} \subset P_{\kappa-2} \tag{2}$$

and

$$P_{\kappa-3} \subset P_{\kappa-1}, \tag{3}$$

then $(i_0, i_1, \cdots, i_\kappa)$ is called an almost chain difference sequence (ACDS).

3 Presentation of the main result

Definition 5 *We call a sequence $(i_0, i_1, \cdots i_\kappa)$ almost chain-permissible if it satisfies*

$$\begin{aligned} i_r &\leq qi_{r-1} - (q+1), \quad \textit{for} \quad 1 \leq r = \kappa - 1, \kappa, & (4) \\ i_r &\leq qi_{r-1}, \quad \textit{for} \quad 1 \leq r \leq \kappa - 2, & (5) \\ i_{\kappa-2} &\leq i_\kappa. & (6) \end{aligned}$$

Clearly, there are only finitely many almost chain-permissible sequences whose first element is i or less. Let $M(i)$ be the number of such sequences, and let $N(i)$ be the number of such sequences which are ACDS.

Theorem 1 *a) If $(i_0, i_1, \cdots, i_\kappa)$ is ACDS, then it is almost chain-permissible.*
b) For $q \geq 3$ and $k \geq 3$ we have

$$M(i) = \frac{q^{k(k-1)/2-4}(q^2-1)^2}{k!}\, i^k + \frac{q^{(k^2-3k-2)/2}(q^2-1)g(q,\kappa)}{2\;(k-1)!}\, i^{k-1} + O(i^{k-2}),$$

where $g(q,\kappa) = q^{k-1} + q^{k-2} - 2q^2 - 4q - 1$,

$$N(i) \geq M(i) - \frac{q^{(k^2-3k-2)/2}(q^2-1)^2(q^{k-3}+q^{k-4}-1)}{(k-1)!}\, i^{k-1} + O(i^{k-2}),$$

and

$$\frac{N(i)}{M(i)} \geq 1 - \frac{k(q+1)}{q}\, i^{-1} + O(i^{-2}).$$

c) Almost all almost chain-permissible sequences are ACDS in the sense that

$$\lim_{i\to\infty} \frac{N(i)}{M(i)} = 1.$$

4 Necessary conditions

Proof of Theorem 1a): Let $(i_0, i_1, \cdots, i_\kappa)$ be an ACDS. By definition, there exists $P_r \in M_r$ for $0 \leq r \leq \kappa - 1$ such that (2) and (3) are satisfied. Let $P_{-1} = \emptyset$ and $P_\kappa = PG(\kappa, q)$. In the sum below, Q runs through the q spaces of dimension $r-1$ such that $P_{r-2} \subset Q \subset P_r$. Let

$$\delta_r = \begin{cases} 1, & \text{if } r = \kappa - 1, \kappa \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq r \leq \kappa$ we get

$$\begin{aligned} \sum_{j=0}^{r} i_j &= m(P_r) \leq \sum_{P_{r-2}\subset Q\subset P_r} m(Q) \\ &\leq (q+1)(m(P_{r-1}) - \delta_r) - qm(P_{r-2}) \\ &= (q+1)(\sum_{j=0}^{r-1} i_j - \delta_r) - q\sum_{j=0}^{r-2} i_j \\ &= (q+1)i_{r-1} + \sum_{j=0}^{r-2} i_j - (q+1)\delta_r. \end{aligned}$$

Therefore we get

$$i_r \leq q i_{r-1} - (q+1)\delta_r,$$

i.e. (4) and (5) are true.

Since the DS is not CDS, $P_{\kappa-2} \not\subset P_{\kappa-1}$. Hence (2) and (3) implies that $P_{\kappa-2} \cap P_{\kappa-1} = P_{\kappa-3}$, and so

$$\begin{aligned} \sum_{j=0}^{\kappa} i_j &= m(P_\kappa) \geq m(P_{\kappa-2} \cup P_{\kappa-1}) \\ &= m(P_{\kappa-2}) + m(P_{\kappa-1}) - m(P_{\kappa-3}) = \sum_{j=0}^{\kappa-1} i_j + i_{\kappa-2}, \end{aligned}$$

i.e. (6) is true. □

5 Upper bound construction

To prove Theorem 1b), we give some sufficient conditions on ACDS. These we obtain by explicit constructions. We use the notations

$$S_{j,l} = (q-1)q^{j-l-1} \text{ for } 0 \leq l \leq j-1, \text{ and } S_{j,j} = 1.$$

From [17, Theorem 2] we have the following Lemma.

Lemma 1 *For any sequence $(i_0, i_1, \cdots, i_\kappa)$ which satisfies*

$$1 \leq i_r \leq q i_{r-1} \text{ for } 1 \leq r \leq \kappa,$$

if there exist nonnegative integers $\alpha_{j,l}$ and $\lambda_{j,l}$ $(0 \leq j \leq \kappa, 0 \leq l \leq j)$ such that the parameters i_j have the expressions:

$$i_j = \sum_{l=0}^{j} (\alpha_{j,l} S_{j,l} + \lambda_{j,l}), 0 \leq j \leq \kappa, \tag{7}$$

where $\lambda_{j,l} < S_{j,l}, \lambda_{j,j} = 0$, and satisfy the monotonicity properties:

$$\alpha_{j,l} \geq \alpha_{j+1,l} + \left\lceil \frac{\lambda_{j+1,l}}{S_{j+1,l}} \right\rceil, \ \alpha_{j,l} \geq \alpha_{j,l+1} + \left\lceil \frac{\lambda_{j,l+1}}{S_{j,l+1}} \right\rceil, \tag{8}$$

then $(i_0, i_1, \cdots, i_\kappa)$ is CDS.

We define some further notations. First we introduce coordinates for the points in $PG(\kappa, q)$. Let

$$e_1 = (0\cdots001), \ e_2 = (0\cdots010), \ \ldots, e_{\kappa+1} = (10\cdots00)$$

be a basis of $PG(\kappa, q)$. Denote by $\langle p_1, p_2, \cdots, p_r \rangle$ the subpace of $PG(\kappa, q)$ spanned by $\{p_i \mid 1 \leq i \leq r\}$. If the points are linearly independent, the space has dimension $r-1$. For example, $\langle p, p' \rangle$ is the line determined by the points $p \neq p'$. Let

$$\begin{aligned} \langle e_1, e_{\kappa+1} \rangle &= \{p(j) \mid 0 \leq j \leq q\}, \\ \langle e_\kappa, e_{\kappa+1} \rangle &= \{\hat{p}(j) \mid 0 \leq j \leq q\}. \end{aligned}$$

For $2 \leq r \leq \kappa - 1$ and integers $j_1, j_2, \ldots, j_{r-1} \in \{0, 1, \ldots, q-1\}$, let

$$\langle p(j_1, j_2, \ldots, j_{r-1}), e_r \rangle = \{p(j_1, j_2, \ldots, j_{r-1}, j_r) \mid 0 \leq j_r \leq q\} \tag{9}$$

and for $1 \leq r \leq \kappa - 2$, let

$$\hat{p}(j, j_1, j_2, \ldots, j_r) = \langle e_r, \hat{p}(j, j_1, j_2, \ldots, j_{r-1})\rangle \cap \langle e_\kappa, p(j_1, j_2, \ldots, j_r)\rangle.$$

Denote by $P_r(t)$ the subspace of $PG(\kappa, q)$ of dimension $r-1$ spanned by $\{e_{\theta \pmod{k}} \mid t \leq \theta \leq t+r-1\}$. Denote by $P_{j,l}$ the set of points in $P_{j-l+1}(l+1)\backslash(P_{j-l}(l+2) \cup P_{j-l}(l+1))$. Then it is easy to verify that

$$|P_{j,l}| = S_{j,l}$$

Let $q^{\kappa-3} \mid (i_{\kappa-2} - 1)$. We define λ and μ by:

$$i_{\kappa-2} - 1 = \lambda q^{\kappa-2} + \mu q^{\kappa-3} \tag{10}$$

where $\lambda \geq 0$, $0 \leq \mu < q$.

For now we let $q \geq 3$, we will consider $q = 2$ in last section.

