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The Institution of Multialgebras

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The Institution of Multialgebras

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1 Introduction

The concept of institutions [3] has become the standard framework for presenting model-theoretic aproaches to logic and, in particular, to algebraic specification.

In this paper we show that the multialgebras form an exact institution $\mathcal{MA}$. We prove that $\mathcal{MA}$ is an institution in section 3.1. In section 3.2 we show that the model functor for multialgebras $\mathsf{Mod}_{\mathcal{MA}}$ sends finite co-limits in $\mathbf{Th}$ (category of specifications) to limits in $\mathbf{Cat}$, i.e. $\mathcal{MA}$ is an exact institution [5] (called institution with composable signatures in [6]). We also mention the well known fact that every exact institution satisfies the amalgamation lemma. The results are not used in this paper but they form a basis for subsequent work on structuring multialgebras and their specifications, in particular, on amalgamation and parameterized multialgebras. In study of parameterizated specifications the co-limits (actually pushouts) are used for defining parameter instantiation. The exactness of the model functor (actually the amalgamation lemma) ensures that corresponding instantiation can be performed at the semantic level.

We begin, in section 2, by presenting the background definitions of multilagebras and collecting the relevant definitions and results about institutions.

2 Preliminaries

2.1 Notation

We use the notation $|\mathbf{C}|$ to denote the objects of a category $\mathbf{C}$. The same notation is used to denote the carrier $|A|$ of an algebra A. (This shouldn't cause any confusion.) Institutions are written with the script font $\mathcal{I}$, categories with bold $\mathbf{Cat}$, and functors with Sans Serif Func.

Since specifications is categories we write $\mathbf{Spec}$ for ordinary specifications. Sequences $s_1, \ldots, s_k$ will be often denoted by $\overline{s}$. Application of functions are then understood to not distribute over the elements, i.e., $f(\overline{s})$ denotes the term $f(s_1, \ldots, s_k)$. Occasionally, a sequence $s_1 \ldots s_k$ may be denoted by s^* – applications of functions are then understood to distribute over the elements, i.e., $f(s^*)$ denotes the sequence $(f(s_1), \ldots, f(s_k))$. We will denote the disjoint union of sets A, B by $A \uplus B$.

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2.2 Algebraic Signatures

Signatures for multialgebras are the same as classical *algebraic signatures.*

Definition 2.1 *The category of signatures* **Sign** *has:*

- *signatures as objects: a signature Σ is a pair of sets $(\mathbf{S}, \Omega)$ of symbols for names of sorts and operations. Each operation symbol $\omega \in \Omega$ is a (k+2)-tuple: $\omega : s_1 \times \cdots \times s_k \to s$, where $s_1, \ldots, s_k, s \in S$ and $k \geq 0$. ω is the name of the operation and $s_1 \times \ldots \times s_k \to s$ its arity. If $k = 0$ then an operation $c :\to s$ is called a constant of sort s.*
- *signature morphisms as arrows: a signature morphism $\mu : \Sigma \to \Sigma'$ is a pair $\mu = (\mu_S, \mu_\Omega)$ of (total) functions: $\mu_S : S \to S'$, $\mu_\Omega : \Omega \to \Omega'$, such that $\mu_\Omega(\omega : s_1 \times \cdots \times s_n \to s) = \omega' : \mu_S(s_1) \times \cdots \times \mu_S(s_n) \to \mu_S(s)$*
- *Identities are the identity signature morphisms and morphism are composed component-wise.*

In the standard way, we extend the signature morphism $\mu : \Sigma \to \Sigma'$ to terms, we use the notation $T(\Sigma, X)$ for the Σ terms with X as variables.

Definition 2.2 *Extension of a signature morphism μ to terms $\tilde{\mu} : T(\Sigma, X) \to T(\Sigma', X')$ is defined by:*

- $\tilde{\mu}(x_s) = x_{\mu(s)}$*, for each variable $x_s \in X_s$*
- $\tilde{\mu}(c) = \mu(c)$
- $\tilde{\mu}(\omega(t_1, \ldots, t_n)) = \mu(\omega)(\tilde{\mu}(t_1), \ldots, \tilde{\mu}(t_n))$

In general variables can be renamed too, but (without loss of generality) we simplify the presentation. We will write $\mu(t)$ instead of $\tilde{\mu}(t)$, for terms $t \in T(\Sigma, X)$.

2.2.1 Algebraic signatures have all finite co-limits

It is well known that the category of algebraic signatures is co-complete, see e.g. [2]. We are merely restating here this standard fact. Limiting our attention to finite co-limits, it is sufficient to consider the existence of initial object, co-products (sums) and co-equlizers (see e.g. [1]). Since multialgebras use algebraic signatures all these results will also hold for multialgebric signatures, $\mathbf{Sign}_{\mathcal{MA}}$.

