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# Partitioned Runge–Kutta methods in Lie-group setting

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**Abstract**

We introduce partitioned Runge–Kutta (PRK) methods as geometric integrators in the Runge–Kutta–Munthe-Kaas (RKMK) method hierarchy. This is done by first noticing that tangent and cotangent bundles are the natural domains for the differential equations to be solved. Next, we equip the (co)tangent bundle of a Lie group with a group structure and treat it as a Lie group. The structure of the differential equations on the (co)tangent-bundle Lie group is such that partitioned versions of the RKMK methods are naturally introduced. Numerical examples are included to illustrate the new methods.

*AMS Subject Classification:* 65L06, 34C40

*Key Words:* Partitioned Runge–Kutta method, RKMK method, Magnus method, Cayley method, Crouch–Grossman method, geometric integration, tangent bundle of Lie group, semidirect product, differential equations on manifolds

## 1 Introduction

The Runge–Kutta–Munthe-Kaas (RKMK) method was introduced in [17] to solve differential equations on homogeneous spaces, that is, differential equations in the form of an infinitesimal generator of the Lie-group action. The idea behind the approach is simple and is essentially to solve a differential equation in the Lie algebra of the Lie group, instead of the original differential equation on the homogeneous space. The differential equation in the Lie algebra evolves in a vector space and any classical Runge–Kutta (RK) method can be used to solve it. Coordinatizing the Lie group with different local coordinates corresponds to different differential equations on the Lie algebra, as shown in [5]. Hence, the choice of coordinates in the RKMK method is a variable to be considered when solving differential equations on a homogeneous space.

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Classical partitioned Runge–Kutta (PRK) methods can of course be introduced in the RKMK method in a straight forward and naive way. Just apply a PRK method directly to the differential equation in the Lie algebra instead of the RK method. We believe, however, that this approach is not the right one for generalizing the classical PRK methods to Lie groups. Recall that PRK methods are used to solve systems of differential equations that naturally separate into two sets of equations [6]. Examples of problems in this form are Hamiltonian problems and second order differential equations[1]. In the context of manifolds these problems generalize to Hamiltonian problems on cotangent bundles, and sprays on tangent bundles, respectively. Thus, we will in this paper consider (co)tangent bundles of Lie groups as an appropriate domain for partitioned differential equations.

The present idea is to equip the (co)tangent bundle of the Lie group with a group structure turning it into another continuous group. Doing this, we can utilize the RKMK approach directly. However, the (co)tangent bundle of a Lie group can be represented in several different ways. In the subsequent section we will investigate three representations, and explore some of their implications.

The rest of the paper is organized as follows: In Section 2 we discuss the different representations of the tangent bundle of a Lie group in detail. Then the subsequent section discusses some examples of differential equations naturally evolving on the tangent bundle. Section 4 introduces the partitioned Runge–Kutta–Munthe-Kaas method as a generalization of the PRK method to cotangent bundles and tangent bundles of Lie groups. In Section 5 the present setting is implemented in the cases of Magnus and Cayley series, and the Crouch–Grossman method. This is followed by numerical experiments in Section 6, and the paper is concluded in Section 7.

# 2 On representing the tangent bundle of a Lie group

First we make precise the meaning of trivialization. The term will refer to how to translate a Lie algebra element out to the tangent space at a specific point on the Lie group. This can be done in two ways; by either the lifted left or right multiplication. Left multiplication is defined as $\mathrm{L}_a(b) = \mu(a, b) = a \cdot b$, and right multiplication is defined as $\mathrm{R}_a(b) = \mu(b, a) = b \cdot a$, where $\mu$ is group multiplication. Hence, we will speak of left and right trivializations of the tangent bundle, and by 'body' and 'space' coordinates we understand coordinates obtained through left and right trivialization, respectively. See [2, p. 15] or [14] for details.

## 2.1 The direct-product group structure

Considered as a manifold, the tangent bundle of a Lie group is diffeomorphic to the direct product of the Lie group with its Lie algebra; that is $\mathrm{T}G \approx G \times \mathfrak{g}$. This representation implies a very simple tangent Lie-group structure; For two pairs of elements $(a, u), (b, v) \in G \times \mathfrak{g}$, just choose group multiplication for the group elements, and vector addition for the algebra elements; i.e.

$$(a, u) \cdot (b, v) = (a \cdot b, u + v). \tag{1}$$

Relevant constructs for this representation of the tangent bundle of a Lie group is summarized in Table 1. In this representation the left and right trivialized versions of the tangent bundle coincide. Following [5], using left multiplication $\mathrm{L}_{(a,u)}(b, v)$ as an action on $G \times \mathfrak{g}$ and calculating its infinites-

[1] Second order differential equations can be solved by a Runge–Kutta–Nyström method, but this method is in fact just a partitioned Runge–Kutta method in disguise [6].

| Representation: $G \times \mathfrak{g}$ | Left/Right |
|---|---|
| Group operation $(a,u)\cdot(b,v)$ | $(ab, u+v)$ |
| Inverse of $(a,u)$ | $(a^{-1}, -u)$ |
| Lie algebra | $\mathfrak{g}\times\mathfrak{g}$ |
| Commutator in Lie algebra | $[(u,v),(x,y)] = ([u,x],0)$ |
| Exponential map | $\exp(u,v) = (\exp(u), v)$ |
| $\mathrm{dexp}^{-1}_{(u,v)}(x,y)$ | $(\mathrm{dexp}^{-1}_u(x), y)$ |

Table 1: Representation of $\mathrm{T}G$ as $G \times \mathfrak{g}$.

imal generator with respect to $\xi, \eta \in \mathfrak{g}$, we conclude that the general form of differential equations on $G \times \mathfrak{g}$ is

$$\begin{aligned} g'(t) &= \xi(g,v)g && (2)\\ v'(t) &= \eta(g,v), && (3) \end{aligned}$$

where $\xi, \eta : G \times \mathfrak{g} \to \mathfrak{g}$. The representation of the tangent bundle Lie group as a direct product $G \times \mathfrak{g}$ does not constitute anything new. This is because the above general form of the differential equation on $G \times \mathfrak{g}$ can be thought of as one differential equation (2) evolving in a Lie group solved in tandem with a classical equation (3). This set of equations can be solved by applying a Lie-group method to (2) and any classical integration technique to (3).

## 2.2 The semidirect-product group structure

To an element $(a,u)$ belonging to $G\times\mathfrak{g}$, we can associate a curve in the Lie group $G$ passing through the point $a$. Using left trivialization, the curve is represented as $a(t) = a + tau + \mathcal{O}(t^2)$, where $au \in \mathrm{T}_aG$ is the tangent. Take two such curves and multiply them together using the group multiplication in $G$:

$$\begin{aligned} a(t)\cdot b(t) &= (a + tau + \mathcal{O}(t^2))\cdot(b + tbv + \mathcal{O}(t^2))\\ &= a\cdot b + ta\cdot b(\mathrm{Ad}_{b^{-1}} u + v) + \mathcal{O}(t^2). \end{aligned}$$

Thus, the group multiplication in the group induces (in the left trivialized case) the following group multiplication on the tangent bundle

$$(a,u)\cdot(b,v) = (a\cdot b, \mathrm{Ad}_{b^{-1}} u + v). \qquad (4)$$

The group multiplication (4) can be interpreted as a lift of the binary operation in the group up to the tangent bundle, since it is induced by the group multiplication in the group. This lifted binary operation is fundamentally different from the 'trivial' group structure (1), and in some sense capture more of the original geometric structure of the Lie group $G$. The lifted multiplication (4) together with $G\times\mathfrak{g}$ is a well-known example of a semidirect product, see [14, p. 325], and we denote it by $G\ltimes_{\mathrm{Ad}}\mathfrak{g}$. Note that the form of the binary operation depends on the trivialization of the tangent bundle.

To verify that (4) is indeed the lift of the binary operation $\mu : G\times G \to G$, is straight forward. Two elements $u, v \in \mathfrak{g}$ are mapped to $\mathrm{T}_e\mathrm{L}_a\, u \in \mathrm{T}_aG$ and $\mathrm{T}_e\mathrm{L}_b\, v \in \mathrm{T}_bG$, respectively, in the left trivialized case. We want to add the two vectors and in order to do that the vectors are transformed to the common tangent space $\mathrm{T}_{ab}G$. By mapping the two vectors back to the Lie algebra, i.e. recover $u$ and $v$, and then transforming them to $\mathrm{T}_{ab}G$ through left multiplication, we infer the group operation (1). On the other hand, if we insist on mapping the tangent vectors at $a$ and $b$ directly to the tangent space at $ab$, we get the vectors $\mathrm{T}_e\mathrm{L}_{ab}\mathrm{Ad}_{b^{-1}}\,u$ and $\mathrm{T}_e\mathrm{L}_{ab}\,v$. Adding these two vectors induces the binary operation (4). In the latter case, the result depends on the trivialization of the tangent bundle.