Construction 1 *Let*

$$i_r = q i_{r-1} - (q+1) \text{ for } r = \kappa - 1, \kappa,$$

and

$$i_{\kappa-2} = \lambda q^{\kappa-2} + \mu q^{\kappa-3} + 1.$$

For the points x in $PG(\kappa, q)$, the values of $m(x)$ are given by Table 1 where $\epsilon = \epsilon(x) = 0$ or 1, are chosen such that

$$\begin{aligned}
m(P_{j,l}) &= \alpha_{j,l} S_{j,l} + \lambda_{j,l} \text{ for } j \leq \kappa - 2, & (11)\\
m(P_{\kappa-1,l}) &= q\, m(P_{\kappa-2}, 2) \text{ for } l \leq \kappa - 4, & (12)\\
m(P_{\kappa-1,\kappa-3}) &= q\, \alpha_{\kappa-2,\kappa-3} S_{\kappa-2,\kappa-3} - 2, & (13)\\
\lambda_{\kappa-2,\kappa-3} &= 0, & (14)\\
m(P_{\kappa-1,\kappa-3} \cap P_\kappa(\kappa)) &= m(P_{\kappa-2,\kappa-3}). & (15)
\end{aligned}$$

$m(x)$	*for x in*	*range of parameters*
$\alpha_{j,l} + \epsilon$ α_{jj}	$P_{j,l}$ $P_{j,j}$	$0 \leq l < j \leq \kappa - 2$ $0 \leq j \leq \kappa - 2$
$\alpha_{\kappa-2,l} + \epsilon$ $\alpha_{\kappa-2,\kappa-3} - \epsilon$ $\alpha_{\kappa-2,\kappa-2} - 1$	$P_{\kappa-1,l}$ $P_{\kappa-1,\kappa-3}$ $P_{\kappa-1,l}$	$0 \leq l \leq \kappa - 4$ $l = \kappa - 2, \kappa - 1$
$\lambda + 1$	$\langle e_2, e_3, \cdots e_{\kappa-1}, \hat{p}(j, j_1)\rangle \backslash P_{\kappa-2}(2)$	$1 \leq j_1 \leq \mu$ $1 \leq j \leq q$
$\lambda - 1$	$\langle e_{\kappa-1}, \hat{p}(j, \mu+1, 1, \cdots, 1)\rangle$ $\cap\ \langle e_\kappa, p(\mu+1, 1, \cdots, 1, j_{\kappa-1})\rangle$ *where the number of* 1 *is* $\kappa - 3$.	$j_{\kappa-1} = q-1, q-2$ $1 \leq j \leq q$
λ	*rest of* $PG(\kappa, q)\backslash P_\kappa(1)$	

Table 1. Values of m for Construction 1.

Using Construction 1, we will prove the following theorem.

Theorem 2 *Let $(i_0, i_1, \cdots, i_\kappa)$ be an almost chain-permissible sequence satisfying*

$$\begin{aligned}
i_r &= q i_{r-1} - (q+1) \text{ for } r = \kappa - 1, \kappa,\\
i_{\kappa-2} &= \lambda q^{\kappa-2} + \mu q^{\kappa-3} + 1, \text{ where } \lambda \geq 1.
\end{aligned}$$

If, for $0 \leq j \leq \kappa - 2$, i_j *can be expressed as*

$$i_j = \sum_{l=0}^{j} (\alpha_{j,l} S_{j,l} + \lambda_{j,l}), \tag{16}$$

where

$$S_{j,l} > \lambda_{j,l}, \quad \lambda_{\kappa-2,l} = \lambda_{jj} = 0, \quad \alpha_{\kappa-2,\kappa-2} \geq 1,$$
$$\alpha_{j,l} \geq \alpha_{j+1,l} + \left\lceil \frac{\lambda_{j+1,l}}{S_{j+1,l}} \right\rceil, \quad \alpha_{j,l} \geq \alpha_{j,l+1} + \left\lceil \frac{\lambda_{j,l+1}}{S_{j,l+1}} \right\rceil + \delta_1 q, \tag{17}$$

and where

$$\delta_1 = \begin{cases} 1, & \text{if } l = 0, \\ 0, & \text{otherwise}, \end{cases}$$

then $(i_0, i_1, \cdots, i_\kappa)$ *is ACDS.*

To prove Theorem 2 we need several lemmas.

Lemma 2 *The subspaces of* $\langle e_r, e_{r+1}, \cdots e_{\kappa-1}, \hat{p}(j, j_1, j_2, \cdots, j_{r-1})\rangle$ *containing* $P_{k-r-2}(r+1)$ *are* $\langle e_{r+1}, e_{r+2}, \cdots, e_{\kappa-1}, \hat{p}(j, j_1, j_2, \cdots, j_r)\rangle$ *for* $0 \leq j_r \leq q$.

Proof: From (9) we get

$$\begin{aligned} &\{\langle e_\kappa, p(j_1, j_2, \cdots, j_r)\rangle \cap \langle e_r, \hat{p}(j, j_1, j_2, \cdots, j_{r-1})\rangle \mid 0 \leq j_r \leq q\} \\ &= \quad \langle e_r, \hat{p}(j, j_1, j_2, \cdots, j_{r-1})\rangle = \{\hat{p}(j, j_1, j_2, \cdots, j_r) \mid 0 \leq j_r \leq q\}. \end{aligned}$$

Hence

$$\begin{aligned} &\bigcup_{0 \leq j_r \leq q} \langle e_{r+1}, e_{r+2}, \cdots, e_{\kappa-1}, \hat{p}(j, j_1, j_2, \cdots, j_r)\rangle \\ &= \quad \langle e_r, e_{r+1}, \cdots, e_{\kappa-1}, \hat{p}(j, j_1, j_2, \cdots, j_{r-1})\rangle. \end{aligned}$$

$\square$

Similarly we have:

Corollary 1 *We have*

$$\begin{aligned} &\langle e_r, e_{r+1}, \cdots e_{\kappa-2}, \hat{p}(j, j_1, \cdots, j_{r-1})\rangle \\ &= \bigcup_{0 \leq j_r \leq q} \langle e_{r+1}, e_{r+2}, \cdots, e_{\kappa-2}, \hat{p}(j, j_1, \cdots, j_r)\rangle. \end{aligned}$$

Lemma 3 *We have*

$$\begin{aligned} &\bigcup_{0 \leq j \leq q} \langle e_{r+1}, e_{r+2}, \cdots, e_{\kappa-1}, \hat{p}(j, j_1, j_2, \cdots j_r)\rangle \\ &= \quad \langle e_{r+1}, e_{r+2}, \cdots, e_\kappa, p(j_1, j_2, \cdots, j_r)\rangle \end{aligned}$$

for $1 \leq r \leq \kappa - 2$.

Proof: Since $\{\hat{p}(j) \mid 0 \leq j \leq q\} = \langle e_\kappa, e_{\kappa+1}\rangle$ and $p(j_1) \in \langle e_1, e_{\kappa+1}\rangle$, so

$$\{\hat{p}(j, j_1) \mid 0 \leq j \leq q\} = \{\langle e_1, \hat{p}(j)\rangle \cap \langle e_\kappa, p(j_1)\rangle \mid 1 \leq j \leq q\} = \langle e_\kappa, p(j_1)\rangle.$$

Suppose that

$$\{\hat{p}(j, j_1, j_2, \cdots j_{r-1}) \mid 0 \leq j \leq q\} = \langle e_\kappa, p(j_1, j_2, \cdots, j_{r-1})\rangle.$$

Then by $p(j_1, j_2, \cdots, j_r) \in \langle e_r, p(j_1, j_2, \cdots, j_{r-1})\rangle$ we have

$$\begin{aligned}
&\{\hat{p}(j, j_1, j_2, \cdots, j_r) \mid 0 \le j \le q\} \\
&\quad = \{\langle e_r, \hat{p}(j, j_1, j_2, \cdots, j_{r-1})\rangle \cap \langle e_\kappa, p(j_1, j_2, \cdots, j_r)\rangle \mid 0 \le j \le q\} \\
&\quad = \langle e_\kappa, p(j_1, j_2, \cdots, j_r)\rangle.
\end{aligned} \tag{18}$$

Therefore, similar to the proof of Lemma 2, Lemma 3 is true. □

Lemma 4 *In Construction 1, if (16) is true, then for $0 \le j \le q$ we have*

$$m(P_\kappa(1)) = m(\langle e_1, e_2, \cdots, e_{\kappa-1}, \hat{p}(j)\rangle) = \sum_{l=0}^{\kappa-1} i_l - 1.$$

Proof: From (11), (12), (13), and Table 1 we have

$$\begin{aligned}
m(P_\kappa(1)) &= m(\langle e_1, e_2, \cdots, e_{\kappa-1}, e_\kappa\rangle) = \sum_{j=0}^{\kappa-1}\sum_{l=0}^{j} m(P_{j,l}) \\
&= \sum_{j=0}^{\kappa-2}\sum_{l=0}^{j}(\alpha_{j,l}S_{j,l} + \lambda_{j,l}) + \sum_{l=0}^{\kappa-4} q\, m(P_{\kappa-2,l}) + \sum_{l=\kappa-3}^{\kappa-1} m(P_{\kappa-1,l}) \\
&= \sum_{j=0}^{\kappa-2} i_j + q\sum_{l=0}^{\kappa-4}(\alpha_{\kappa-2,l}S_{\kappa-2,l} + \lambda_{\kappa-2,l}) \\
&\quad + q(\alpha_{\kappa-2,\kappa-3}S_{\kappa-2,\kappa-3} + \lambda_{\kappa-2,\kappa-3}) - 2 + q(\alpha_{\kappa-2,\kappa-2} - 1) \\
&= \sum_{j=0}^{\kappa-2} i_j + q i_{\kappa-2} - q - 2 = \sum_{j=0}^{\kappa-1} i_j - 1.
\end{aligned}$$

From Table 1, (10), (15), and Lemma 2, for $1 \le j \le q$ we have

$$\begin{aligned}
&m(\langle e_1, e_2, \cdots, e_{\kappa-1}, \hat{p}(j)\rangle) \\
&\quad = m(P_{\kappa-1}(1)) + \lambda q^{\kappa-1} - 2 + \sum_{j_1=1}^{\mu} |\langle e_2, e_3, \cdots, e_{\kappa-1}, \hat{p}(j, j_1)\rangle \backslash P_{\kappa-2}(2)| \\
&\quad = \sum_{j=0}^{\kappa-2} i_j + \lambda q^{\kappa-1} - 2 + \mu q^{\kappa-2} = \sum_{j=0}^{\kappa-2} i_j + q(i_{\kappa-2} - 1) - 2 = \sum_{j=0}^{\kappa-1} i_j - 1.
\end{aligned}$$