Fact 2.3 *The empty signature, $\Sigma_\emptyset$ is initial in* **Sign**

Fact 2.4 *The sum of two signatures; $\Sigma + \Sigma'$ is the disjoint union (of sorts and operations), with the natural injections.*

Fact 2.5 *Given two signature morphims $\mu, \nu : \Sigma \to \Sigma'$, let $\sim$ be the least equivalence on Σ' induced by the relation with components:*

- *Sorts:* $\sim_{S'} = \{\langle \mu(s), \nu(s) \rangle : s \in \Sigma\}$,
- *Operations* $\sim_{\Omega'} = \{\langle \mu(\omega), \nu(\omega) \rangle : \omega \in \Omega\}$

Then $\Sigma'/_\sim$ is a co-equalizer object, with canonical signature morphim $\iota : \Sigma' \to \Sigma'/_\sim$, and we have that $\mu; \iota = \nu; \iota$, by construction.

Note that if $\sigma : \Sigma' \to Z$ is a signature morphism such that $\mu; \sigma = \nu; \sigma$, then the kernel of σ has to include $\sim$, so the signature morphism $u_\sigma : \Sigma'/_\sim \to Z$, defined by $u_\sigma([s']_\sim) = \sigma(s')$ and $u_\sigma([\omega']_\sim) = \sigma(\omega')$ is the unique factorization arrow.

Fact 2.6 *[2] The category* **Sign** *of algebraic signatures has all (finite) co-limits.*

2.3 Multialgebras

We will now summarize the relevant notions about multialgebras (for an overview, see [9, 7]).

A multialgebra for a signature Σ is an algebra where operations may be set-valued. $\mathcal{P}(y)$ denotes the power set of set y.

Definition 2.7 *(Multialgebra) A multialgebra A for Σ is given by:*

- *a set s^A, the carrier set, for each sort symbol $s \in \mathbf{S}$*
- *a subset $c^A \in \mathcal{P}(s^A)$, for each constant, $c :\rightarrow s$*
- *an operation $\omega^A : s_1^A \times \cdots \times s_k^A \rightarrow \mathcal{P}(s^A)$ for each symbol $\omega : s_1 \times \cdots \times s_k \rightarrow s \in \Omega$*

One sometimes demands that constants and operations are total [9, 8], i.e. never return empty set and take values only in $\mathcal{P}^+(s^A)$, the nonempty subsets of s^A. We will not make this assumption.

Note that for a constant $c \in \Omega$, c^A denotes a (sub)set of the carrier s^A. This allows one to use constants as predicates as was done, for instance, in [4].

As homomorphisms of multialgebras, we will use weak homomorphisms (see [7] for alternative notions).

Definition 2.8 *Given two multialgebras A and B, a function $h : |A| \rightarrow |B|$ is a (weak) homomorphism if:*

1. *$h(c^A) \subseteq c^B$, for each constant $c :\rightarrow s$*
2. *$h(\omega^A(a_1, \ldots, a_n)) \subseteq \omega^B(h(a_1), \ldots, h(a_n))$, for each operation $\omega : s_1 \times \cdots \times s_n \rightarrow s \in \Omega$ and for all $a_i \in s_i^A$.*

Saying "homomorphism" we will always mean weak homomorphism.

Definition 2.9 *The category of Σ-multialgebras, $\mathbf{MAlg}_\Sigma$, has Σ-multialgebras as objects and homomorphisms as arrows. The identity arrows are the identity homomorphisms and composition of arrows is obvious composition of homomorphisms.*

Multialgebraic specifications are written using the following formulae:

Definition 2.10 *Formulae of multialgebraic specifications are of the following forms:*

1. *Atomic formulae, suppose $t, t' \in T(\Sigma, X)$:*
   - *$t \doteq t'$ (equality), t and t' denote the same one-element set.*
   - *$t \prec t'$ (inclusion), the set interpreting t is included in the set interpreting t'.*
2. *$a_1, \ldots, a_n \Rightarrow b_1, \ldots, b_m$, where either $n > 0$ or $m > 0$ and each a_i and b_j is atomic.*

Given a set of variables X, an assignment is a function $\alpha : X \rightarrow |A|$ assigning *individual* elements of the carrier of A to variables. It induces a unique interpretation $\overline{\alpha} : T(\Sigma(X)) \rightarrow A$ of every term t (with variables from X) in A.

Definition 2.11 *Given a Σ-multialgebra A, an assignment to variables X is a function $\alpha : X \rightarrow |A|$. An assignment induces a unique interpretation $\overline{\alpha}(t)$ in A of any term $t \in T(\Sigma, X)$ as follows:*

- $\overline{\alpha}(x) = \{\alpha(x)\}$
- $\overline{\alpha}(c) = c^A$
- $\overline{\alpha}(\omega(t_1, \ldots, t_n)) = \bigcup_{a_i \in \overline{\alpha}(t_i)} \omega^A(a_1, \ldots, a_n)$

Keep in mind that variables are assigned not sets but individual elements of the carrier. We will write $\alpha(t)$ instead of $\overline{\alpha}(t)$.

Satisfaction of formulae in a multialgebra is defined as follows:

Definition 2.12 *Given an assignment $\alpha : X \to |A|$:*

1. $A \models_\alpha t \doteq t'$ iff $\overline{\alpha}(t) = \overline{\alpha}(t') = \{e\}$, fore someelement $e \in |A|$
2. $A \models_\alpha t \prec t'$ iff $\overline{\alpha}(t) \subseteq \overline{\alpha}(t')$
3. $A \models_\alpha a_1, \ldots, a_n \Rightarrow b_1, \ldots, b_m$ iff $\exists i : 1 \leq i \leq n : A \not\models_\alpha a_i$ or $\exists j : 1 \leq j \leq m : A \models_\alpha b_j$
4. $A \models \varphi$ iff $A \models_\alpha \varphi$ for all α

Putting these definitions together, we show (in section 3.1) that the multialgebras form an institution $\mathcal{MA}$. Before that we recall the standard concepts of institution and some results which will be relevant for us in investigating the institution of multialgebras.