Table 2 lists group operation, left and right multiplication, inner automorphism, etc., for the tangent Lie group $\mathrm{T}G$ represented as a semidirect product $G\ltimes_{\mathrm{Ad}}\mathfrak{g}$. For later reference we also list the infinitesimal generators of the left and right multiplicative actions in Table 2.

| Representation: $G\ltimes_{\mathrm{Ad}}\mathfrak{g}$ | Left | Right |
|---|---|---|
| Group operation $(a,u)\cdot(b,v)$ | $(ab, \mathrm{Ad}_{b^{-1}}\,u+v)$ | $(ab, u+\mathrm{Ad}_a\,v)$ |
| Inverse of $(a,u)$ | $(a^{-1}, -\mathrm{Ad}_a\,u)$ | $(a^{-1}, -\mathrm{Ad}_{a^{-1}}\,u)$ |
| $\mathrm{L}_{(a,u)}(b,v)$ | $(ab, \mathrm{Ad}_{b^{-1}}\,u+v)$ | $(ab, u+\mathrm{Ad}_a\,v)$ |
| $\mathrm{R}_{(a,u)}(b,v)$ | $(ba, \mathrm{Ad}_{a^{-1}}\,v+u)$ | $(ba, v+\mathrm{Ad}_b\,u)$ |
| $\mathrm{I}_{(a,u)}(b,v)$ | $(\mathrm{I}_a\,b, \mathrm{Ad}_{ab^{-1}}\,u+\mathrm{Ad}_a(v-u))$ | $(\mathrm{I}_a\,b, u-\mathrm{Ad}_{\mathrm{I}_a\,b}\,u+\mathrm{Ad}_a\,v)$ |
| Lie algebra | $\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$ | $\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$ |
| $[(u,v),(x,y)]$ | $(\mathrm{ad}_u\,x, -\mathrm{ad}_x\,v+\mathrm{ad}_u\,y)$ | $(\mathrm{ad}_u\,x, -\mathrm{ad}_x\,v+\mathrm{ad}_u\,y)$ |
| $\mathrm{Ad}_{(a,u)}(x,y)$ | $(\mathrm{Ad}_a\,x, \mathrm{Ad}_a(-\mathrm{ad}_x\,u+y))$ | $(\mathrm{Ad}_a\,x, -\mathrm{ad}_{\mathrm{Ad}_a\,x}\,u+\mathrm{Ad}_a\,y)$ |
| Exponential map $\exp(u,v)$ | $(\exp_G(u), \mathrm{dexp}_u(v))$ | $(\exp_G(u), \mathrm{dexp}_u(v))$ |
| Logarithm map $\log(g,u)$ | $(\log_G(g), \mathrm{dexp}^{-1}_{\log(g)}(u))$ | $(\log_G(g), \mathrm{dexp}^{-1}_{\log(g)}(u))$ |
| $\mathrm{dexp}^{-1}_{(u,v)}(x,y)$ | $\sum_{j=0}^{\infty}\frac{B_j}{j!}(-\mathrm{ad})^j_{(u,v)}(x,y)$ | $\sum_{j=0}^{\infty}\frac{B_j}{j!}\,\mathrm{ad}^j_{(u,v)}(x,y)$ |
| $(x,y)^{\mathrm{L}}_{G\ltimes_{\mathrm{Ad}}\mathfrak{g}}(b,v)$ | $(\mathrm{T}_e\mathrm{R}_b\,x, \mathrm{Ad}_{b^{-1}}\,y)$ | $(\mathrm{T}_e\mathrm{R}_b\,x, y+\mathrm{ad}_x\,v)$ |
| $(x,y)^{\mathrm{R}}_{G\ltimes_{\mathrm{Ad}}\mathfrak{g}}(b,v)$ | $(\mathrm{T}_e\mathrm{L}_b\,x, y-\mathrm{ad}_x\,v)$ | $(\mathrm{T}_e\mathrm{L}_b\,x, \mathrm{Ad}_b\,y)$ |

Table 2: Representation of $\mathrm{T}G$ as $G\ltimes_{\mathrm{Ad}}\mathfrak{g}$.

The Lie algebra is denoted as $\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$, which is to be thought of as $\mathfrak{g}\times\mathfrak{g}$, but equipped with a more complex commutator structure than simply the $\mathfrak{g}$-bracket in the first component and the zero bracket in the second. The bracket on $\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$, which is the same for both trivialized cases, is calculated by differentiating the group adjoint action. To accomplish this, we only need knowledge of the exponential mapping, which disregards and do not use the additional commutator structure on the Lie algebra. In the RKMK method we need to calculate the trivialized differential of the coordinate mapping and its inverse. We need the commutator in the Lie algebra $\mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}$ to do this.

Infering what the exponential mapping is can be done in at least two different ways. Similar to the lifting of the binary operation $\mu$ on $G$ up to the trivialized tangent bundle, we can also lift $\exp_G : \mathfrak{g}\to G$ up to $\exp : \mathfrak{g}\ltimes_{\mathrm{ad}}\mathfrak{g}\to G\ltimes_{\mathrm{Ad}}\mathfrak{g}$; $(u,v)\mapsto(\exp_G(u),\mathrm{dexp}_u(v))$, where $\mathrm{dexp}_u$ is the trivialized differential of $\exp_G$ (different in the left and right cases). Alternatively, the same result can be obtained through solving the differential equation

$$\sigma'(t) = X_w(\sigma(t)) = \mathrm{T}_e\,\mathrm{L}_{\sigma(t)}\,w,$$

with initial condition $\sigma(0)=e$. In the right trivialized case this ODE looks like

$$\begin{aligned} a'(t) &= a(t)u \\ v'(t) &= \mathrm{Ad}_a(v), \end{aligned}$$

where $(a(0), v(0)) = (e, 0)$, and $w(t) = (a(t), v(t))$.

The general form of differential equations on $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ depends on the trivialization of the tangent bundle. For a left trivialized tangent bundle, any ODE can be phrased (matrix notation) as

$$\begin{aligned} g'(t) &= a(g, v, t)g \\ v'(t) &= \mathrm{Ad}_{g(t)^{-1}}(\gamma(g, v, t)), \end{aligned}$$

for suitable functions $a, \gamma : G \times \mathfrak{g} \times \mathbb{R} \to \mathfrak{g}$. For the right trivialized tangent bundle, with the same functions $a$ and $\gamma$ as above, any ODE is of the form

$$\begin{aligned} g'(t) &= a(g, v, t)g \\ v'(t) &= \mathrm{ad}_{a(g,v,t)}\, v + \gamma(g, v, t). \end{aligned}$$

### 2.3 The Cartesian-product representation

In the literature on the discretization of variational principles, e.g. [18], we find that it is convenient to represent the tangent bundle of the Lie group $G$ as the Cartesian product $G \times G$. This representation of the tangent bundle might at first sight appear awkward, since the tangent (velocity) of a group element is now represented by another group element. But on the other hand, this is really the same as finite differences. The benefit of this approach is that one can construct *discrete versions* of the continuous Euler–Lagrange and Euler–Poincaré equations that preserve discrete analogous of the geometric properties of the continuous equations [13].

The representation of the tangent bundle of a Lie group $G$ as a Cartesian product $G \times G$ implies the operations listed in Table 3. Defining the left and right actions and other types of hybrid actions is straight forward, as finding their infinitesimal generators. We will see in Section 3.3 that the symmetric rigid body equations is an example where the representation of $\mathrm{T}G$ as $G \times G$ occurs naturally.

| Representation: $G \times G$ | Operation |
|---|---|
| Group operation $(a_1, a_2) \cdot (b_1, b_2)$ | $(a_1 b_1, a_2 b_2)$ |
| Inverse of $(a, b)$ | $(a^{-1}, b^{-1})$ |
| Lie algebra | $\mathfrak{g} \times \mathfrak{g}$ |
| Commutator in Lie algebra | $[(u, v), (x, y)] = ([u, x], [v, y])$ |
| Exponential map | $\exp(u, v) = (\exp(u), \exp(v))$ |
| $\mathrm{dexp}^{-1}_{(u,v)}(x, y)$ | $(\mathrm{dexp}_u^{-1}(x), \mathrm{dexp}_v^{-1}(y))$ |

Table 3: Representing $\mathrm{T}G$ as $G \times G$.

## 3 Differential equations on the tangent bundle of a Lie group

The purpose of this section is to give a few examples of differential equations that naturally evolve on tangent bundles. The equations will be written out in trivialized form to emphasize their partitioned structure. Note that the equations are defined irrespectively of the underlying group structure modeling the tangent bundle. Hence, there are choices in what underlying representation to use for the solution of a given equation. In the following we tacitly assume that the underlying manifold is a Lie group. For background notation and further details, see [1, 14].