Lemma 5 *In Construction 1, if (16) is true, then*

$$m(P_\kappa(\kappa)) = \sum_{j=0}^{\kappa-1} i_j.$$

Proof: Since $\langle e_{\kappa-1}, \hat{p}(j)\rangle \cap \langle e_\kappa, e_{\kappa+1}\rangle = \hat{p}(j)$, so

$$\langle e_1, e_2, \cdots, e_{\kappa-2}, e_{\kappa-1}, \hat{p}(j)\rangle \cap P_\kappa(\kappa) = \langle e_1, e_2, \cdots, e_{\kappa-2}, \hat{p}(j)\rangle. \tag{19}$$

By Corollary 1 we have

$$\langle e_1, e_2, \cdots, e_{\kappa-2}, \hat{p}(j)\rangle \backslash P_{\kappa-2}(1) = \bigcup_{j_1=1}^{q} (\langle e_2, e_3, \cdots, e_{\kappa-2}, , \hat{p}(j, j_1)\rangle \backslash P_{\kappa-3}(2)).$$

Hence, by Table 1 we have

$$\begin{aligned} & m(\langle e_1, e_2, \cdots, e_{\kappa-2}, \hat{p}(j)\rangle \backslash P_{\kappa-2}(1)) \\ = & \lambda q^{\kappa-2} + \sum_{j_1=1}^{\mu} |\langle e_2, e_3, \cdots, e_{\kappa-2}, p(j.j_1)\rangle \backslash P_{\kappa-3}(2)| \\ = & \lambda q^{\kappa-2} + \mu q^{\kappa-3} = i_{\kappa-2} - 1. \end{aligned}$$

Therefore, from (15), (19), and Table 1 we get

$$\begin{aligned} & m(P_\kappa(\kappa)) \\ = & \sum_{j=1}^{q} m(\langle e_1, e_2, \cdots, e_{\kappa-2}, \hat{p}(j)\rangle \backslash P_{\kappa-2}(1)) + m(\langle e_1, e_2, \cdots, e_{\kappa-2}, e_\kappa\rangle) \\ = & q(i_{\kappa-2} - 1) + (\sum_{j=0}^{\kappa-2} i_j - 1) = \sum_{j=0}^{\kappa-1} i_j. \end{aligned}$$

$\square$

Proof of Theorem 2. From (16) and $\lambda_{\kappa-2,l} = 0$ we have

$$\begin{aligned} i_{\kappa-1} - 1 &= q i_{\kappa-2} - q - 2 \\ &= q \sum_{l=0}^{\kappa-2} \alpha_{\kappa-2,l} S_{\kappa-2,l} - q - 2 \\ &= \sum_{l=0}^{\kappa-4} \alpha_{\kappa-2,l} S_{\kappa-1,l} + ((\alpha_{\kappa-2,\kappa-3} - 1) S_{\kappa-1,\kappa-3} + q(q-1) - 2) \\ & \quad + (\alpha_{\kappa-2,\kappa-2} - 1) S_{\kappa-1,\kappa-2} + (\alpha_{\kappa-2,\kappa-2} - 1) S_{\kappa-1,\kappa-1} \\ &=: \sum_{l=0}^{\kappa-1} (\alpha_{\kappa-1,l} S_{\kappa-1,l} + \lambda_{\kappa-1,l}), \end{aligned}$$

where $\lambda_{\kappa-1,l} = 0$ for $l \neq \kappa-3$, $\lambda_{\kappa-1,\kappa-3} = q(q-1)-2$; $\alpha_{\kappa-1,l} = \alpha_{\kappa-2,l}$ for $l \leq \kappa-4$; $\alpha_{\kappa-1,l} = \alpha_{\kappa-2,l} - 1$ for $l = \kappa-3, \kappa-2, \kappa-1$. Hence

$$\alpha_{\kappa-2,l} \geq \alpha_{\kappa-1,l} + \left\lceil \frac{\lambda_{\kappa-1,l}}{\alpha_{\kappa-1,l}} \right\rceil$$

and

$$\alpha_{\kappa-1,l} \geq \alpha_{\kappa-1,l+1} + \left\lceil \frac{\lambda_{\kappa-1,l+1}}{\alpha_{\kappa-1,l+1}} \right\rceil.$$

Therefore, by (16) and (17) for the sequence $(i_0, i_1, \cdots i_{\kappa-2}, i_{\kappa-1} - 1)$, we get (7) (with κ changed to $\kappa - 1$) and (8). So, from Lemma 1, $(i_0, i_1, \cdots i_{\kappa-2}, i_{\kappa-1} - 1)$ is CDS. Now the values $m(x)$ on $P_\kappa(1)$ in Table 1 are similar to the values $m(x)$ in [17, Structure 1]. Therefore in $P_\kappa(1)$ we have

$$\sum_{j=0}^{r-1} i_j = m(P_r(1)) \geq m(S_r) \text{ for } 1 \leq r \leq \kappa - 1, \tag{20}$$

where S_r is any subspace in $P_\kappa(1)$ of dimension $r - 1$.

Note that

$$P_\kappa(\kappa) = \langle e_\kappa, e_{\kappa+1}, e_1, e_2, \cdots, e_{\kappa-2}\rangle \supset \langle e_1, e_2, \cdots, e_{\kappa-2}\rangle = P_{\kappa-2}(1), \tag{21}$$

i.e. (2) is true. By Lemma 4 we have

$$
\begin{aligned}
m(PG(\kappa,q)) &= m(P_\kappa(1)) + \sum_{j=1}^{q} m(\langle e_1, e_2, \cdots, e_{\kappa-1}, \hat{p}(j)\rangle \backslash P_{\kappa-1}(1)) \\
&= \sum_{j=0}^{\kappa-1} i_j - 1 + q(i_{\kappa-1}-1) = \sum_{j=0}^{\kappa} i_j. \qquad (22)
\end{aligned}
$$

Hence from Lemmas 4 and 5, (20), (21), and (22) we only need prove that

$$\sum_{j=0}^{l-1} i_j > m(S_l) \text{ for } 1 \le l \le \kappa,$$

where S_l is any subspace of dimension $l-1$ satisfying $S_l \not\subset P_\kappa(1)$.

If $S_l \not\subset \langle e_2, e_3, \cdots, e_\kappa, p(j_1)\rangle$ for any j_1 and $l < \kappa$, then by Lemma 3, (4), (5), (20), and Table 1 we have

$$
\begin{aligned}
m(S_l) &= \sum_{j_1=1}^{q} m(S_l \cap (\langle e_2, e_3, \cdots, e_\kappa, p(j_1))\backslash P_{\kappa-1}(2)\rangle + m(S_l \cap P_\kappa(1)) \\
&\le \sum_{j_1=1, j_1\ne\mu+1}^{q} \sum_{j=1}^{q} m(S_l \cap (\langle e_2, e_3, \cdots, e_{\kappa-1}, \hat{p}(j,j_1)\rangle \backslash P_{\kappa-2}(2))) \\
&\quad +\lambda q^{l-2} + m(S_l \cap P_\kappa(1)) \\
&= (\lambda+1)\mu q^{l-2} + \lambda(q-1-\mu)q^{l-2} + \lambda q^{l-2} + m(S_l \cap P_\kappa(1)) \\
&\le \lambda q^{l-1} + \mu q^{l-2} + \sum_{j=0}^{l-2} i_j = \frac{i_{\kappa-2}-1}{q^{\kappa-1-l}} + \sum_{j=0}^{l-2} i_j < \sum_{j=0}^{l-1} i_j.
\end{aligned}
$$

For $l = \kappa$, if $S_l \cap \langle e_{\kappa-1}, e_\kappa\rangle = \{e_{\kappa-1}\}$, then $S_l \supset \{x_1, x_2\}$, where $m(x_1) = m(x_2) = \lambda - 1$. Hence

$$m(S_l) \le \lambda q^{l-1} + \mu q^{l-2} - 2 + \sum_{j=0}^{l-2} i_j = q(i_{\kappa-2}-1) - 2 + \sum_{j=0}^{l-2} i_j = \sum_{j=0}^{l-1} i_j - 1.$$

If $S_l \cap \langle e_{\kappa-1}, e_\kappa\rangle \ne \{e_{\kappa-1}\}$, then

$$m(S_l \cap P_\kappa(1)) \le \sum_{j=0}^{l-1} i_j - 1.$$

Hence

$$m(S_l) \le q(i_{\kappa-2}-1) + \sum_{j=0}^{l-2} i_j - 1 = \sum_{j=0}^{l-1} i_j,$$

and if $S_{l-1} \subset S_l$, then

$$m(S_{l-1}) < \sum_{j=0}^{l-1} i_j.$$

If $S_l \subset \langle e_2, e_3, \cdots, e_\kappa, p(j_1^*)\rangle$ for some j_1^*, then $(S_l \cap P_\kappa(1)) \subset \langle e_2, e_3, \cdots, e_\kappa\rangle$. Denote $S'_{l-1} := S_l \cap P_\kappa(1)$. By Table 1, (16) and (17), similar to the proof of [17, Theorem 2], we get $m(S'_{l-1}) \le m(P_{l-1}(2))$. From (17) and Table 1 we get:

$$m(P_{l-1}(2)) = \sum_{j=1}^{l-1} \sum_{t=1}^{j} m(P_{j,t}) = \sum_{j=1}^{l-1} \sum_{t=1}^{j} (\alpha_{j,t} S_{j-1,t-1} + \lambda_{j,t})$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{l-2}\sum_{t=0}^{j}(\alpha_{j,t}S_{j,t}+\lambda_{j,t}) - q\sum_{j=0}^{l-2}S_{j,0} - q\epsilon_1 \\
&= \sum_{j=0}^{l-2} i_j - q^{l-1} - q\epsilon_1,
\end{aligned}
$$

where

$$
\epsilon_1 = \begin{cases} 1 & \text{if } l=\kappa, \\ 0 & \text{otherwise.} \end{cases}
$$