2.4 Institutions

Definition 2.13 *[3] An institution is a quadruple $\mathcal{I} = (\mathbf{Sign}, \mathsf{Sen}, \mathsf{Mod}, \models)$, where:*

- $\mathbf{Sign}$ *is a category of signatures.*
- $\mathsf{Sen} : \mathbf{Sign} \to \mathbf{Set}$ *is a functor which associates a set of sentences to each signature.*
- $\mathsf{Mod} : \mathbf{Sign}^{op} \to \mathbf{Cat}$ *is a functor which associates a category of models, whose morphisms are called Σ-morphisms, to each signature Σ*
- $\models$ *is a satisfaction relation – for each signature Σ, a relation $\models_\Sigma \subseteq |\mathsf{Mod}(\Sigma)| \times \mathsf{Sen}(\Sigma)$, such that the following satisfaction condition holds: for any $M' \in \mathsf{Mod}(\Sigma'), \mu : \Sigma \to \Sigma', \phi \in \mathsf{Sen}(\Sigma)$*

$$M' \models_{\Sigma'} \mathsf{Sen}(\mu)(\phi) \textit{ iff } \mathsf{Mod}(\mu)(M') \models_\Sigma \phi$$

The definition can be represented as the following diagram:

$$\begin{array}{ccccc} \Sigma & \mathsf{Mod}(\Sigma) & \models_\Sigma & \mathsf{Sen}(\Sigma) \\ \Big\downarrow{\scriptstyle \mu} & \Big\uparrow{\scriptstyle \mathsf{Mod}(\mu)} & & \Big\downarrow{\scriptstyle \mathsf{Sen}(\mu)} \\ \Sigma' & \mathsf{Mod}(\Sigma') & \models_{\Sigma'} & \mathsf{Sen}(\Sigma') \end{array}$$

The following subsections review institution independend concepts and results which will be used in the later section.

2.4.1 Category of specifications

Based on the above satisfaction relation for institutions (definition 2.13) we write: $\Gamma \models_\Sigma \varphi$ iff $\forall M \in \mathsf{Mod}(\Sigma) : M \models_\Sigma \Gamma \Rightarrow M \models_\Sigma \varphi$. With this in mind we write $\Gamma^\bullet$ for the semantical consequences of Γ i.e. $\Gamma^\bullet = \{\varphi : \Gamma \models \varphi\}$

A *theory* (specification) in an institution is any pair $Th = (\Sigma, \Gamma)$ where $\Sigma \in |\mathbf{Sign}|$ and $\Gamma \subseteq \mathsf{Sen}(\Sigma)$. For a given institution $\mathcal{I}$, we have the corresponding category of theories $\mathbf{Th}$ (possibly indexed by the institution, $\mathbf{Th}_\mathcal{I}$) with theories as objects and theory morphisms $\mu : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$, where $\mu : \Sigma \to \Sigma'$, is a signature morphism such that: $\Gamma' \models_{\Sigma'} \mathsf{Sen}(\mu)(\Gamma)$. The models for the theory $Th = (\Sigma, \Gamma)$ is the full sub category $\mathsf{Mod}_\models(\Sigma, \Gamma)$ of $\mathsf{Mod}(\Sigma)$ where $M \in \mathsf{Mod}_\models(\Sigma, \Gamma)$ iff $M \models_\Sigma \varphi, \forall \varphi \in \Gamma$, we will write $\mathsf{Mod}(\Sigma, \Gamma)$ instead of $\mathsf{Mod}_\models(\Sigma, \Gamma)$. The satisfaction condition gives that $\mathsf{Mod}(\mu)(\mathsf{Mod}(\Sigma', \Gamma')) \subseteq \mathsf{Mod}(\Sigma, \Gamma)$, for each morphism $\mu : (\Sigma, \Gamma) \to (\Sigma', \Gamma') \in \mathbf{Th}$. This means that the functor Mod can be extended to a functor $\mathsf{Mod}_\models : \mathbf{Th}^{op} \to \mathbf{Cat}$. There is a canonic projection (forgetful) functor $\mathsf{Sign} : \mathbf{Th} \to \mathbf{Sign}$ and there is an embedding functor $\mathsf{th} : \mathbf{Sign} \to \mathbf{Th}$ defined by $\mathsf{th}(\Sigma) = (\Sigma, \emptyset)$.

2.4.2 Construction of co-limits of specifications

The following institution independent result ensures that the category of specifications has all co-limits the signature category has. This result is used to create co-limits of specifications by first creating the co-limit for the coresponding signatures.

Theorem 2.14 *[3] The functor* $\mathsf{Sign} : \mathbf{Th} \to \mathbf{Sign}$ *reflects co-limits, in any institution* $\mathcal{I}$.