### 3.1 Second-order vector fields

Classic second order differential equations in the form

$$y'' = f(y, y'), \quad y(0) = y_0, \quad y'(0) = u_0,$$

are generalized to vector fields on the tangent bundle in the coordinate-free setting. A vector field $V$ on $\mathrm{T}\mathcal{M}$ is called a second-order equation if $\mathrm{T}\tau_{\mathcal{M}} \circ V = Id$, where $\tau_{\mathcal{M}} : \mathrm{T}\mathcal{M} \to \mathcal{M}$ is the canonical projection.

In the case of $\mathcal{M}$ being a Lie group $G$, and where the tangent bundle of $G$ is left-trivialized as $G \times \mathfrak{g}$, a second-order vector field takes the form:

$$\begin{aligned} g' &= \mathrm{T}_e \, \mathrm{L}_g \, v, \\ v' &= a(g, v), \end{aligned}$$

where $a : G \times \mathfrak{g} \to \mathfrak{g}$.

### 3.2 Trivialized Euler–Lagrange equations

Given a Lagrangian $L(g, v)$, where $g \in G$ and $v \in \mathfrak{g}$, the Euler–Lagrange equations in the left trivialized case are:

$$\dot{g} = \mathrm{T}_e \, \mathrm{L}_g \, v, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L}{\delta v} = \frac{\delta L}{\delta g} + \mathrm{ad}_v^* \frac{\delta L}{\delta v}. \tag{6}$$

In the right trivialized case, $\mathrm{L}_g$ in (5) is substituted with $\mathrm{R}_g$, and the plus sign in front of $\mathrm{ad}_v^*$ in (6) is changed to a minus sign. The functional derivatives of the Lagrangian function are defined as

$$dL(g, v)(\mathrm{T}_e \, \mathrm{L}_g \, u, w) = \langle \frac{\delta L}{\delta g}, u \rangle + \langle \frac{\delta L}{\delta v}, w \rangle,$$

where $\frac{\delta L}{\delta g}, \frac{\delta L}{\delta v} : G \times \mathfrak{g} \to \mathfrak{g}^*$. In the case of a regular Lagrangian function $L$, the Euler–Lagrange equations are a second-order vector field.

**The unreduced rigid body equations:** Choosing $L(g, \Omega) = \frac{1}{2}\Omega \cdot I\Omega$ as our Lagrangian function, we obtain the unreduced version of the rigid body equations on the full tangent bundle $\mathrm{T}G$. Furthermore, we consider left trivialization of the tangent bundle, since we want the equations represented in body coordinates, where $g$ is the current configuration and $\Omega$ is the body angular velocity.

The functional derivatives are easily checked to equal

$$\frac{\delta L}{\delta g} = 0, \qquad \frac{\delta L}{\delta \Omega} = \Pi,$$

where $\Pi = I\Omega$ is the body angular momentum. Putting this into (5-6), *the unreduced rigid body equations in body coordinates* are

$$\begin{aligned} g'(t) &= \mathrm{T}_e \, \mathrm{L}_g \, \Omega \\ \Pi'(t) &= \mathrm{ad}_\Omega^* \, \Pi. \end{aligned}$$

Specifying $G = \mathrm{SO}(3)$, the equations reduce to

$$\begin{aligned} g'(t) &= g\Omega \\ \Pi'(t) &= \Pi \times \Omega, \end{aligned}$$

which are the familiar unreduced Euler equations.

### 3.3 The symmetric rigid body equations

An interesting rewriting of the rigid body equations in the case $G = \mathrm{SO}(n)$ give rise to a set of equivalent equations on $\mathrm{SO}(n) \times \mathrm{SO}(n)$ called the symmetric rigid body equations [3]. The body version of these equations are

$$\begin{aligned} Q' &= Q\Omega \\ P' &= P\Omega, \end{aligned} \tag{7}$$

where $Q, P \in \mathrm{SO}(n)$, $\Omega = I^{-1}M \in \mathfrak{so}(n)$, and $M = Q^TP - P^TQ$ is the momentum. By differentiation the symmetric rigid body equations reduce to the unreduced rigid body equation on $\mathrm{SO}(n)$. That is,

$$\begin{aligned} Q' &= Q\Omega \\ M' &= [M, \Omega]. \end{aligned}$$

In [4] it is shown that a discretization of the symmetric rigid body equations coincide with the discrete Moser–Veselov rigid body equations [15]. Integrators derived from this approach conserve momentum and are symplectic.

## 4 Partitioned Runge–Kutta methods in Lie group setting

We will now adapt the classical partitioned Runge–Kutta methods to the setting developed in Section 2. We briefly recall the classical methods before introducing the new methods.

### 4.1 Classical partitioned Runge–Kutta methods

Consider a partitioned ordinary differential equation in $\mathbb{R}^{2n}$ of the form

$$\frac{dp}{dt} = f(p, q), \qquad \frac{dq}{dt} = g(p, q), \tag{8}$$

where $p, q \in \mathbb{R}^n$ and $f, g : \mathbb{R}^{2n} \to \mathbb{R}^n$. The classical partitioned Runge–Kutta method consists of choosing the method coefficients $a_{ij}, b_i$ and $\hat{a}_{ij}, \hat{b}_i$ of two Runge–Kutta methods. Use the first Runge–Kutta method to advance the solution of $p$ and the second Runge–Kutta method to advance the solution of $q$ in (8). This can be stated in algorithmic form as

**Algorithm 4.1 (Classical $r$'th order partitioned Runge–Kutta method)**
*Given the method coefficients $a_{ij}, b_i$ and $\hat{a}_{ij}, \hat{b}_i$ of two $r$'th order $s$-stage Runge–Kutta methods that also satisfy the necessary coupling conditions [7]. The following is an $r$'th order update of the solution of (8) from $t = t_n$ to $t = t_{n+1}$:*

*Given* $p_n = p(t_n)$ *and* $q_n = q(t_n)$.
for $i = 1, 2, \ldots, s$
$u_i = p_n + h \sum_{j=1}^{s} a_{ij} k_j$
$v_i = q_n + h \sum_{j=1}^{s} \hat{a}_{ij} l_j$
$k_i = f(u_i, v_i)$
$l_i = g(u_i, v_i)$
end
$p_{n+1} = p_n + h \sum_{i=1}^{s} b_i k_i$
$q_{n+1} = q_n + h \sum_{i=1}^{s} \hat{b}_i l_i$

Partitioned Runge–Kutta methods generalize the Runge–Kutta methods. Further, the PRK methods show favorable features when applied to separable Hamiltonian problems with Hamiltonian function $H(p, q) = K(p) + V(q)$. Methods can be constructed that are both explicit and symmetric, or both explicit and symplectic. The celebrated Störmer–Verlet method and the symplectic Euler method are examples of explicit and symplectic partitioned Runge–Kutta methods. A natural question to investigate is if we can construct methods in the new proposed family – to be introduced in the next section – that share the same desirable features of the classical versions.

## 4.2 Partitioned Runge–Kutta–Munthe-Kaas methods

In Section 2 we equipped the tangent bundle of the Lie group $G$ with a group structure, thus, turning it into a continuous group. As we saw, the natural representation of any differential equation on the trivialized tangent bundle of a Lie group, as a pair of two differential equations, should motivate the introduction and use of partitioned Runge–Kutta methods in the Lie group setting.

Next, we apply the approach of [17] to transform the differential equation on the Lie group $\mathrm{T}G$ into a differential equation in the Lie algebra $\mathrm{T}_e(\mathrm{T}G)$ of $\mathrm{T}G$. The transformed equation $\mathrm{d}f^{-1}_{(u,w)}(\xi, \eta)$ in the Lie algebra is naturally in partitioned form, and we apply a classical PRK method to solve this differential equation. The situation is summarized in the following commutative diagram [5].

$$\begin{array}{ccc} \mathrm{T}\mathrm{T}_e(\mathrm{T}G) & \xrightarrow{\mathrm{T}f} & \mathrm{T}(\mathrm{T}G) \\ {\scriptstyle \mathrm{d}f^{-1}_{(u,w)}(\xi,\eta)} \uparrow & & \uparrow {\scriptstyle \mathrm{T}_e\mathrm{R}_{f(u,w)}(\xi,\eta)} \\ \mathrm{T}_e(\mathrm{T}G) & \xrightarrow[f]{} & \mathrm{T}G \end{array}$$

In the above diagram $f : \mathrm{T}_e(\mathrm{T}G) \to \mathrm{T}G$ represents coordinates on the Lie group $\mathrm{T}G$. The differential equation on the Lie group is the infinitesimal generator of the left multiplicative action with respect to *two* elements $\xi, \eta \in \mathfrak{g}$. The diagram is still valid for a general homogeneous manifold, as long as the vector field is in the form of an infinitesimal generator [5].

We now state the proposed method in algorithmic form.