Hence

$$
\begin{aligned}
m(S_l) &\leq (\lambda+1)q^{l-1} + m(S'_{l-1}) \leq (\lambda+1)q^{l-1} + \sum_{j=0}^{l-2} i_j - q^{l-1} - q\epsilon_1 \\
&\leq \frac{i_{\kappa-2}-1}{q^{\kappa-1-l}} + \sum_{j=0}^{l-2} i_j - q\epsilon_1 < \sum_{j=0}^{l-1} i_j.
\end{aligned}
$$

6 Some sets of lines or planes in $PG(\kappa, q)$

In order to get a general construction from the upper bound construction, we study some sets of lines and planes in $PG(\kappa, q)$. We define some notations. Define two kinds of points:

$$
\begin{aligned}
p_1(j_1, j_2, \cdots, j_\tau) &:= (1 j_1 0 j_2 \cdots j_\tau 0 \cdots 0), \\
p_2(j_1, j_2, \cdots, j_\tau) &:= (0 1 j_1 j_2 \cdots j_\tau 0 \cdots 0),
\end{aligned}
$$

where $0 \leq j_r < q$ for $1 \leq r \leq \tau$. Let $L(j_1, j_2, \cdots, j_t, j)$ denote the set of the following $q^{\kappa-1-t}$ lines with one point omitted:

$$
\langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle \backslash \{p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\}
$$

where $0 \leq j_r < q$ for $t < r \leq \kappa - 1$.

Denote

$$
\begin{aligned}
S_r(p) &:= \langle e_1, e_2, \cdots, e_{r-1}, p\rangle, \\
S_r(p, p') &:= \langle e_1, e_2, \cdots, e_{r-2}, p, p'\rangle.
\end{aligned}
$$

For a collection X of sets of points, $\bigcup X$ denotes the union of the sets in X (and similarly for multisets).

Lemma 6 *For $j > 0$ and $0 < t \leq \kappa - 2$, we have*

$$
\begin{aligned}
&\bigcup L(j_1, j_2, \cdots j_t, j) \\
&\quad = S_{k-t}(p_1(j_1, j_2, \cdots, j_t), p_2(j, j_1, j_2 \cdots, j_{t-1})) \backslash S_{k-t-1}(p_2(j, j_1, j_2 \cdots, j_{t-1})).
\end{aligned}
$$

Proof: Since $e_1 = (0 \cdots 01)$, we have

$$
\{p_1(j_1, \cdots, j_{\kappa-1}) \mid 0 \leq j_{\kappa-1} < q\} \cup \{e_1\} = \langle e_1, p_1(j_1, \cdots, j_{\kappa-2})\rangle.
$$

Hence by $\langle e_1, p_2(j, j_1, \cdots, j_{\kappa-2})\rangle = \langle e_1, p_2(j, j_1, \cdots, j_{\kappa-3})\rangle$ we get

$$
\begin{aligned}
&\bigcup L(j_1, \cdots, j_{\kappa-2}, j) \cup \langle e_1, p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle \\
&\quad = \bigcup_{0 \leq j_{\kappa-1} < q} \langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle \backslash \{p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\} \\
&\qquad \cup \langle e_1, p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle \\
&\quad = \langle e_1, p_1(j_1, \cdots, j_{\kappa-2}), p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle \\
&\quad = \langle e_1, p_1(j_1, \cdots, j_{\kappa-2}), p_2(j, j_1, j_2, \cdots, j_{\kappa-3})\rangle.
\end{aligned}
$$

This proves Lemma 6 for $t = \kappa - 2$. Suppose that Lemma 6 is true for $t = m$, i.e. we have

$$\begin{aligned}\bigcup L(j_1, j_2, \cdots, j_m, j) &= S_{k-m}(p_1(j_1, \cdots, j_m), p_2(j, j_1, \cdots, j_{m-1})) \\ &\quad \backslash S_{k-m-1}(p_2(j, j_1, \cdots, j_{m-1})). \end{aligned} \tag{23}$$

Since

$$\{p_1(j_1, \cdots, j_m) \mid 0 \le j_m < q\} \cup \{e_{k-m-1}\} = \langle e_{k-m-1}, p_1(j_1, \cdots, j_{m-1})\rangle,$$

and the space containing $S_{k-m-1}(p_2(j, j_1, \cdots, j_{m-1}))$ and one point on the line $\langle e_{k-m-1}, p_1(j_1, \cdots, j_{m-1})\rangle$ is contained in $S_{k-m+1}(p_1(j_1, \cdots, j_{m-1}), p_2(j, j_1, \cdots, j_{m-1}))$, by (23) we have

$$\begin{aligned}
&\bigcup L(j_1, \cdots, j_{m-1}, j) \cup S_{k-m}(p_2(j, j_1, \cdots, j_{m-1})) \\
&\quad = \bigcup_{0 \le j_m < q} \bigcup L(j_1, \cdots, j_m, j) \cup S_{k-m}(p_2(j, j_1, \cdots, j_{m-1})) \\
&\quad = \bigcup_{0 \le j_m < q} S_{k-m}(p_1(j_1, \cdots, j_m), p_2(j, j_1, \cdots, j_{m-1})) \backslash S_{k-m-1}(p_2(j, j_1, \cdots, j_{m-1})) \\
&\qquad \cup S_{k-m}(p_2(j, j_1, \cdots, j_{m-1})) \\
&\quad = S_{k-m+1}(p_1(j_1, \cdots, j_{m-1}), p_2(j, j_1, \cdots, j_{m-1})).
\end{aligned}$$

Hence by $\langle e_{k-m-1}, p_2(j, j_1, \cdots, j_{m-1})\rangle = \langle e_{k-m-1}, p_2(j, j_1, \cdots, j_{m-2})\rangle$, we prove the lemma for $t = m - 1$. □

Corollary 2 *For $j > 0$ we have*

$$\bigcup_{0 \le j_1 < q} \bigcup L(j_1, j) = P_k(1) \backslash P_\kappa(1).$$

Proof: Let $t = 1$. By Lemma 6 we have

$$\bigcup L(j_1, j) = S_\kappa(p_1(j_1), p_2(j)) \backslash S_{\kappa-1}(p_2(j)).$$

Similar to the proof of Lemma 6 we get

$$\begin{aligned}
&\bigcup_{0 \le j_1 < q} \bigcup L(j_1, j) \cup P_\kappa(1) \\
&\quad = \bigcup_{0 \le j_1 < q} S_\kappa(p_1(j_1), p_2(j)) \backslash S_{\kappa-1}(p_2(j)) \cup S_\kappa(p_2(j)) = P_k(1).
\end{aligned}$$

□

Denote by cS the multiset of set S, where each element in S appears c times. Let $l(j_1, j)$ denote the set of lines

$$\langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1, j_2, \cdots, j_{\kappa-2})\rangle$$

where $0 \le j_r < q$ for $1 < r \le \kappa - 1$. Then from Corollary 2 we get the following corollary.

Corollary 3 *Let*

$$L := \{l(j_1, j) \mid 0 \le j_1 < q,\ 0 < j < q\}.$$

Then

$$\bigcup L = (q-1)(P_k(1) \backslash P_\kappa(1)) \cup q((P_\kappa(1) \backslash P_{\kappa-1}(1)) \backslash P_\kappa(\kappa)).$$

Define two kinds of points:

$$\tilde{p}_1(j_1, j_2, \cdots, j_\tau) := (10 j_1 j_2 \cdots j_\tau 0 \cdots 0),$$
$$\tilde{p}_2(j_1, j_2, \cdots, j_\tau) := (010 j_1 j_2 \cdots j_\tau 0 \cdots 0),$$

where $0 \le j_r < q$ for $1 \le r \le \tau$. Let $\tilde{L}(j_1, j_2, \cdots, j_t, j)$ denote the following set of lines with one point omitted:

$$\langle \tilde{p}_1(j_1, \cdots, j_{\kappa-1}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-2})\rangle \backslash \tilde{p}_2(j, j_2, \cdots, j_{\kappa-2})$$

where $0 \le j_r < q$ for $t < r \le \kappa - 1$.