As a particular case, the theorem means that: Given specifications $\mathbf{X} = (\Sigma, \Phi)$, $\mathbf{X^1} = (\Sigma^1, \Phi^1)$, $\mathbf{X^2} = (\Sigma^2, \Phi^2)$, and specification morphisms $\mu_1 : \mathbf{X} \to \mathbf{X^1}$ and $\mu_2 : \mathbf{X} \to \mathbf{X^2}$. If the diagram to the left is a pushout of signatures than the diagram to the right is a pushout of specifications:

$$\begin{tikzcd}
\Sigma \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] & \Sigma^1 \arrow[d, "{\mu'}_2"] & & \mathbf{X} \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] & \mathbf{X^1} \arrow[d, "{\mu'}_2"] \\
\Sigma^2 \arrow[r, "{\mu'}_1"'] & \Sigma' & \arrow[l, phantom, "\overset{\mathsf{Sign}}{\Longleftarrow}"] & \mathbf{X^2} \arrow[r, "{\mu'}_1"'] & \mathbf{X'}
\end{tikzcd}$$

where $\mathbf{X'} = (\Sigma', \Phi')$ and $\Phi' = {\mu'}_1(\Phi^2) \cup {\mu'}_2(\Phi^1)$.

2.4.3 Continuity of Mod and amalgamation

Construction on specifications can be "carried over" to the respective constructions on their model classes provided that the Mod functor has some desired properties. Typical constructions on specifications are co-limits and the desired property of Mod is that it transforms co-limits in $\mathbf{Th}$ to limits in $\mathbf{Cat}$.

Definition 2.15 *An institution* $\mathcal{I}$ *is*

1. semi exact *iff* $\mathbf{Sign}$ *has pushouts and* Mod *sends pushouts in* $\mathbf{Sign}$ *to pullbacks in* $\mathbf{Cat}$,
2. exact *iff* $\mathbf{Sign}$ *has finite co-limits and* Mod *sends finite co-limits in* $\mathbf{Sign}$ *to limits in* $\mathbf{Cat}$.

Of course, any exact institution is also semi exact. The importance of this notion is exemplified by the following lemma which indicates the construction for instantiation of parameterized specifications.

Lemma 2.16 *(Amalgamation Lemma).*
In any semi exact institution $\mathcal{I}$*, for every pushout of signatures (on the left):*

$$\begin{tikzcd}
\Sigma \arrow[r, "\mu_1"] \arrow[d, "\mu_2"'] & \Sigma^1 \arrow[d, "{\mu'}_2"] & & \mathsf{Mod}(\Sigma) & \mathsf{Mod}(\Sigma^1) \arrow[l, "_|_{\mu_1}"'] \\
\Sigma^2 \arrow[r, "{\mu'}_1"'] & \Sigma' & \arrow[r, phantom, "\overset{\mathsf{Mod}}{\Longrightarrow}"] & \mathsf{Mod}(\Sigma_2) \arrow[u, "_|_{\mu_2}"] & \mathsf{Mod}(\Sigma') \arrow[l, "_|_{\mu'_1}"] \arrow[u, "_|_{\mu'_2}"']
\end{tikzcd}$$

we have that: for any two models $M_1 \in \mathsf{Mod}(\Sigma^1)$ *and* $M_2 \in \mathsf{Mod}(\Sigma^2)$ *satisfying* $M_1|_{\mu_1} = M_2|_{\mu_2}$*, there exists a unique model* $M' \in \mathsf{Mod}(\Sigma')$*, such that* $M'|_{{\mu'}_1} = M_2$ *and* $M'|_{{\mu'}_2} = M_1$.

The corresponding amalgamation property holds also for homomorphisms. In fact, the amlgamation lemma tells that the model class of a pushout Σ' of signatures along μ_1, μ_2 is a pullback (in $\mathbf{Cat}$) of the respective morphisms $\_|_{\mu_1}$, $\_|_{\mu_2}$. The model M' is the *amalgamated union* of M_1 and M_2.

By theorem 2.14, the amalgamation lemma holds then also for pushouts of specifications, since these are constructed from pushouts of signatures.

3 Institution of multialgebras, $\mathcal{MA}$

We now define and prove that the multialgebras form an institution $\mathcal{MA}$ (subsection 3.1) and that this institution is exact (subsection 3.2).

3.1 Multialgebras form an institution

First we apply the standard concept of reduct to multialgebras.

Definition 3.1 *Let $\mu : \Sigma \to \Sigma'$ be a signature morphism.*
Reduct of an algebra: The μ-reduct of a Σ'-multialgebra A', is the Σ-multialgebra $A'|_\mu$ defined by:

$$\begin{array}{ll} s^{A'|_\mu} = \mu(s)^{A'} & \text{, for all } s \in S, \\ \omega^{A'|_\mu} = \mu(\omega)^{A'} & \text{, for all } \omega \in \Omega, \end{array}$$

Reduct of assignment: For a set of variables X, t a Σ' term, A' a Σ' algebra and $\alpha' : \mu(X) \to A'$ an assignment for A', the μ-reduct of α', $\alpha'|_\mu : X \to A'|_\mu$ is defined by:

$$(\alpha'|_\mu)_s(x) = \alpha'_{\mu(s)}(\mu(x))$$

Reduct of a homomorphism: The μ reduct of a weak Σ' homomorphism $h' : A' \to B'$, is the weak Σ homomorphism $h'|_\mu : A'|_\mu \to B'|_\mu$ defined by:

$$(h'|_\mu)_s = h'_{\mu(s)}$$

If one wants to allow possible renamings of variables X along the signature morphisms, the definition would be entirely analogous – we omit this technicality.