**Algorithm 4.2 (Partitioned Runge–Kutta–Munthe-Kaas method)**
*Assume that the vector field* $F$ *over the tangent bundle* $\mathrm{T}G$ *is the infinitesimal generator of the left multiplicative action* $\mathrm{T}G$ *on itself. That is;* $F(g, v) = \mathrm{T}_e\mathrm{R}_{(g,v)}(\xi, \eta)$ *for some maps* $\xi, \eta : \mathrm{T}G \to \mathfrak{g}$

*which generate the differential equation. The following is an $r$'th order partitioned Runge–Kutta-based numerical method for the solution $(g(t), v(t)) \in \mathrm{T}G$:*

- *Choose the initial point $(u_n, w_n) = (0, 0)$ in $\mathrm{T}_e(\mathrm{T}G)$. For any initial point $(g_n, v_n) \in \mathrm{T}G$ we have $(g_n, v_n) = f(u_n, w_n) \cdot (g_n, v_n)$, where $\cdot$ is the group multiplication in $\mathrm{T}G$.*
- *Solve $(u, w)' = \mathrm{d}f^{-1}_{(u,w)}(\xi(g, v), \eta(g, v)))$, $(u(0), w(0)) = (u_n, w_n)$ with $(g, v) = f(u, w) \cdot (g_n, v_n)$, by taking one step with any $r$'th order partitioned Runge–Kutta method. Denote the solution by $(u_{n+1}, w_{n+1})$.*
- *Advance the solution on $\mathrm{T}G$ from $(g_n, v_n)$ to $(g_{n+1}, v_{n+1}) = f(u_{n+1}, w_{n+1}) \cdot (g_n, v_n)$.*

This is an $r$'th order method on $\mathrm{T}G$ since $f$ is smooth, and an $r$'th order modification on $\mathrm{T}_e(\mathrm{T}G)$ will result in a corresponding modification on $\mathrm{T}G$ with at least order $r$. We have given Algorithm 4.2 in the case of the tangent bundle $\mathrm{T}G$, but we note that the algorithm is applicable for the cotangent bundle $\mathrm{T}^*G$ with obvious modifications.

### 4.3 Some examples

We will now show a few simple examples of numerical methods based on Algorithm 4.2. We do this by showing how the classical symplectic Euler method [6, p. 2] looks like in the three different representations of $\mathrm{T}G$. For the reminder of this section we treat Algorithm 4.2 in exponential coordinates, only. The first order symplectic Euler method is given by the two Butcher tableaus

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}. \tag{9}$$

**The direct product $G \times \mathfrak{g}$:** Recall equations (2-3):

$$\begin{aligned} g' &= \xi(g, w)g, \qquad & g(0) &= g_0, \\ w' &= \eta(g, w), \qquad & w(0) &= w_0. \end{aligned} \tag{10}$$

Then (10) transforms to the equivalent equation

$$(u', v') = \mathrm{dexp}^{-1}_{(u,v)}(\xi(g, w), \eta(g, w)), \qquad (u(0), v(0)) = (0, 0), \tag{11}$$

in the Lie algebra $\mathfrak{g} \times \mathfrak{g}$. This equation separates into

$$\begin{aligned} u' &= \mathrm{dexp}^{-1}_u(\xi(g, w)), \qquad & u(0) &= 0, \\ v' &= \eta(g, w), \qquad & v(0) &= 0, \end{aligned} \tag{12}$$

where $(g, w) = (\exp(u)g_0, v+w_0)$. The symplectic Euler method is of first order, hence the equations in (12) are truncated to yield the following set of equations.

$$\begin{aligned} u' &= \xi(\exp(u)g_0, v + w_0), \qquad & u(0) &= 0, \\ v' &= \eta(\exp(u)g_0, v + w_0), \qquad & v(0) &= 0. \end{aligned} \tag{13}$$

Applying the symplectic Euler method (9) to (13) yields the following PRKMK-SE1 method. Solve

$$\begin{aligned} u_1 &= h\xi(\exp(u_1)g_0, w_0) \\ v_1 &= h\eta(\exp(u_1)g_0, w_0) \end{aligned} \tag{14}$$

for $u_1$ and $v_1$, and update the solution in the group by

$$\begin{aligned} g_1 &= \exp(u_1)g_0 \\ w_1 &= v_1 + w_0. \end{aligned} \tag{15}$$

This method is in general implicit for the $u$-variable, and explicit for the $v$-variable.

**The semidirect product $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$:** In the following we will only consider the *left trivialized case*, but remark that everything can be appropriately rephrased for the right trivialized case. The starting point now is the following set of equations on the group $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$.

$$\begin{aligned} g' &= \xi(g, w)g, & g(0) &= g_0, \\ w' &= \mathrm{Ad}_{g^{-1}}\,\eta(g, w), & w(0) &= w_0. \end{aligned}$$

In the Lie algebra $\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ we formally get the same equation (11) as in the above case. But, now since the commutator is different, (11) separates into the two equations

$$\begin{aligned} u' &= \mathrm{dexp}_u^{-1}(\xi(g, w)), & u(0) &= 0, \\ v' &= \mathrm{ddexp}_{(u,v)}^{-1}(\xi(g, w), \eta(g, w)), & v(0) &= 0, \end{aligned} \tag{16}$$

where $(g, w) = (\exp(u)g_0, \mathrm{Ad}_{g_0^{-1}}(\mathrm{dexp}_u(v)) + w_0)$, and $\mathrm{ddexp}^{-1}$ can be thought of as the inverse differential of $\mathrm{dexp}$, as defined in equations (30-31) in Appendix A. Next, the equations (16) are truncated to yield the following set of equations.

$$\begin{aligned} u' &= \xi(\exp(u)g_0, \mathrm{Ad}_{g_0^{-1}}(\mathrm{dexp}_u(v)) + w_0), & u(0) &= 0, \\ v' &= \eta(\exp(u)g_0, \mathrm{Ad}_{g_0^{-1}}(\mathrm{dexp}_u(v)) + w_0), & v(0) &= 0. \end{aligned}$$

The PRKMK-SE1 method is now formulated as follows. Solve

$$\begin{aligned} u_1 &= h\xi(\exp(u_1)g_0, w_0) \\ v_1 &= h\eta(\exp(u_1)g_0, w_0) \end{aligned} \tag{17}$$

for $u_1$ and $v_1$, and update the solution in the group by

$$\begin{aligned} g_1 &= \exp(u_1)g_0 \\ w_1 &= \mathrm{Ad}_{g_0^{-1}}(\mathrm{dexp}_{u_1}(v_1)) + w_0. \end{aligned} \tag{18}$$

This method is in general implicit for the $u$-variable, and explicit for the $v$-variable.

**The Cartesian product $G \times G$:** Finally, we will now consider the Cartesian product formulation $G \times G$. The general form of differential equations on this group are

$$\begin{aligned} g' &= \xi(g, p)g, & g(0) &= g_0 \in G, \\ p' &= \eta(g, p)p, & p(0) &= p_0 \in G. \end{aligned}$$

These two equations formally transform into (11), but separate into different equations than in the previous cases because the commutator is different.

$$\begin{aligned} u' &= \operatorname{dexp}_u^{-1}(\xi(g,p)), \qquad && u(0) = 0, \\ v' &= \operatorname{dexp}_v^{-1}\eta(g,p)), \qquad && v(0) = 0, \end{aligned} \tag{19}$$

where $(g, p) = (\exp(u)g_0, \exp(v)p_0)$. Again, truncating (19) we obtain

$$\begin{aligned} u' &= \xi(\exp(u)g_0, \exp(v)p_0), \qquad && u(0) = 0, \\ v' &= \eta(\exp(u)g_0, \exp(v)p_0), \qquad && v(0) = 0. \end{aligned}$$

The PRKMK-SE1 method obtained this way is: Solve

$$\begin{aligned} u_1 &= h\xi(\exp(u_1)g_0, p_0) \\ v_1 &= h\eta(\exp(u_1)g_0, p_0) \end{aligned} \tag{20}$$

for $u_1$ and $v_1$, and update the solution in the group by

$$\begin{aligned} g_1 &= \exp(u_1)g_0 \\ p_1 &= \exp(v_1)p_0. \end{aligned} \tag{21}$$

This method is in general implicit for the $u$-variable, and explicit for the $v$-variable.

**General remarks:** The reader should have the followings facts in mind when studying the above examples of methods: The coordinate map is chosen to be the exponential mapping in all the examples, and we have used the same first order method in the discretization of the equations in the algebra. The reader must not be misled to conclude that the truncated equations are always equal for the three different representations, albeit that is the case above (See equations (14), (17), and (20).) This is a coincidence, and is a consequence of the first order method used in the discretization. A higher order method will in general give rise to quite different truncated equations in the three representations. The difference of the above three methods becomes clear in the updating of the group element, which is apparent from (15), (18), and (21).