Lemma 7 *For $j \ge 0$, we have*

$$\begin{aligned} & \bigcup \tilde{L}(j_1, j_2, \cdots, j_t, j) \\ = \quad & S_{k-t}(\tilde{p}_1(j_1, j_2, \cdots, j_t), \tilde{p}_2(j, j_2, \cdots, j_t)) \backslash S_{k-t-1}(\tilde{p}_2(j, j_2, \cdots, j_t)). \end{aligned}$$

Proof: Similar to the proof of Lemma 6 we have

$$\begin{aligned} & \bigcup \tilde{L}(j_1, j_2, \cdots, j_{\kappa-2}, j) \cup \langle e_1, \tilde{p}_2(j, j_2, \cdots, j_{\kappa-2})\rangle \\ = \quad & \langle e_1, \tilde{p}_1(j_1, j_2, \cdots, j_{\kappa-2}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-2})\rangle. \end{aligned}$$

Hence the lemma is true for $t = \kappa - 2$. Suppose that Lemma 7 is true for $t = m$. Let $t = m - 1$. Similar to the proof of Lemma 6 and by

$$\langle e_{k-m-2}, \tilde{p}_2(j, j_2, \cdots, j_m)\rangle = \langle e_{k-m-2}, \tilde{p}_2(j, j_2, \cdots, j_{m-1})\rangle,$$

and

$$\langle e_{k-m-1}, \tilde{p}_1(j_1, \cdots, j_m)\rangle = \langle e_{k-m-1}, \tilde{p}_1(j_1, \cdots, j_{m-1})\rangle$$

we have

$$\begin{aligned} & \bigcup \tilde{L}(j_1, \cdots, j_{m-1}, j) \cup S_{k-m}(\tilde{p}_2(j, j_2, \cdots, j_{m-1})) \\ = \quad & \bigcup_{0 \le j_m < q} S_{k-m}(\tilde{p}_1(j_1, \cdots, j_m), \tilde{p}_2(j, j_2, \cdots, j_m)) \backslash S_{k-m-1}(\tilde{p}_2(j, j_2, \cdots, j_m)) \\ & \cup S_{k-m}(\tilde{p}_2(j, j_2, \cdots, j_{m-1})) \\ = \quad & \bigcup_{0 \le j_m < q} S_{k-m}(\tilde{p}_1(j_1, \cdots, j_m), \tilde{p}_2(j, j_2, \cdots, j_{m-1})) \backslash S_{k-m-1}(\tilde{p}_2(j, j_1, \cdots, j_{m-1})) \\ & \cup S_{k-m}(\tilde{p}_2(j_1', j_2, \cdots, j_{m-1})) \\ = \quad & S_{k-m+1}(\tilde{p}_1(j_1, \cdots, j_m), \tilde{p}_2(j, j_2, \cdots, j_{m-1})) \\ = \quad & S_{k-m+1}(\tilde{p}_1(j_1, \cdots, j_{m-1}), \tilde{p}_2(j, j_2, \cdots, j_{m-1})). \end{aligned}$$

□

Corollary 4 *Let $\tilde{l}(j_1, j)$ denote the set of lines*

$$\langle \tilde{p}_1(j_1, \cdots, j_{\kappa-1}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-2})\rangle$$

where $0 \le j_r < q$ for $2 \le r \le \kappa - 1$, and let

$$\tilde{L} := \{\tilde{l}(j_1, j) \mid 1 \le j_1 < q,\ 0 \le j < q\}.$$

Then we have

$$\bigcup_{1 \le j_1 < q} \bigcup \tilde{L}(j_1, j) = P_k(1) \backslash (P_\kappa(1) \cup P_\kappa(\kappa))$$

and

$$\bigcup \tilde{L} = q\Big(P_k(1) \backslash (P_\kappa(1) \cup P_\kappa(\kappa))\Big) \cup q(q-1)\Big(P_\kappa(1) \cap P_\kappa(\kappa)\Big).$$

The proof is omitted.

Next, we consider sets of planes. By a similar method we can get the same kind of results as we did for the sets of lines. We omit the details of the proofs. Define three kinds of points:

$$p_1 = (001j_1j_2\cdots j_{\kappa-2}),$$
$$p_2 = (010jj_1j_2\cdots j_{\kappa-3}),$$
$$p_3 = (100j^*jj_1j_2\cdots j_{\kappa-4}).$$

Let

$$p(j,j^*) := \{\langle p_1,p_2,p_3\rangle\backslash\langle p_2,p_3\rangle \mid 0\le j_r<q, 1\le r\le\kappa-2\},$$
$$L^*(j^*) := \{\langle p_2,p_3\rangle \mid 0\le j_r<q, 1\le r\le\kappa-3, 0\le j<q\}.$$

We have

$$\begin{aligned}\bigcup p(j,j^*) &= P_k(1)\backslash P_\kappa(\kappa),\\ \bigcup L^*(j^*) &= (P_\kappa(\kappa)\backslash P_{\kappa-1}(k))\cup q(S_{\kappa-2}(100j^*0\cdots 0)).\end{aligned}$$

Hence

$$\bigcup_{0\le j^*<q}\bigcup L^*(j^*) = q(P_\kappa(\kappa)\backslash P_{\kappa-2}(1)),$$

$$\bigcup_{0\le j<q, 0\le j^*<q}\bigcup p(j,j^*)\ \cup\ q\bigcup_{0\le j^*<q}\bigcup L^*(j^*) = q^2(P_k(1)\backslash P_{\kappa-2}(1)).$$

Therefore we get following lemma.

Lemma 8 *If*

$$P := \{\langle p_1,p_2,p_3\rangle \mid 0\le j_r<q,\ \textit{for}\ 1\le r\le\kappa-2,\ 0\le j<q,\ 0\le j^*<q\},$$

then

$$\bigcup P = q^2(P_k(1)\backslash P_{\kappa-2}(1)).$$

7 A general Construction

Denote

$$\pi_r := (1-q)\sum_{j=0}^{r-1} i_j + i_r = \sum_{j=1}^{r}(i_j - qi_{j-1}) + i_0,$$

for $0\le r\le\kappa$. From (5) and (7) of [17], or [5], we have the following lemma.

Lemma 9 *If $i_j\le qi_{j-1}$ for $1\le j\le\kappa$, then we have*

$$\pi_0\ge\pi_1\ge\cdots\ge\pi_\kappa \tag{24}$$

and

$$i_r = \sum_{j=0}^{r}\pi_j S_{r,j}\ \textit{for}\ 0\le r\le\kappa. \tag{25}$$

Lemma 10 *Define*

$$\alpha_{r,j} := \pi_j^* := \pi_j - q + 1\ \textit{for}\ 1\le j\le\kappa-2, j\le r\le\kappa-2,$$
$$\alpha_{r,0} := \pi_0^* := \pi_0 + 1\ \textit{for}\ 0\le r\le\kappa-2,$$
$$\lambda_{r,j} := 0\ \textit{for}\ 0\le r\le j\le\kappa-2.$$

Then $\alpha_{r,j},\lambda_{r,j}$ satisfy (16) and (17) (but i_0 is changed to i_0+1 in (16)).

Proof: By (25) of Lemma 9, $S_{r,j} = qS_{r,j+1}$ for $r > j+1$, and we have

$$\begin{aligned} i_r &= \sum_{j=0}^{r} \pi_j S_{r,j} = (\pi_0 + 1)S_{r,0} + \sum_{j=1}^{r} (\pi_j - q + 1)S_{r,j} \\ &= \sum_{j=0}^{r} \pi_j^* S_{r,j} = \sum_{j=0}^{r} (\alpha_{r,j} S_{r,j} + \lambda_{r,j}), \end{aligned}$$

for $0 < r \le \kappa - 2$, and

$$i_0 + 1 = \pi_0^*.$$

From (24) we get

$$\begin{aligned} &\alpha_{r,0} = \pi_0^* = \pi_0 + 1 \ge (\pi_1 - q + 1) + q = \pi_1^* + q = \alpha_{r,1} + q, \\ &\alpha_{r,j} = \pi_j^* = \pi_j - q + 1 \ge \pi_{j+1} - q + 1 = \pi_{j+1}^* = \alpha_{r,j+1}, \text{ for } 1 \le j \le \kappa - 3. \\ &\alpha_{r,j} = \alpha_{r+1,j} = \pi_j^*. \end{aligned}$$

□

For $0 \le r \le \kappa$, denote

$$m_r = \left\lfloor \frac{i_r}{q^r} \right\rfloor, \; p_r = i_r - m_r q^r.$$

For $0 \le r \le \kappa - 1$, let

$$\delta_r = \begin{cases} 0 & \text{if} \quad p_{r+1} \le p_r q, \\ 1 & \text{if} \quad p_{r+1} > p_r q. \end{cases}$$

Define

$$f(q) := 1 + \sum_{r=0}^{\kappa-3} (\delta_r (q^{r+1} - 1) + qp_r - p_{r+1}). \tag{26}$$

We note that $f(q)$ depends on the sequence $(i_0, i_1, \ldots, i_\kappa)$. However,

$$0 \le \delta_r (q^{r+1} - 1) + qp_r - p_{r+1} \le q^{r+1} - 2$$

and so

$$1 \le f(q) \le F(q) := \frac{q}{q-1}(q^{\kappa-2} - 1) - 2\kappa + 5, \tag{27}$$

where $F(q)$ only depends on q and κ, not the sequence.

Theorem 3 *For any almost chain permissible DS $(i_0, i_1, \cdots, i_\kappa)$, if it satisfies*

$$\begin{aligned} i_1 &\le qi_0 - q, \\ i_r &= qi_{r-1} - (q+1) \textit{ for } r = \kappa - 1, \kappa, \\ i_{\kappa-2} &= \lambda q^{\kappa-2} + (\mu - \epsilon) q^{\kappa-3} + 1, \end{aligned} \tag{28}$$

where λ and μ are defined by

$$i_{\kappa-3} = \lambda q^{\kappa-3} + \mu q^{\kappa-4} + \nu, \; 0 \le \mu < q, \, 0 \le \nu < q^{\kappa-4}, \tag{29}$$

$$\epsilon = \begin{cases} 1, & \textit{if } \nu = 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{30}$$

and

$$\pi_{\kappa-2} \ge q, \tag{31}$$

then $(i_0, i_1, \cdots, i_\kappa)$ is ACDS. Condition (31) can be changed to

$$m_{\kappa-2} \ge q + f(q) - 1. \tag{32}$$

Note: $\pi_{\kappa-2}$, $m_{\kappa-2}$ are defined by $(i_0 - 1, i_1, \cdots, i_\kappa)$.