Fact 3.2 *The reduct of a multialgebra is a multialgebra, the reduct of an assignment is an assignment to the reduct algebra, the reduct of a weak homomorphism is a weak homomorphism between the reduct of two algebras.*

The proof of this fact is analogous to the classical case.

We are now ready to define the model functor $\mathsf{Mod}_{\mathcal{MA}} : \mathbf{Sign}^{op} \to \mathbf{Cat}$ which maps each $\Sigma \in |\mathbf{Sign}|$ to the category of all Σ-multialgebras $\mathbf{MAlg}_\Sigma$.

Definition 3.3 *The functor $\mathsf{Mod}_{\mathcal{MA}} : \mathbf{Sign}^{op} \to \mathbf{Cat}$ is defined by:*

- *objects:* $\mathsf{Mod}_{\mathcal{MA}}(\Sigma) = \mathbf{MAlg}_\Sigma$
- *arrows:* $\mathsf{Mod}_{\mathcal{MA}}(\mu : \Sigma \to \Sigma') = \mathsf{Mod}_{\mathcal{MA}\mu} : \mathbf{MAlg}_{\Sigma'} \to \mathbf{MAlg}_\Sigma$, *where the functor* $\mathsf{Mod}_{\mathcal{MA}\mu}$ *is given by:*
  1. $\mathsf{Mod}_{\mathcal{MA}\mu}(A') = A'|_\mu$
  2. $\mathsf{Mod}_{\mathcal{MA}\mu}(h') = h'|_\mu$

Lemma 3.4 *(Reduct theorem) If $\mu : \Sigma \to \Sigma'$ is a signature morphism, X a set of variables, t a Σ' term, A' a Σ' algebra and $\alpha' : \mu(X) \to A'$ an assignment for A' then we have that:*

$$\alpha'|_\mu(t)^{A'|_\mu} = \alpha'(\mu(t))^{A'}$$

Proof. The proof goes by induction on the complexity of the term t.

1. $t = x \in X_s$.

$$\begin{array}{ll} & \alpha'|_\mu(x)^{A'|_\mu} \\ = & \{ \text{ assignment} \} \\ & (\alpha'|_\mu)_s(x) \\ = & \{ \text{ def. } \alpha'|_\mu \} \\ & (\alpha'_{\mu(s)})(\mu(x)) \\ = & \{ \text{ assignment} \} \\ & \alpha'(\mu(x))^{A'} \end{array}$$

2. $t = c, (c :\rightarrow s)$

$$\begin{array}{ll} & \alpha'|_\mu(c)^{A'|_\mu} \\ = & \{ \text{ no assignment for constants} \} \\ & c^{A'|_\mu} \\ = & \{ \text{ def. } A'|_\mu \} \\ & \mu(c)^{A'} \\ = & \{ \text{ no assignment, for constants} \} \\ & \alpha'(\mu(c))^{A'} \end{array}$$

3. $t = \omega(t_1, \dots, t_n), (\omega : s_1 \times \dots \times s_n \rightarrow s)$

$$\begin{array}{ll} & \alpha'|_\mu(\omega(t_1, \dots, t_n)^{A'|_\mu} \\ = & \{ \text{ assignment on function} \} \\ & \bigcup_{a_i \in \alpha'|_\mu(t_i)} \omega^{A'|_\mu}(a_1, \dots, a_n) \\ = & \{ \text{ ind. hyp. and def reduct} \} \\ & \bigcup_{a_i \in \alpha'(\mu(t_i))} \mu(\omega)^{A'}(a_1, \dots, a_n) \\ = & \{ \text{ assignment on function} \} \\ & \alpha'(\mu(\omega)(\mu(t_1), \dots, \mu(t_n))^{A'} \end{array}$$

□

The following lemma leads immediately to the satisfaction condition.

Lemma 3.5 *For any signature morphism $\mu : \Sigma \rightarrow \Sigma'$ and Σ' algebra A', given assignment*

- $\alpha : X \rightarrow A'|_\mu$, *define* $\alpha' : \mu(X) \rightarrow A'$ *by:* $\alpha'_{\mu(s)}(\mu(x)) = \alpha_s(x)$, *and*
- $\alpha' : \mu(X) \rightarrow A'$, *define* $\alpha : X \rightarrow A'|_\mu$ *by:* $\alpha_s(x) = \alpha'_{\mu(s)}(\mu(x))$, *i.e.* $\alpha = \alpha'|_\mu$

Then for any Σ-formula φ and for any Σ' multialgebra A' we have that:

$$(A'|_\mu) \models_\alpha \varphi \Longleftrightarrow A' \models_{\alpha'} \mu(\varphi)$$

Proof. By induction on the formulas:

1. For atomic formulae $t_1 \doteq t_2$ and $t_1 \prec t_2$, the reduct theorem gives $\alpha'|_\mu(t_i)^{A'|_\mu} = \alpha'(\mu(t_i))^{A'}$, which yields the result.