All the three proposed methods are implicit in the solution of the first truncated equation in the algebra[2]. We get a similar situation as for classical separable Hamiltonian problems if the two generator maps $\xi$ and $\eta$ depend on the group variables as

$$\begin{aligned} \xi &= \xi(w) \quad && \text{— function of “velocity” only,} \\ \eta &= \eta(g) \quad && \text{— function of “position” only.} \end{aligned} \tag{22}$$

Plugging these generator maps into either of the above three methods will produce explicit schemes. The same effect is also obtained if one discretizes the equations in the algebra with the Störmer–Verlet method.

## 4.4 Other types of coordinates

As shown in [5] the continuous group can be coordinatized by different coordinates yielding different vector fields $\mathrm{d}f^{-1}$ in the algebra. Hence, we can obtain different realizations of the RKMK method

[2]It is essential that one considers the truncated equations, because the full equations will generally introduce dependences on both group variables through e.g. the $\operatorname{dexp}^{-1}$ function. One way to circumvent this dependence in the full equations, is in addition to (22), to impose a commutativity condition on $\xi$ and/or $\eta$.

based on different coordinate mappings. Illustrating this point in the current partitioned setting we will coordinatize the group by the Cayley mapping $\mathrm{cay} : \mathfrak{g} \to G$, defined for quadratic Lie groups [8];

$$\mathrm{cay}(u) = \frac{1 + \frac{1}{2}u}{1 - \frac{1}{2}u} = 1 + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} u^i.$$

We emphasize that in the rest of this section we are restricting our attention to matrices only, and that the multiplications below are really nothing else than ordinary matrix multiplications.

What we need to find is expressions for the Cayley mapping and

$$\mathrm{dcay}_{(u,v)}^{-1}(x, y)$$

in the different representations of the tangent bundle group. In order to do this it is important to realize how to multiply two elements in the Lie algebra. Think of the following multiplication of two algebra elements as an infinitesimal version of the group multiplication[3].

$$(u, v) \cdot (x, y) = \left. \frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}s} \right|_{t=s=0} \exp(tu, tv) \cdot \exp(sx, sy). \tag{23}$$

**The direct product $G \times \mathfrak{g}$:** Simple calculations give the following not surprising results, where we have used the multiplication rule $(u, v) \cdot (x, y) = (ux, 0)$:

$$\begin{aligned} \mathrm{cay}(u, v) &= \{(e, 0) + \frac{1}{2}(u, v)\} \cdot \{(e, 0) - \frac{1}{2}(u, v)\}^{-1} \\ &= (e, 0) + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} (u, v)^i. \\ &= (\mathrm{cay}(u), v), \end{aligned}$$

and

$$\begin{aligned} \mathrm{dcay}_{(u,v)}^{-1}(x, y) &= \{(e, 0) - \frac{1}{2}(u, v)\} \cdot (x, y) \cdot \{(e, 0) + \frac{1}{2}(u, v)\} \\ &= ((e - \frac{1}{2}u)x(e + \frac{1}{2}u), y) \\ &= (\mathrm{dcay}_u^{-1}(x), y). \end{aligned}$$

Compared to (12), replacing the exponential coordinates with the above Cayley coordinates will result in the simplification that the infinite sum in the first equation is replaced by a three term sum. The second equation remains the same. It is now straight forward to construct an integrator in Cayley coordinates based on e.g. the symplectic Euler method (9).

**The Cartesian product $G \times G$:** For this representation a natural duplication of the operations occur since the multiplication in the algebra is $(u, v) \cdot (x, y) = (ux, vy)$, and hence we get

$$\begin{aligned} \mathrm{cay}(u, v) &= \{(e, e) + \frac{1}{2}(u, v)\} \cdot \{(e, e) - \frac{1}{2}(u, v)\}^{-1} \\ &= (\mathrm{cay}(u), \mathrm{cay}(v)) \end{aligned}$$

[3] We note that we are using the existence of a universal enveloping algebra to the Lie algebra, and all the elements $(u, v)^n$ are really elements of this algebra. For more information, see e.g. [16].

for the Cayley map, and the vector field

$$\begin{aligned}
\operatorname{dcay}^{-1}_{(u,v)}(x,y) &= \{(e,e)-\frac{1}{2}(u,v)\}\cdot(x,y)\cdot\{(e,e)+\frac{1}{2}(u,v)\} \\
&= ((e-\frac{1}{2}u)x(e+\frac{1}{2}u),(e-\frac{1}{2}v)y(e+\frac{1}{2}v)) \\
&= (\operatorname{dcay}^{-1}_u(x),\operatorname{dcay}^{-1}_v(y))
\end{aligned}$$

in the algebra $\mathfrak{g}\times\mathfrak{g}$.

**The semidirect product $G\ltimes_{\mathrm{Ad}}\mathfrak{g}$:** Starting from the left trivialized group multiplication $(a,u)\cdot(b,v) = (ab,\operatorname{Ad}_{b^{-1}}u+v)$, we end up with $(u,v)\cdot(x,y)=(ux,-\operatorname{ad}_x v)$ as multiplication in the algebra. A simple induction argument yields $(u,v)^n=(u^n,(-\operatorname{ad}_u)^{n-1}v$. To prove correctness we now find the exponential mapping from the algebra to the group.

$$\begin{aligned}
\exp(u,v) &= \sum_{n=0}^{\infty}\frac{(u,v)^n}{n!} \\
&= \left(\sum_{n=0}^{\infty}\frac{u^n}{n!},\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}\operatorname{ad}_u^n v\right) \\
&= (\exp(u),\frac{\exp(-\operatorname{ad}_u)-I}{-\operatorname{ad}_u}v) \\
&= (\exp(u),\operatorname{dexp}_u(v)).
\end{aligned}$$

In a similar way we can now find the expression for the Cayley coordinates.

$$\begin{aligned}
\operatorname{cay}(u,v) &= (I,0)+(u,v)+\frac{1}{2}(u^2,-\operatorname{ad}_u v)+\frac{1}{4}(u^3,(-\operatorname{ad}_u)^2v)+\cdots \\
&= \left(\operatorname{cay}(u),\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\operatorname{ad}_u^n v\right) \\
&= (\operatorname{cay}(u),\frac{\operatorname{cay}(-\operatorname{ad}_u)-I}{-\operatorname{ad}_u}v) \\
&= (\operatorname{cay}(u),\frac{I}{I+\frac{1}{2}\operatorname{ad}_u}v).
\end{aligned}$$

In the above formulae, notice that

$$\operatorname{dcay}_u(v)=(I+u/2)^{-1}v(I-u/2)^{-1}\neq\frac{\operatorname{cay}(-\operatorname{ad}_u)-I}{-\operatorname{ad}_u}v.$$

For the exponential coordinates, however, we have the equality

$$\operatorname{dexp}_u(v)=\frac{\exp(-\operatorname{ad}_u)-I}{-\operatorname{ad}_u}v.$$

The vector field $\operatorname{dcay}^{-1}_{(u,v)}(x,y)$ is given as

$$\begin{aligned}
\operatorname{dcay}^{-1}_{(u,v)}(x,y) &= (x,y)-\frac{1}{2}\operatorname{ad}_{(u,v)}(x,y)-\frac{1}{4}(u,v)\cdot(x,y)\cdot(u,v) \\
&= (\operatorname{dcay}^{-1}_u(x),y-\frac{1}{2}([u,y]+[v,x])-\frac{1}{4}[u,[x,v]]).
\end{aligned}$$

Further avenues to explore would be canonical coordinates of the second kind, higher order Padé approximants, etc.

# 5 Application to other Lie-group methods

In this section we will comment upon other Lie-group methods. For a general introduction, see [9]. Since we in the above have focused on finding a continuous group structure on the tangent bundle, the above setting is "ready-for-use" directly by e.g. Magnus and Cayley series (and also their collocation versions) and the Crouch–Grossman method.

## 5.1 Magnus and Cayley methods

In the Magnus and Cayley methods one is concerned with linear ODEs in Lie groups[4]. That is, one looks at problems of the form

$$y' = \mathrm{T}_e \mathrm{R}_y\, a(t), \quad a(t) \in \mathfrak{g}, \quad y(0) = y_0 \in G.$$

In case of the Magnus method one is seeking solutions of the form $y(t) = \exp(u(t))y_0$, and for the Cayley method; $y(t) = \mathrm{cay}(u(t))y_0$.

The appropriate and generic problem in the partitioned setting is then

$$(g', w') = \mathrm{T}_e \mathrm{R}_{(g,w)}(\xi(t), \eta(t)), \quad (g(0), w(0)) = (g_0, w_0) \quad \xi, \eta : \mathbb{R} \to \mathfrak{g},$$

where we seek solutions of the form $(g(t), w(t)) = f(u(t), v(t)) \cdot (g_0, w_0)$, for a coordinate map $f : \mathrm{T}_e(\mathrm{T}G) \to \mathrm{T}G$. Choosing $f = \exp$ we obtain the partitioned Magnus series, whereas for $f = \mathrm{cay}$ we get the partitioned Cayley series.