Proof: By (28) we know that $(i_0-1, i_1, \cdots, i_\kappa)$ is chain permissible. Since $\pi_{\kappa-2} \geq q$, by the definition of Lemma 10 we get $\alpha_{r,j}, \lambda_{r,j}$ satisfying (16) and (17). From (32) we have

$$qi_{\kappa-3} \geq i_{\kappa-2} \geq q^{\kappa-1}.$$

Hence by (29)

$$\lambda = \lfloor i_{\kappa-3}/q^{\kappa-3} \rfloor \geq q \geq 1. \tag{33}$$

From Theorem 2, $(i_0, i_1, \cdots, i_\kappa)$ is ACDS.

If $(i_0-1, i_1, \cdots, i_{\kappa-2})$ satisfy

$$m_r = m_{r+1} + \delta_r, \text{ for } 0 \leq r \leq \kappa - 3, \tag{34}$$

then by [5] we have

$$m_{\kappa-2} = \pi_{\kappa-2} + \sum_{r=0}^{\kappa-3} (\delta_r(q^{r+1}-1) + qp_r - p_{r+1}). \tag{35}$$

Similar to [1, Lemma 6] and [17, Lemma 3] (or [5, Lemma 2]), for any chain permissible $(i'_0-1, i'_1, \cdots, i'_{\kappa-2})$ we can get $(i_0-1, i_1, \cdots, i_{\kappa-2})$ satisfying (34) by

$$i'_r = i_r + \alpha_r q^r \text{ for } 0 \leq r \leq \kappa - 3, \tag{36}$$

$$i'_{\kappa-2} = i_{\kappa-2},$$

where

$$\alpha_r := \sum_{i=r}^{\kappa-3} (m'_i - (m'_{i+1} + \delta'_i)) \text{ for } 0 \leq r \leq \kappa - 3;$$

m'_i and δ'_i are defined by $(i'_0-1, i'_1, \cdots, i'_{\kappa-2})$.

Hence for $(i'_0, i'_1, \cdots, i'_{\kappa-2})$ we define

$$\alpha_{r,j} = \pi^*_j + \alpha_r \text{ for } 0 \leq r \leq \kappa - 3,$$

where π^*_j are defined by Lemma 10, π_j are defined by $(i_0-1, i_1, \cdots, i_\kappa)$. Since $(i'_0-1, i'_1, \cdots, i'_{\kappa-2})$ is chain permissible, we have

$$\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\kappa-3} \geq 0.$$

Therefore $\alpha_{r,j}$, $0 \leq j \leq r \leq \kappa-2$, satisfy (17) by Lemma 10. From (36) and Lemma 10 we also get

$$i'_r = \sum_{j=0}^{r} \alpha_{r,j} S_{r,j} + \lambda_{r,j} \text{ for } 0 \leq r \leq \kappa - 2.$$

Hence by (35), condition (32) can be changed to condition (31), and by Theorem 2, $(i'_0, i'_1, \ldots, i'_{\kappa-2}, i_{\kappa-1}, i_\kappa)$ of Theorem 3 is ACDS.

Theorem 4 *For any almost chain permissible DS $(i_0, i_1, \cdots, i_\kappa)$, if it satisfies (28) and*

$$i_r = qi_{r-1} - (q+1) \textit{ for } r = \kappa, \kappa - 1, \tag{37}$$

$$U_{\kappa-2} - \omega q^\kappa \leq i_{\kappa-2} \leq U_{\kappa-2} := \lambda q^{\kappa-2} + (\mu - \epsilon) q^{\kappa-3} + 1 - q\delta, \tag{38}$$

where $\omega = \lfloor (\pi^*_{\kappa-2} - 1)/q^2 \rfloor$, and $\delta = 1$ for $k = 4$, $\delta = 0$ otherwise,

$$\pi^*_{\kappa-2} := \lfloor U_{\kappa-2}/q^{\kappa-2} \rfloor - q + 2 - f(q) \geq 1, \tag{39}$$

then $(i_0, i_1, \cdots, i_\kappa)$ is ACDS, where λ, μ, ϵ, $f(q)$ are defined by (29), (30), (31), and (26).

Proof: If $i_{\kappa-2} = U_{\kappa-2}$ and (34) is true, then by (37), (39), and Theorem 3, $(i_0, i_1, \cdots, i_\kappa)$ is ACDS. From the proof of Lemma 4, and by Lemma 10, (14) and (17) we have

$$\begin{aligned}
&\lambda q^{\kappa-1} + (\mu - \epsilon)q^{\kappa-2} - 2 - q^2\delta \\
= \;& q\sum_{l=0}^{\kappa-4}(\alpha_{\kappa-2,l}S_{\kappa-2,l} + \lambda_{\kappa-2,l}) + q(\alpha_{\kappa-2,\kappa-3}S_{\kappa-2,\kappa-3} + \lambda_{\kappa-2,\kappa-3}) \\
& -2 + q(\alpha_{\kappa-2,\kappa-2} - 1) \\
= \;& q\sum_{l=0}^{\kappa-2}\alpha_{\kappa-2,l}S_{\kappa-2,l} - q - 2 \\
\geq \;& q(\alpha_{\kappa-2,1} + q)S_{\kappa-2,0} + q\sum_{l=1}^{\kappa-2}\alpha_{\kappa-2,l}S_{\kappa-2,l} - q - 2 \\
\geq \;& \alpha_{\kappa-2,\kappa-2}q^{\kappa-1} + (q-1)q^{\kappa-1} - q - 2
\end{aligned}$$

Hence for $k \geq 3$ we get

$$\lambda \geq \alpha_{\kappa-2,\kappa-2}.$$

Therefore in Construction 1 of Lemma 10 we have

$$\pi^*_{\kappa-2} - 1 = \alpha_{\kappa-2,\kappa-2} - 1 = \min\{m(x) \mid x \in PG(\kappa, q)\}. \tag{40}$$

If (34) is true and $\omega > 0$, by Lemma 8 from Construction 1 we can give a modified new constructions as following: Define m' by

$$m'(x) = \begin{cases} m(x) - 1 \text{ if } x \in \langle p_1, p_2, p_3\rangle, \\ m(x), \text{ otherwise,} \end{cases}$$

where $\langle p_1, p_2, p_3\rangle$ is a plane in the set P of Lemma 8, $m(x)$ is defined by Table 1 and Lemma 10. Since $\langle p_1, p_2, p_3\rangle \cap P_{\kappa-2}(1) = \phi$, we have

$$\sum_{j=0}^{r-1} i_j = m'(P_r(1)) \geq m'(S_r), 1 \leq r \leq \kappa - 2$$

where S_r is any subspace in $PG(\kappa, q)$ of dimension $r - 1$. Since $\langle p_1, p_2, p_3\rangle \cap S_{\kappa-1}$ contains at least one point, and $\langle p_1, p_2, p_3\rangle \cap P_{\kappa-1}(1)$ contains exactly one point, we get

$$m'(P_{\kappa-1}(1)) \geq m'(S_{\kappa-1});$$

since $\langle p_1, p_2, p_3\rangle \cap S_\kappa$ contains at least a line, and $\langle p_1, p_2, p_3\rangle \cap P_\kappa(\kappa)$ is a line, we get

$$m'(P_\kappa(\kappa)) \geq m'(S_\kappa).$$

Hence $(i_0, i_1, \cdots, i_{\kappa-3}, U_{\kappa-2} - 1, i_{\kappa-1}, i_\kappa)$ satisfying (28), (34) and (37) is ACDS. We can repeat this modification until $i_{\kappa-2} = U_{\kappa-2} - q^2 \cdot q^{\kappa-2}$, when $\langle p_1, p_2, p_3\rangle$ are selected from the set P one by one in some order. We can also repeat the all modifications above ω times. In the end, by (40) and Lemma 8 we get:

$$\min\{m'(x) \mid x \in PG(\kappa, q)\} = \pi^*_{\kappa-2} - 1 - \omega q^2 \geq 0.$$

Hence any $(i_0, i_1, \cdots, i_\kappa)$ satisfying (28), (34), (37) (38), and (39) is ACDS. Similar to the proof above that (32) can be changed to (31), we can omit the condition (34).