2. $a_1, \dots, a_n \rightarrow b_1, \dots b_m$
   There are two cases:
   a) for some $a_i : (A'|_\mu) \not\models_\alpha a_i \overset{IH}{\Longleftrightarrow} A' \not\models_{\alpha'} \mu(a_i)$, or
   b) for some $b_k : (A'|_\mu) \models_\alpha b_k \overset{IH}{\Longleftrightarrow} A' \models_{\alpha'} \mu(b_k)$

□

Lemma 3.6 *(Satisfaction condition) The satisfaction condition is fullfiled for multialgebras, i.e. for any signature morphism $\mu : \Sigma \rightarrow \Sigma'$, for any Σ-formula φ and for any Σ' multialgebra A' we have that:*

$$(A'|_\mu) \models_\Sigma \varphi \Longleftrightarrow A' \models_{\Sigma'} \mu(\varphi)$$

Proof. Let φ be an arbitrary formula.
"$\Leftarrow$": let $\alpha : X \to A'|_\mu$ be arbitrary and let α' be as in lemma 3.5. Then, by assumption, $A' \models_{\Sigma'} \mu(\varphi)$ and, in particular, $A' \models_{\alpha'} \mu(\varphi)$. By lemma 3.5, $(A'|_\mu) \models_\alpha \varphi$. Since α was arbitrary, we obtain $(A'|_\mu) \models_\Sigma \varphi$.
"$\Rightarrow$": let $\alpha' : \mu(X) \to A'$ be arbitrary, and let α be as in lemma 3.5. By assumption, $(A'|_\mu) \models_\Sigma \varphi$, in particular, $(A'|_\mu) \models_\alpha \varphi$. By lemma 3.5, $A' \models_{\alpha'} \mu(\varphi)$. Since α' was arbitrary, we obtain $A' \models_{\Sigma'} \mu(\varphi)$. □

Finally, the functor assigning to each signature the set of sentences is defined as in definition 2.10:

Definition 3.7 *The sentences functor* $\mathsf{Sen}_{\mathcal{MA}} : \mathbf{Sign} \to \mathbf{Set}$ *is given by:*

- *objects:* $\mathsf{Sen}_{\mathcal{MA}}(\Sigma) =$ *the set of all* Σ *formulae (def. 2.10)*
- $\mathsf{Sen}_{\mathcal{MA}}(\mu : \Sigma \to \Sigma') = \mathsf{Sen}_{\mathcal{MA}_\mu} : \mathsf{Sen}_{\mathcal{MA}} \to \mathsf{Sen}_{\mathcal{MA}}(\Sigma')$ *defined by:*
  1. $\mathsf{Sen}_{\mathcal{MA}_\mu}(t \doteq t') = \tilde{\mu}(t) \doteq \tilde{\mu}(t')$
  2. $\mathsf{Sen}_{\mathcal{MA}_\mu}(t \prec t') = \tilde{\mu}(t) \prec \tilde{\mu}(t')$
  3. $\mathsf{Sen}_{\mathcal{MA}_\mu}(a_1, \ldots, a_n \to b_1, \ldots, b_n)$
     $= \mathsf{Sen}_{\mathcal{MA}_\mu}(a_1), \ldots, \mathsf{Sen}_{\mathcal{MA}_\mu}(a_n) \to \mathsf{Sen}_{\mathcal{MA}_\mu}(b_1), \ldots, \mathsf{Sen}_{\mathcal{MA}_\mu}(b_n)$

With these definitions, lemma 3.6 yields the following:

Fact 3.8 *The multialgebras form the institution* $\mathcal{MA}$ *with:*

- *the category* **Sign** *as signatures (def. 2.1),*
- *the model functor* $\mathsf{Mod}_{\mathcal{MA}}$ *(def. 3.3),*
- *the sentence functor* $\mathsf{Sen}_{\mathcal{MA}}$ *(def. 3.7),*
- $\models$ *from def. 2.12 as the satisfaction relation.*

3.2 $\mathcal{MA}$ is an exact institution

It is well known that the category of algebraic signatures is co-complete, see e.g. [2], where it also is proved (by use of comma categories) that the model functor for classical total algebras is continuous. Using the constructions of the required co-limits of algebraic signatures from 2.2.1, we now show that that $\mathcal{MA}$ is an exact institution – that the model functor for multialgebras $\mathsf{Mod}_{\mathcal{MA}} : \mathbf{Sign}^{op} \to \mathbf{Cat}$ is continuous, i.e. it maps finite co-limits in **Sign** into limits in **Cat**. (Note that this is different from showing that the category $\mathsf{Mod}_{\mathcal{MA}}(\Sigma)$ has all limits (resp. co-limits), which is proved in [7].)

Lemma 3.9 *The model of the empty signature is the unit category, that is final in* **Cat**.