The way to proceed is to find a differential equation for $u$ and $v$, which we already know equals

$$(u', v') = \mathrm{d}f^{-1}_{(u,v)}(\xi(t), \eta(t)), \quad (u(0), v(0)) = (0, 0).$$

This equation can e.g. be solved by Picard iteration. There is not known a generic way of solving this for an arbitrary $f$, so in the following we specify $f = \exp$. One obtains

$$\begin{aligned}
(u(t), v(t)) \;=\;& \int_0^t (\xi(\zeta_1), \eta(\zeta_1))\mathrm{d}\zeta_1 - \frac{1}{2}\int_0^t \left[\int_0^{\zeta_1} (\xi(\zeta_2), \eta(\zeta_2))\mathrm{d}\zeta_2, (\xi(\zeta_1), \eta(\zeta_1))\right]\mathrm{d}\zeta_1 \\
&+ \frac{1}{12}\int_0^t \left[\int_0^{\zeta_1} (\xi(\zeta_2), \eta(\zeta_2))\mathrm{d}\zeta_2, \left[\int_0^{\zeta_1} (\xi(\zeta_2), \eta(\zeta_2))\mathrm{d}\zeta_2, (\xi(\zeta_1), \eta(\zeta_1))\right]\right]\mathrm{d}\zeta_1 \\
&+ \frac{1}{4}\int_0^t \left[\int_0^{\zeta_1} \left[\int_0^{\zeta_2} (\xi(\zeta_3), \eta(\zeta_3))\mathrm{d}\zeta_3, (\xi(\zeta_2), \eta(\zeta_2))\right]\mathrm{d}\zeta_2, (\xi(\zeta_1), \eta(\zeta_1))\right]\mathrm{d}\zeta_1 \\
&+ \cdots .
\end{aligned}$$

In the case $f = \mathrm{cay}$ the result is

$$\begin{aligned}
(u(t), v(t)) \;=\;& \int_0^t (\xi(\zeta_1), \eta(\zeta_1))\mathrm{d}\zeta_1 - \frac{1}{2}\int_0^t \left[\int_0^{\zeta_1} (\xi(\zeta_2), \eta(\zeta_2))\mathrm{d}\zeta_2, (\xi(\zeta_1), \eta(\zeta_1))\right]\mathrm{d}\zeta_1 \\
&+ \frac{1}{4}\int_0^t \left[\int_0^{\zeta_1} \left[\int_0^{\zeta_2} (\xi(\zeta_3), \eta(\zeta_3))\mathrm{d}\zeta_3, (\xi(\zeta_2), \eta(\zeta_2))\right]\mathrm{d}\zeta_2, (\xi(\zeta_1), \eta(\zeta_1))\right]\mathrm{d}\zeta_1 \\
&- \frac{1}{4}\int_0^t \left(\int_0^{\zeta_1} (\xi(\zeta_2), \eta(\zeta_2))\mathrm{d}\zeta_3\right) \cdot (\xi(\zeta_1), \eta(\zeta_1)) \cdot \left(\int_0^{\zeta_1} (\xi(\zeta_3), \eta(\zeta_3))\mathrm{d}\zeta_2\right)\mathrm{d}\zeta_1 \\
&+ \cdots .
\end{aligned}$$

[4]Linear ODEs generalize the classical quadrature problem $y'(t) = f(t)$, $y(0) = a$.

Just like in the case of the PRKMK methods, we are now faced with the choice how to model $\mathrm{T}G$. Depending on our choice, the partitioned Magnus and Cayley series will take on quite different forms. E.g. in the case $G \times \mathfrak{g}$ the algebra multiplication (23) is $(u, v) \cdot (x, y) = (ux, 0)$, and the partitioned Magnus series collapses down to an ordinary Magnus series for the $u$-variable, and for the $v$-variable we get

$$v(t) = \int_0^t \eta(\zeta)\mathrm{d}\zeta. \tag{24}$$

Hence, we can solve for the $u$-variable with an ordinary Magnus method, and discretize (24) to the same order. In the case where we model $\mathrm{T}G$ as $G \times G$ we obtain two Magnus series, one for each of the two variables. In this case, one applies two Magnus methods of the same order, one on each variable.

The most interesting case is $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$, since the algebra multiplication is nontrivial in the second component, i.e. $(u, v) \cdot (x, y) = (ux, -\operatorname{ad}_x v)$. As usual we get an ordinary Magnus series for the $u$-variable, but for the $v$-variable we get a series with the first few terms equal to

$$\begin{aligned} v(t) \;=\; & \int_0^t \eta(\zeta_1)\mathrm{d}\zeta_1 - \frac{1}{2}\int_0^t \left[\int_0^{\zeta_1} \eta(\zeta_2)\mathrm{d}\zeta_2, \xi(\zeta_1)\right] \mathrm{d}\zeta_1 - \frac{1}{2}\int_0^t \left[\int_0^{\zeta_1} \xi(\zeta_2)\mathrm{d}\zeta_2, \eta(\zeta_1)\right] \mathrm{d}\zeta_1 \\ & + \frac{1}{12}\int_0^t \left[\int_0^{\zeta_1} \xi(\zeta_2)\mathrm{d}\zeta_2, \left[\int_0^{\zeta_1} \xi(\zeta_2)\mathrm{d}\zeta_2, \eta(\zeta_1)\right]\right] \mathrm{d}\zeta_1 \\ & \quad + \frac{1}{12}\int_0^t \left[\int_0^{\zeta_1} \eta(\zeta_2)\mathrm{d}\zeta_2, \left[\int_0^{\zeta_1} \xi(\zeta_2)\mathrm{d}\zeta_2, \xi(\zeta_1)\right]\right] \mathrm{d}\zeta_1 \\ & \qquad + \frac{1}{12}\int_0^t \left[\int_0^{\zeta_1} \xi(\zeta_2)\mathrm{d}\zeta_2, \left[\int_0^{\zeta_1} \eta(\zeta_2)\mathrm{d}\zeta_2, \xi(\zeta_1)\right]\right] \mathrm{d}\zeta_1 + \cdots, \end{aligned}$$

which solves

$$v' = \operatorname{ddexp}^{-1}_{(u,v)}(\xi, \eta), \quad v(0) = 0.$$

This above discussion also applies to the partitioned Cayley series. More can be said about partitioned Magnus and Cayley series, but we leave the discussion at the present time, in order to also introduce the partitioned Crouch–Grossman method.

## 5.2 Crouch–Grossman method

In the same manner as we introduced the partitioned versions of Magnus and Cayley series, we can also introduce the partitioned Crouch–Grossman method. The starting point for the Crouch–Grossman method is given a set of frame vector fields $E_1, \ldots, E_d$ over the domain manifold, which at each point on the manifold span the tangent space. One also assumes that the vector field defining the ODE can be written in terms of this frame as $F(y) = \sum_{i=1}^d f_i(y)E_i$, where $f_i : \mathcal{M} \to \mathbb{R}$ are smooth functions. Further, let $F_p = \sum_{i=1}^d f_i(p)E_i$ denote the vector field with coefficients frozen at $p$. We define the partitioned Crouch–Grossman method as follows: We want to solve the ODE

$$\begin{bmatrix} g' \\ w' \end{bmatrix} = \begin{bmatrix} F^1(g, w) \\ F^2(g, w) \end{bmatrix},$$

where $F^i$ is expressed as a sum of frame vector fields. Given two set of coefficients

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \qquad \begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b}^T \end{array},$$

one step of the method proceeds as:

$$\begin{aligned}(Y_i, V_i) &= \exp(ha_{i,s}F^1_{(Y_s,V_s)}, h\tilde{a}_{i,s}F^2_{(Y_s,V_s)}) \\ &\qquad \exp(ha_{i,s-1}F^1_{(Y_{s-1},V_{s-1})}, h\tilde{a}_{i,s-1}F^2_{(Y_{s-1},V_{s-1})}) \qquad\qquad i = 1, \dots, s, \\ &\qquad \cdots \exp(ha_{i,1}F^1_{(Y_1,V_1)}, h\tilde{a}_{i,1}F^2_{(Y_1,V_1)})(g_n, w_n), \\ (g_{n+1}, w_{n+1}) &= \exp(hb_sF^1_{(Y_s,V_s)}, h\tilde{b}_sF^2_{(Y_s,V_s)}) \cdots \exp(hb_1F^1_{(Y_1,V_1)}, h\tilde{b}_1F^2_{(Y_1,V_1)})(g_n, w_n),\end{aligned}$$

where $h$ is the stepsize. This method will reduce to the classical partitioned Runge–Kutta method in the case of an Abelian group.