Theorem 5 *For any almost chain permissible DS* $(i_0, i_1, \cdots, i_\kappa)$, *if it satisfies (28), (38), (39), and*

$$i_{\kappa-1} \geq U_{\kappa-1} - (\omega_1 + \omega_2) q(q-1) q^{\kappa-2}, \tag{41}$$

where

$$\begin{aligned} U_{\kappa-1} &= q i_{\kappa-2} - q - 1, \\ \omega' &= \left\lceil \frac{U_{\kappa-2} - i_{\kappa-2}}{q^\kappa} \right\rceil, \\ \omega_1 &= \left\lfloor \frac{\pi^*_{\kappa-2} - 1 - \omega' q^2}{q} \right\rfloor, \\ \omega_2 &= \min\left\{ \left\lfloor \frac{\pi^*_{\kappa-2} - 1 - \omega' q^2}{q(q-1)} \right\rfloor, \left\lfloor \frac{\lambda - 1 - \omega' q^2 - \omega_1 (q-1)}{q} \right\rfloor \right\}, \end{aligned}$$

and $U_{\kappa-2}$, $\pi^*_{\kappa-2}$, λ *are define by (38), (39), and (29), then* $(i_0, i_1, \cdots, i_\kappa)$ *is ACDS.*

Proof: If (28), (37), (38), and (39) are true, then by Theorem 4, $(i_0, i_1, \cdots, i_\kappa)$ is ACDS. Denote the corresponding value function by $m(x)$. Define m' by

$$m'(x) = \begin{cases} m(x) - 1 \text{ if } x \in \langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1, \cdots, j_{\kappa-2}) \rangle, \\ m(x) \text{ otherwise}, \end{cases}$$

where $\langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1 \cdots, j_{\kappa-2}) \rangle$ is a line from the set L in Corollary 3. Similar to the proof of Theorem 4, we get that $(i_0, i_1, \cdots, i_{\kappa-2}, i_{\kappa-1} - 1, i_\kappa)$ is ACDS. If $\langle p_1(j_1, \cdots, j_{\kappa-1}), p_2(j, j_1 \cdots, j_{\kappa-2}) \rangle$ are selected from L one by one in some order, then we can repeat this modification until $i_{\kappa-1} - q(q-1)q^{\kappa-2}$. We can also repeat all modifications above ω_1 times. By Theorem 4 for $i_{\kappa-1} = q i_{\kappa-2} - q - 1$ we have

$$\min\{m(x) \mid x \in PG(\kappa, q)\} \geq \pi^*_{\kappa-2} - 1 - \omega' q^2 \geq 0,$$

Hence, by Corollary 3, we get

$$\min\{m'(x) \mid x \in PG(\kappa, q)\} \geq 0, \tag{42}$$

$$\min\{m'(x) \mid x \in P_k(1) \backslash P_\kappa(1)\} \geq \lambda - 1 - \omega' q^2 - \omega_1 (q-1), \tag{43}$$

$$\min\{m'(x) \mid x \in (P_\kappa(1) \backslash P_{\kappa-1}(1)) \backslash P_\kappa(\kappa)\} \geq \pi^*_{\kappa-2} - 1 - \omega' q^2 - \omega_1 q, \tag{44}$$

$$\min\{m'(x) \mid x \in P_\kappa(1) \cap P_\kappa(\kappa)\} \geq \pi^*_{\kappa-2} - 1 - \omega' q^2, \tag{45}$$

where m' is the value function of the last modification. Similarly we can again give a further modified new constructions as follows: Define m'' by

$$m''(x) = \begin{cases} m'(x) - 1, \text{ if } x \in \langle \tilde{p}_1(j_1, \cdots, j_{\kappa-1}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-1}) \rangle, \\ m'(x), \text{ otherwise}, \end{cases}$$

where $\langle \tilde{p}_1(j_1, \cdots, j_{\kappa-1}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-1}) \rangle$ is a line from the set $\tilde{L}$ in Corollary 4. Similarly the DS corresponding to m'' is ACDS. By Corollary 4 we can repeat this modification $q(q-1)$ times if $\langle \tilde{p}_1(j_1, \cdots, j_{\kappa-1}), \tilde{p}_2(j, j_2, \cdots, j_{\kappa-1}) \rangle$ are selected from $\tilde{L}$ in Corollary 4 one by one in some order. We can also repeat all the modifications above ω_2 times. Hence by Corollary 4 and (42)–(45) we get

$$\min\{m''(x) \mid x \in PG(\kappa, q)\} \geq 0.$$

Therefore, similarly any almost chain permissible DS satisfying (28), (38), (39), (41) and $i_\kappa = i_{\kappa-1} q - q - 1$ is ACDS. By the core method in [6] we can modify the construction to make it valid for all $i_\kappa \leq q\, i_{\kappa-1} - q - 1$. □

8 Estimating $M(i)$ and $N(i)$

By definition, $M(i)$ is the number of sequences $(i_0, i_1, \ldots, i_\kappa)$ such that $i_0 \le i$ and

$$i_{\kappa-2} \le i_\kappa, \tag{46}$$
$$1 \le i_\kappa \le q\, i_{\kappa-1} - q - 1, \tag{47}$$
$$1 \le i_{\kappa-1} \le q\, i_{\kappa-2} - q - 1, \tag{48}$$
$$1 \le i_r \le q\, i_{r-1} \quad \text{for} \quad 1 \le r \le \kappa - 2. \tag{49}$$

For $3 \le t \le \kappa$, let σ_t be the number of sequences $(i_{\kappa-t}, i_{\kappa-t+1}, \ldots, i_\kappa)$ satisfying (46)–(49) for some fixed $i_{\kappa-t}$.

Lemma 11 *We have*

$$\begin{aligned}\sigma_3 &= \frac{q^2(q^2-1)^2}{6}\, i_{\kappa-3}^3 - \frac{q(q^2-1)(q^2+3q+1)}{4}\, i_{\kappa-3}^2 \\ &\quad + \frac{q(q+1)(q^2+8q+9)}{12}\, i_{\kappa-3} - (q+2).\end{aligned}$$

Proof: Combining (46) and (47) we get

$$i_{\kappa-1} \ge \frac{i_\kappa + q + 1}{q} \ge \frac{i_{\kappa-2} + q + 1}{q}.$$

From (48) we get $i_{\kappa-2} > 1$. Hence

$$\begin{aligned}\sigma_3 &= \sum_{i_{\kappa-2}=2}^{q\, i_{\kappa-3}} \sum_{i_{\kappa-1}=\lceil \frac{i_{\kappa-2}+q+1}{q} \rceil}^{q\, i_{\kappa-2}-q-1} \sum_{i_\kappa = i_{\kappa-2}}^{q\, i_{\kappa-1}-q-1} 1 \\ &= \sum_{i_{\kappa-2}=2}^{q\, i_{\kappa-3}} \sum_{i_{\kappa-1}=\lceil \frac{i_{\kappa-2}+q+1}{q} \rceil}^{q\, i_{\kappa-2}-q-1} (q\, i_{\kappa-1} - q - i_{\kappa-2}) \\ &= \sum_{i_{\kappa-2}=2}^{q-1} \sum_{i_{\kappa-1}=2}^{q\, i_{\kappa-2}-q-1} (q\, i_{\kappa-1} - q - i_{\kappa-2}) \\ &\quad + \sum_{a=1}^{i_{\kappa-3}-1} \sum_{i_{\kappa-2}=aq}^{aq+q-1} \sum_{i_{\kappa-1}=a+2}^{q\, i_{\kappa-2}-q-1} (q\, i_{\kappa-1} - q - i_{\kappa-2}) \\ &\quad + \sum_{i_{\kappa-1}=i_{\kappa-3}+2}^{q^2\, i_{\kappa-3}-q-1} (q\, i_{\kappa-1} - q - q\, i_{\kappa-3}) \\ &= \frac{q^2(q^2-1)^2}{6}\, i_{\kappa-3}^3 - \frac{q(q^2-1)(q^2+3q+1)}{4}\, i_{\kappa-3}^2 \\ &\quad + \frac{q(q+1)(q^2+8q+9)}{12}\, i_{\kappa-3} - (q+2).\end{aligned}$$

For $4 \le t \le \kappa$ we have

$$\sigma_t = \sum_{i_{\kappa-t+1}=1}^{q\, i_{\kappa-t}} \sigma_{t-1}. \tag{50}$$

Let

$$g(q,t) = q^{t-1} + q^{t-2} - 2q^2 - 4q - 1.$$

In particular, $g(q,3) = -(q^2+3q+1)$. From (50) and Lemma 11, using induction, we get the following lemma.

Lemma 12 *For $3 \le t \le \kappa$ and $q \ge 3$, we have*

$$\sigma_t = \frac{q^{t(t+1)/2-4}(q^2-1)^2}{t!}\, i_{\kappa-t}^t + \frac{q^{t(t-1)/2-2}(q^2-1)g(q,t)}{2\ (t-1)!}\, i_{\kappa-t}^{t-1} + O(i_{\kappa-t}^{t-2}).$$

In particular, $\sigma_\kappa = \sigma_\kappa(i_0)$ is given by Lemma 12. Since $M(i) = \sum_{i_0=1}^{i} \sigma_\kappa(i_0)$, we get the following lemma.

Lemma 13 *We have*

$$M(i) = \frac{q^{k(k-1)/2-4}(q^2-1)^2}{k!}\, i^k + \frac{q^{(k^2-3k-2)/2}(q^2-1)g(q,k)}{2\ (k-1)!}\, i^{k-1} + O(i^{k-2}).$$

Now we estimate $N(i)$. In addition to the conditions (4)–(6) defining almost chain-permissible sequences, we have introduced some extra conditions on i_1, $i_{\kappa-2}$, and $i_{\kappa-1}$ for the constructions, namely (28), (38), (39), and (41).