Proof. The model of the empty signature is an algebra $\emptyset$ with no carrier (and no operations), i.e. $\mathsf{Mod}_{\mathcal{MA}}(\Sigma_\emptyset) = \{\emptyset\}$. There is only one function (homomorphism) $h : \emptyset \to \emptyset$ – the identity homomorphism. This is obviously a final object in **Cat**. □

Lemma 3.10 Mod *sends sums to products:*

$$\begin{tikzcd}
\Sigma \arrow[dr, "\iota_\Sigma"'] & & \Sigma' \arrow[dl, "\iota_{\Sigma'}"] & \arrow[r, Rightarrow, "\mathsf{Mod}"] & & \mathsf{Mod}_{\mathcal{MA}}(\Sigma) & & \mathsf{Mod}_{\mathcal{MA}}(\Sigma') \\
& \Sigma + \Sigma' & & & & & \mathsf{Mod}_{\mathcal{MA}}(\Sigma + \Sigma') \arrow[ul, "|_{\iota_\Sigma}"] \arrow[ur, "|_{\iota_{\Sigma'}}"'] &
\end{tikzcd}$$

Proof. Since the reduct $\_|_\mu$ is a functor for any signature morphism μ the diagram to the right is a cone in $\mathbf{Cat}$ – we have to show that it is a product cone.

Suppose that $(\mathbf{C}, \mathsf{F} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma), \mathsf{G} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma'))$ is a cone in $\mathbf{Cat}$.

$$\begin{tikzcd}
 & \mathbf{C} \arrow[d, dashed, "!u_{(\mathsf{F},\mathsf{G})}"] \arrow[ddl, bend right, "\mathsf{F}"'] \arrow[ddr, bend left, "\mathsf{G}"] & \\
 & \mathsf{Mod}_{\mathcal{MA}}(\Sigma + \Sigma') \arrow[dl, "|_{\iota_\Sigma}"] \arrow[dr, "|_{\iota_{\Sigma'}}"'] & \\
\mathsf{Mod}_{\mathcal{MA}}(\Sigma) & & \mathsf{Mod}_{\mathcal{MA}}(\Sigma')
\end{tikzcd}$$

Given two multialgebras $A \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma)$ and $A' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma')$, we get an algebra $A \oplus A' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma + \Sigma')$ by taking the Σ-part from A and the Σ'-part from A' – it is defined as follows: for any sort symbol $s \in \Sigma : s^{A \oplus A'} = s^A$ (and $s \in \Sigma' : s^{A \oplus A'} = s^{A'}$), and for any operation symbol $f \in \Sigma : f^{A \oplus A'} = f^A$ (and $f \in \Sigma' : f^{A \oplus A'} = f^{A'}$). This works because $\Sigma + \Sigma'$ is disjoint union.

Likewise, given a Σ-homomorphism $h : A \to B$ and a Σ'-homomorphism $h' : A' \to B'$, the $\Sigma \oplus \Sigma'$-homomorphism $h \oplus h' : A \oplus A' \to B \oplus B'$, is defined by $(h \oplus h')_s = h_s$ if $s \in \Sigma$, and h'_s otherwise (when $s \in \Sigma'$). Then $\oplus$ yields the unique objects/morphisms satisfying:

$$\begin{array}{ccc} (A \oplus A')|_{\iota_\Sigma} = A & \textit{and} & (A \oplus A')|_{\iota_{\Sigma'}} = A' \\ (h \oplus h')|_{\iota_\Sigma} = h & \textit{and} & (h \oplus h')|_{\iota_{\Sigma'}} = h' \end{array} \tag{1}$$

We define the functor $u_{(\mathsf{F},\mathsf{C})} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma + \Sigma')$ by $u_{(\mathsf{F},\mathsf{C})}(C) = \mathsf{F}(C) \oplus \mathsf{G}(C)$ (and analogously for morphisms in $\mathbf{C}$). It is a factorization, i.e., $u_{(\mathsf{F},\mathsf{G})}; |_{\iota_\Sigma} = \mathsf{F}$, similarly for $|_{\iota_{\Sigma'}}$ and G.

$u_{(\mathsf{F},\mathsf{G})}$ is unique since each pair of algebras (resp. homomorphisms) $A \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma)$ and $A' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma')$, has a unique preimage $(A \oplus A') \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma + \Sigma')$, satisfying (1).

Thus the image of a co-product diagram from $\mathbf{Sign}$ is a product diagram in $\mathbf{Cat}$. $\square$

Lemma 3.11 Mod *sends co-equalizers to equalizers.*

Proof. Let $\mu, \nu : \Sigma \to \Sigma'$ be two morphisms in $\mathbf{Sign}$ and $\Sigma'/_\sim$, $\iota : \Sigma' \to \Sigma'/_\sim$ their co-equalizer. We have to show that $\mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$, $|_\iota$ is an equlizer for $|_\mu$ and $|_\nu$, in $\mathbf{Cat}$. So assume that for any σ'' such that $\mu; \sigma'' = \nu; \sigma''$, there is a unique $u_{\sigma''}$ with $\sigma'' = \iota; u_{\sigma''}$:

$$\begin{tikzcd}
\Sigma \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] & \Sigma' \arrow[r, "\iota"] \arrow[dr, "\sigma''"'] & \Sigma'/_\sim \arrow[d, dashed, "u_{\sigma''}"] \\
 & & \Sigma''
\end{tikzcd}$$

Let $\mathbf{C}$ and $\mathsf{F} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma')$ be arbitrary in $\mathbf{Cat}$ such that $\mathsf{F}; |_\mu = \mathsf{F}; |_\nu$. We have to show the existence of a unique u_F satisfying $u_\mathsf{F}; |_\iota = \mathsf{F}$:

$$\begin{tikzcd}
\mathbf{C} \arrow[d, dashed, "!u_\mathsf{F}"] \arrow[dr, "\mathsf{F}"] & & \\
\mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim) \arrow[r, "|_\iota"'] & \mathsf{Mod}_{\mathcal{MA}}(\Sigma') \arrow[r, shift left, "|_\mu"] \arrow[r, shift right, "|_\nu"'] & \mathsf{Mod}_{\mathcal{MA}}(\Sigma)
\end{tikzcd}$$

1. First, we show that $|_\iota; |_\mu = |_\iota; |_\nu$, i.e. $\mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ is a cone.