The following is the partitioned Crouch–Grossman method based on the Störmer–Verlet scheme.

$$\begin{aligned}(Y_1, V_1) &= \exp(0, \frac{h}{2}F^2_{(Y_1,V_1)})(g_n, w_n) \\ (Y_2, V_2) &= \exp(\frac{h}{2}F^1_{(Y_2,V_2)}, \frac{h}{2}F^2_{(Y_2,V_2)}) \exp(\frac{h}{2}F^1_{(Y_1,V_1)}, 0)(g_n, w_n) \\ (g_{n+1}, w_{n+1}) &= \exp(\frac{h}{2}F^1_{(Y_2,V_2)}, \frac{h}{2}F^2_{(Y_2,V_2)}) \exp(\frac{h}{2}F^1_{(Y_1,V_1)}, \frac{h}{2}F^2_{(Y_1,V_1)})(g_n, w_n).\end{aligned}$$

# 6 Numerical examples

In the numerical examples below we tested two PRKMK methods. One method is based on the coefficients of the symplectic Euler method (9), while the second is based on the second order Störmer–Verlet method:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \qquad\qquad \begin{array}{c|cc} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

## 6.1 Symmetric rigid body in body coordinates

Recall the symmetric rigid body equations (7). The solution evolves on the direct product $\mathrm{SO}(3) \times \mathrm{SO}(3)$, given initial conditions in $\mathrm{SO}(n)$. We have solved equations (7) by first generating a random matrix $Q_0$ in $\mathrm{SO}(3)$. The body angular momentum vector is initially

$$M_0 = [0.250 \;\; 0.625 \;\; 0.875]^T, \tag{25}$$

and the inertia tensor in body coordinates is

$$I = \mathrm{diag}(7/8, 5/8, 1/4). \tag{26}$$

Finding the second initial condition $P_0$, given $Q_0$ and $M_0$ is done by exploiting the identity

$$M = Q^T P - P^T Q = 2\sinh(\sinh^{-1}(M/2)) = \exp(\sinh^{-1}(M/2)) - \exp(-\sinh^{-1}(M/2)),$$

where we locally get the following expression

$$P_0 = Q_0 \exp(\sinh^{-1}(M_0/2)).$$

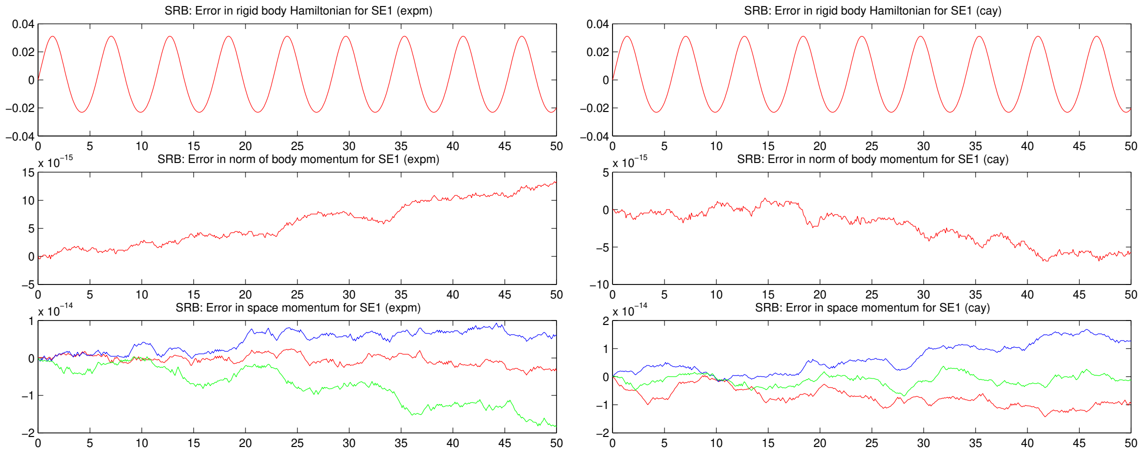


Figure 1: Symmetric rigid body: Integrating over $[0, 50]$ using symplectic Euler in exponential (left) and Cayley (right) coordinates.

Further, the body angular velocity $\Omega$ is calculated as

$$\Omega = I^{-1}M = I^{-1}(Q^T P - P^T Q).$$

In all the numerical experiments we have used the stepsize $h = 1/10$.

In Figure 1 we have shown the results for applying the symplectic Euler method to the equations in the algebra. By construction all the iterates $P_n$ and $Q_n$ are elements of the group SO(3), and the numerical method respects the underlying domain manifold by default. What is interesting is to check how well this method preserves other known conserved quantities. In the upper plot of Figure 1 we have plotted the error of the Hamiltonian. It is clear that the error oscillates in a constant band, and does not increase in magnitude. The trend is the same for integration over longer time intervals. This feature of a numerical method is typical for self-adjoint (symmetric) methods applied to Hamiltonian problems. Self-adjoint Lie group methods are discussed at length in [19].

The middle and lower plots of Figure 1 displays the norm of the angular momentum in body coordinates, and the three components of the angular momentum vector in space coordinates, respectively. It is known that the norm of the momentum vector in body coordinates is constant, which is numerically displayed in the middle plot where the norm is preserved to machine precision. Further, the space angular momentum vector is a constant, which is reproduced numerically in the lower plot, since the three components are all rendered to machine precision.

In Figure 2 we have shown the results for applying the Störmer–Verlet method to the equations in the algebra. In this case the method does not display the favorable property of adjointness. However, the lack of this feature might be corrected for by centering the coordinate system [19]. The Störmer–Verlet based PRKMK method further shows the same good behavior on the retention of the angular momentum in both body and space coordinates, since in both cases we obtain retention to machine precision.

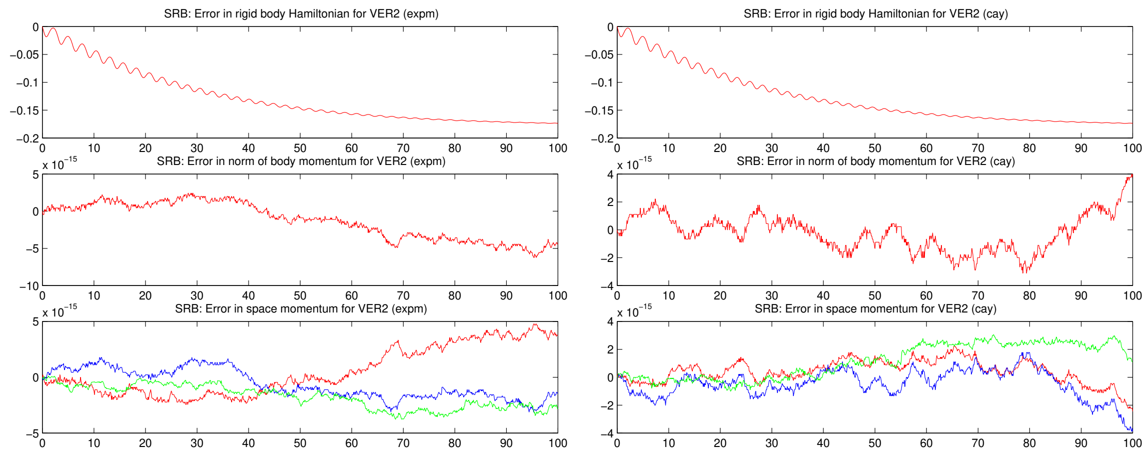


Figure 2: Symmetric rigid body: Integrating over $[0, 100]$ using Störmer–Verlet in exponential (left) and Cayley (right) coordinates.

### 6.2 Unreduced rigid body in body coordinates

We now consider the unreduced rigid body in body (local) coordinates. The configuration space is the cotangent bundle of the Lie group $SO(3)$, which we model as the semidirect product $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$ (Everything in Section 2 naturally carries over to the cotangent bundle case). We refer to Appendix B and C for details relating to the representation of $\mathrm{T}^*G$ as $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$, and the Hamilton equations on $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$.

The initial configuration is a random matrix in $SO(3)$. Further, we have chosen the same initial momentum (25) and inertia tensor (26) as for the symmetric rigid body. In all the numerical experiments we have used the stepsize $1/100$.