We note that if c_1 and c_2 are constants and $r \ge 1$, then

$$\begin{aligned}\sum_{m=c_1}^{qn-c_2} m^r &= \frac{(qn-c_2)^{r+1}}{r+1} + \frac{(qn-c_2)^r}{2} + O(n^{r-1})\\ &= \frac{q^{r+1}}{r+1}\, n^{r+1} - q^r\left(c_2 - \frac{1}{2}\right) n^r + O(n^{r-1}).\end{aligned}$$

In particular, the two main terms does not depend on c_1. Similarly, the two main terms of $\sum_{m=n/q+c_1}^{qn-c_2} m^r$ will not depend on c_1.

We first have to determine or estimate the quantities in the bounds (28), (38), (39), and (41).

From (29), (30), and (38) we have

$$qi_{\kappa-3} - U_{\kappa-2} = \begin{cases} q^{\kappa-3} - 1 + q\delta & \text{if } \nu = 0,\\ q\nu - 1 + q\delta & \text{if } \nu > 0.\end{cases} \tag{51}$$

In particular,

$$U_{\kappa-2} \ge qi_{\kappa-3} - q^{\kappa-3} + 1. \tag{52}$$

By (27), (38) and (39) we have

$$\begin{aligned}\omega &= \lfloor (\pi^*_{\kappa-2} - 1)/q^2 \rfloor \ge \frac{\pi^*_{\kappa-2}}{q^2} - 1\\ &= \frac{\lfloor U_{\kappa-2}/q^{\kappa-2} \rfloor - q + 2 - f(q)}{q^2} - 1\\ &> \frac{U_{\kappa-2}}{q^\kappa} - \frac{1}{q^2}(F(q) + q^2 + q - 1).\end{aligned}$$

Hence

$$U_{\kappa-2} - \omega q^\kappa < F_1(q) := q^{\kappa-2}(F(q) + q^2 + q - 1). \tag{53}$$

By (41) and (39)

$$\begin{aligned}\omega_1 &\ge \frac{\pi^*_{\kappa-2} - \omega' q^2}{q} - 1\\ &\ge \frac{\pi^*_{\kappa-2} - 1 - q^2(U_{\kappa-2} - i_{\kappa-2})/q^\kappa}{q} - 1\\ &\ge \frac{i_{\kappa-2}}{q^{\kappa-1}} - \frac{F(q)}{q} - q - 2.\end{aligned} \tag{54}$$

Similarly

$$\Big\lfloor\frac{\pi^*_{\kappa-2}-1-\omega' q^2}{q(q-1)}\Big\rfloor \geq \frac{i_{\kappa-2}}{(q-1)q^{\kappa-1}} - \frac{F(q)}{(q-1)q} - \frac{2q}{q-1}. \tag{55}$$

By (38), (41), and (54) we have

$$\begin{aligned}
&\Big\lfloor\frac{\lambda-1-\omega' q^2-\omega_1(q-1)}{q}\Big\rfloor \\
\leq\;& \frac{1}{q}\Big(-1-\frac{(\mu-\epsilon)q^{\kappa-3}-q\delta+1}{q^{\kappa-2}}+\frac{i_{\kappa-2}}{q^{\kappa-2}}-\omega_1(q-1)\Big) \\
\leq\;& \frac{i_{\kappa-2}}{q^{\kappa}}+\frac{F(q)(q-1)}{q^2}+\frac{(q+2)(q-1)}{q}-\frac{1}{q}\Big(\frac{\mu+\epsilon}{q}+\frac{1-q\delta}{q^{\kappa-2}}+1\Big) \\
=:\;& \frac{i_{\kappa-2}}{q^{\kappa}}+F_2(q).
\end{aligned} \tag{56}$$

Hence if

$$i_{\kappa-2} \geq F_3(q) := (q-1)q^{\kappa}\Big(F_2(q)+\frac{F(q)}{q(q-1)}+\frac{2q}{q-1}\Big), \tag{57}$$

then by (55) and (56) we get

$$\omega_2 = \Big\lfloor\frac{\lambda-1-\omega' q^2-\omega_1(q-1)}{q}\Big\rfloor \geq \frac{i_{\kappa-2}}{q^{\kappa}}+F_2(q)-\Big(2q+1-\frac{1}{q}\Big). \tag{58}$$

From (41), (54), and (58), if (57) is true, then we have

$$\begin{aligned}
&U_{\kappa-1}-(\omega_1+\omega_2)(q-1)q^{\kappa-1} \\
\leq\;& \frac{1}{q}i_{\kappa-2}+\Big(\frac{F(q)}{q}-F_2(q)+3q+3-\frac{1}{q}\Big)(q-1)q^{\kappa-1}-q-1 \\
=:\;& \frac{1}{q}i_{\kappa-2}+F_4(q).
\end{aligned} \tag{59}$$

If

$$i_{\kappa-3} \geq q^{\kappa-3}(q+2+F(q)), \tag{60}$$

then

$$\Big\lfloor\frac{i_{\kappa-3}}{q^{\kappa-3}}\Big\rfloor \geq q+1+F(q),$$

and so, by (38) and (29), we have

$$\Big\lfloor\frac{U_{\kappa-2}}{q^{\kappa-2}}\Big\rfloor \geq \lambda-2 = \Big\lfloor\frac{i_{\kappa-3}}{q^{\kappa-3}}\Big\rfloor - 2 \geq q+f(q)-1,$$

i.e. (39) is true. We summarize this discussion as follows: if $(i_0, i_1, \ldots, i_\kappa)$ is a chain permissible sequence such that

$$i_1 \leq q i_0 - q,$$

$$i_{\kappa-3} \geq q^{\kappa-3}(q+2+F(q)),$$

$$\max\{F_1(q), F_3(q)\} \leq i_{\kappa-2} \leq q i_{\kappa-3} - q^{\kappa-3}+1,$$

$$\frac{1}{q}i_{\kappa-2}+F_4(q) \leq i_{\kappa-1} \leq q i_{\kappa-2}-q-1,$$

then by (28); (60); (38), (39), (52), and (53); (41) and (59), respectively, $(i_0, i_1, \ldots, i_\kappa)$ is ACDS. Note that $F_1(q)$–$F_4(q)$ all are constants independent of the sequence $(i_0, i_1, \ldots, i_\kappa)$.

Let $N_u(i)$ denote the number of such sequences with $i_0 \leq i$. Then $N(i) \geq N_u(i)$. We will determine the two main terms of $N_u(i)$. Since $F(q)$, $F_1(q)$, $F_3(q)$, and $F_4(q)$ are constants, they will not affect the two main terms.

In analogy to the estimate of $M(i)$, for $3 \leq t \leq \kappa$, let ρ_t be the number of sequences $(i_{\kappa-t}, i_{\kappa-t+1}, \ldots, i_\kappa)$ satisfying the conditions. The computations are similar to the computations for $M(i)$ and we only give the results.

Lemma 14 *For $3 \leq t \leq \kappa - 1$ and $q \geq 3$, we have*

$$\begin{aligned} \rho_t &= \frac{q^{t(t+1)/2-4}(q^2-1)^2}{t!}\, i_{\kappa-t}^t \\ &\quad + \frac{q^{t(t-1)/2-2}(q^2-1)(g(q,t)-2(q^2-1)(q^{\kappa-3}-1))}{2\,(t-1)!}\, i_{\kappa-t}^{t-1} \\ &\quad + O(i_{\kappa-t}^{t-2}). \end{aligned}$$

Further,

$$\begin{aligned} \rho_\kappa &= \frac{q^{\kappa(\kappa+1)/2-4}(q^2-1)^2}{\kappa!}\, i_0^\kappa \\ &\quad + \frac{q^{\kappa(\kappa-1)/2-2}(q^2-1)(g(q,\kappa)-2(q^2-1)(q^{\kappa-2}+q^{\kappa-3}-1))}{2\,(\kappa-1)!}\, i_0^{\kappa-1} \\ &\quad + O(i_0^{\kappa-2}) \end{aligned}$$

and

$$\begin{aligned} N_u(i) &= \frac{q^{k(k-1)/2-4}(q^2-1)^2}{k!}\, i^k \\ &\quad + \frac{q^{(k^2-3k-2)/2}(q^2-1)(g(q,k)-2(q^2-1)(q^{k-3}+q^{k-4}-1)}{2\,(k-1)!}\, i^{k-1} \\ &\quad + O(i^{k-2}). \end{aligned}$$

Remark. If $q = 2$, then we change Table 1 a little: Only if $j_{\kappa-1} = q-1$, we take $m(x) = \lambda - 2$. Since $P_{\kappa-1,\kappa-3}\backslash P_\kappa(\kappa)$ contains a simgle point x^*, we take $m(x^*) = \alpha_{\kappa-2,\kappa-3} - 1$. We take $m(P_{\kappa-1,\kappa-2}) = \alpha_{\kappa-2,\kappa-2} - 2$. We require that (11–15) are true. If $m(P_{\kappa-1,\kappa-2}) = \alpha_{\kappa-2,\kappa-2} - 1$, then we may use Lemma 1. Now $m(P_{\kappa-1,\kappa-2})$ decrease by 1 and $\sum_{j=0}^{l} i_j$ remains unchanged for $0 \le l \le \kappa-2$. So we can prove that Theorem 2 is true for $\lambda \ge 2$ and $\alpha_{\kappa-2,\kappa-2} \ge 2$. By (33), Theorems 3, 4, and 5 are true if the right sides of (31), (32), and (39) are increased by 1. Therefore, Theorem 1 a) and c) is true also for $q = 2$. A result similar to Theorem 1 b) can be found, but we omit the details here.

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