Let $A'' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ be arbitrary and let s be any sort symbol in $\Sigma'/_\sim$. We have $s^{(A''|_\iota)|_\mu} = \mu(s)^{(A''|_\iota)} = \iota(\mu(s))^{A''}$, and similarly $s^{(A''|_\iota)|_\nu} = \nu(s)^{(A''|_\iota)} = \iota(\nu(s))^{A''}$. But since ι is equalizing

μ and ν, we have $\iota(\mu(s)) = \iota(\nu(s))$, so that $\iota(\mu(s))^{A''} = \iota(\nu(s))^{A''}$. In the same way, we show the equality $\iota(\mu(\omega))^{A''} = \iota(\nu(\omega))^{A''}$ for any operation symbol $\omega \in \Sigma'/_\sim$.

2. We now show that $\mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ is a limit cone.

a. Given an $A' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma')$ with $A'|_\mu = A'|_\nu$ we construct an $A^- \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ satisfying $A^-|_\iota = A'$.

Since ι is surjective, any symbol $x'' \in \Sigma'/_\sim$ is in its image, i.e., $x'' = \iota(x')$ for some $x' \in \Sigma'$. We then let $\iota(x')^{A^-} = x'^{A'}$. This is in fact well defined algebra. For suppose that there are two different symbols $s' \neq t' \in \Sigma'$ such that $\iota(s') = \iota(t') = x''$. Then, from the construction of ι and $\Sigma'/_\sim$, we know that there exists an $x \in \Sigma$ such that $s' = \mu(x)$ and $t' = \nu(x)$. But then, since $A'|_\mu = A'|_\nu$, we obtain the middle of the following equalities: $s'^{A'} = x^{A'|_\mu} = x^{A'|_\nu} = t'^{A'}$. Thus $\iota(s') = \iota(t') \Rightarrow s'^{A'} = t'^{A'}$, and A^- is well defined.

Obviously, $A^-|_\iota = A'$, since for each symbol $x' \in \Sigma'$ we have $x'^{A^-|_\iota} = \iota(x')^{A^-} = x'^{A'}$.

b. In fact A^- is the unique $\Sigma'/_\sim$-algebra satisfying $A^-|_\iota = A'$.

For if $B'' \in \mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ is such that $B''|_\iota = A'$ then we must have for any symbol $\iota(x') = x'' \in \Sigma'/_\sim : x''^{B''} = \iota(x')^{B''} = x'^{B|_\iota} = x'^{A'} = x'^{A^-|_\iota} = \iota(x')^{A^-} = x''^{A^-}$, i.e., $B'' = A^-$.

c. We extend the definition of A^- from 2a. to homomorphisms between the respective objects. That is, for a (homo)morphism $h' : A' \to B'$ in $\mathsf{Mod}_{\mathcal{MA}}(\Sigma')$ where both $A'|_\mu = A'|_\nu$ and $B'|_\mu = B'|_\nu$, $h^- : A^- \to B^-$ is a (homo)morphism in $\mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$ defined by $h^-_{\iota(s')}(a) = h'_{s'}(a)$ for all sort symbols $\iota(s') = s'' \in \Sigma'/_\sim$ and $a \in s'^{A'}$. Obviously, this is a unique h^- such that $h|_\iota = h'$.

d. Now, given an $\mathsf{F} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma')$ satisfying $\mathsf{F}; |_\mu = \mathsf{F}; |_\nu$, we define $u_\mathsf{F} : \mathbf{C} \to \mathsf{Mod}_{\mathcal{MA}}(\Sigma'/_\sim)$:

- for any object $C \in \mathbf{C} : u_\mathsf{F}(C) = (\mathsf{F}(C))^-$ (as in point 2a.)
- for any morphism $h : C \to D \in \mathbf{C} : u_\mathsf{F}(h) = (\mathsf{F}(h))^-$ (as in point 2c.)

It is trivial to verify that u_F is a functor and, indeed, one that makes $u_\mathsf{F}; |_\iota = \mathsf{F}$. By uniqueness of A^- and h^-, this is also a unique functor satisfying this equality. □

Summing upp we get the following result.

Proposition 3.12 $\mathsf{Mod}_{\mathcal{MA}}$ *is a finitely continuous functor.*

Proof. We have that $\mathsf{Mod}_{\mathcal{MA}}$ is finitely continous on signatures by lemma 3.9, lemma 3.10 and lemma 3.11. So the result follows by teorem 2.14. □

Corollary 3.13 $\mathcal{MA}$ *is an exact institution.*

Corollary 3.14 *The amalgamation lemma holds for* $\mathcal{MA}$*.*

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