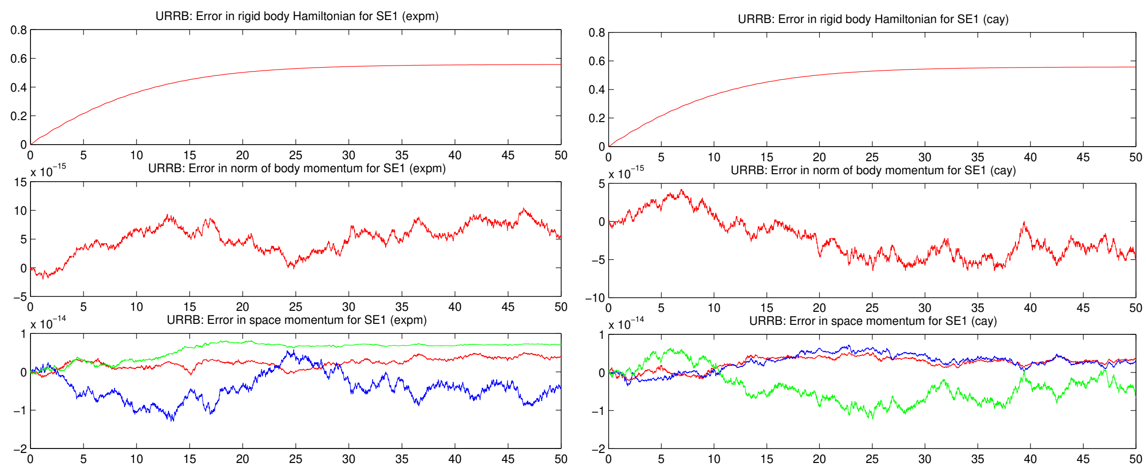


Figure 3: Unreduced rigid body: Integrating over $[0, 50]$ using symplectic Euler in exponential (left) and Cayley (right) coordinates.

As is clear from Figures 3 and 4 both the symplectic Euler scheme and the Störmer–Verlet method preserves the momentum to within machine accuracy. However, neither method does any good on conserving the Hamiltonian. What we see is that the intrinsic numerical error on the manifold of the two methods produce an incorrect trajectory ending up at one of the 'Poles' of the momentum sphere. The same thing is also happening in Figure 2, but here the trajectory ends up at the opposite 'Pole'.

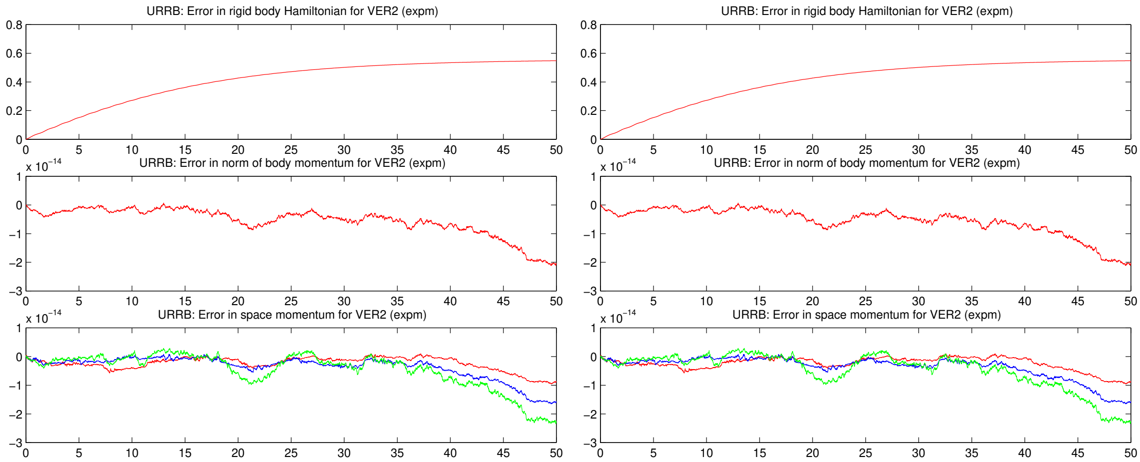


Figure 4: Unreduced rigid body: Integrating over $[0, 50]$ using Störmer–Verlet in exponential (left) and Cayley (right) coordinates.

As a closing remark we emphasize that the above numerical experiments are only simple illustrations of the proposed numerical methods. Extensive numerical testing and comparisons are needed in order to uncover the behavior of the methods, accompanied by further theoretical investigations to determine further properties of the methods.

## 7 Conclusion

In the present paper we have introduced partitioned versions of the RKMK method. This was accomplished by first noticing that most examples of partitioned differential equations naturally occur on bundles. Since we are working in the Lie group setting, we introduce a continuous group structure on the bundle. However, this group structure is not unique, and the choices lead to at least three different ways of thinking of the bundle of a Lie group as a continuous group. We carefully investigate the implications of the three representations of the bundle for the partitioned RKMK method, and show examples of how the same numerical method manifests for each choice. Two numerical examples are included to illustrate the new methods. As implicitly indicated in Section 6, a more detailed understanding of the PRKMK methods in the context of preserving time-symmetry, momentum quantities, and the symplectic structure, is needed. We leave these questions as problems for future research.

Since the present setting introduces a group structure on the bundle, things are set up for immediate use by other Lie group methods like Magnus and Cayley series, and the Crouch–Grossman method. These partitioned Magnus and Cayley series should be investigated further, along with determining their properties. For the partitioned Crouch–Grossman method the next natural and interesting thing to do is to work out the order conditions for this type of non-commutative partitioned Runge–Kutta method.

## Acknowledgment

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## A On $\mathrm{dexp}_{(u,v)}$ and $\mathrm{dexp}_{(u,v)}^{-1}$ for $\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$

In Table 2 the formal expression for $\mathrm{dexp}_{(u,v)}^{-1}$ is given as

$$\mathrm{dexp}_{(u,v)}^{-1}(x,y) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \mathrm{ad}_{(u,v)}^n(x,y).$$

A similar expression for $\mathrm{dexp}_{(u,v)}$ is

$$\mathrm{dexp}_{(u,v)}(x,y) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \mathrm{ad}_{(u,v)}^n(x,y).$$

To resolve what these expressions really mean, we need to know an expression for $\mathrm{ad}_{(u,v)}^n(x,y)$.

**Lemma A.1**

$$\mathrm{ad}_{(u,v)}^n(x,y) = (\mathrm{ad}_u^n x, \mathrm{ad}_u^n y + \sum_{k=1}^{n} \mathrm{ad}_u^{n-k} \mathrm{ad}_v \mathrm{ad}_u^{k-1} x) \tag{27}$$

The proof of this lemma is by induction on the commutator $\mathrm{ad}_{(u,v)}(x,y) = (\mathrm{ad}_u x, \mathrm{ad}_u y + \mathrm{ad}_v x)$ from Table 2. The sum in (27) can be interpreted as a derivative;

$$\sum_{k=1}^{n} \mathrm{ad}_u^{n-k} \mathrm{ad}_v \mathrm{ad}_u^{k-1} = \left( \frac{\partial}{\partial \, \mathrm{ad}_u} \mathrm{ad}_u^n \right) \mathrm{ad}_v,$$

since the operators involved do not commute.

Now, the expression for $\mathrm{dexp}_{(u,v)}$ is found to be

$$\mathrm{dexp}_{(u,v)}(x,y) \quad = \quad (\mathrm{dexp}_u x, \mathrm{dexp}_u y + \partial\mathrm{dexp}_u(v)(x)), \tag{28}$$

where $\partial\mathrm{dexp}_u(v)(x)$ is defined as

$$\partial\mathrm{dexp}_u(v)(x) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left( \frac{\partial}{\partial \, \mathrm{ad}_u} \mathrm{ad}_u^n \right) (\mathrm{ad}_v)(x). \tag{29}$$

Likewise, for $\mathrm{dexp}_{(u,v)}^{-1}$ we get

$$\mathrm{dexp}_{(u,v)}^{-1}(x,y) \quad = \quad (\mathrm{dexp}_u^{-1} x, \mathrm{dexp}_u^{-1} y + \partial\mathrm{dexp}_u^{-1}(v)(x)), \tag{30}$$

where $\partial\mathrm{dexp}_u^{-1}(v)(x)$ is defined as

$$\partial\mathrm{dexp}_u^{-1}(v)(x) = \sum_{n=1}^{\infty} \frac{B_n}{n!} \left( \frac{\partial}{\partial \, \mathrm{ad}_u} \mathrm{ad}_u^n \right) (\mathrm{ad}_v)(x). \tag{31}$$

### A.1 Formulaes for $\mathfrak{so}(3)$

The expressions (28)-(31) considerably simplify in the case of the Lie algebra $\mathfrak{so}(3)$. It is known that for instance the exponential mapping from $\mathfrak{so}(3)$ to SO(3) is expressible as the Rodrigues' formulae [14, p. 291]:

$$\exp(\hat{u}) = I + \frac{\sin(\theta)}{\theta}\hat{u} + \frac{1-\cos(\theta)}{\theta^2}\hat{u}^2,$$

where we have used the hat-map isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, and $\theta^2 = u^T u$, $u \in \mathbb{R}^3$. This formulae is proved by considering the exponential series of $\hat{u}$, using the identity $\hat{u}^3 = -\theta^2\hat{u}$, and reordering the terms. In an equal manner, see e.g. [9], one can show

$$\begin{aligned} \operatorname{dexp}_u &= I + \frac{1-\cos(\theta)}{\theta^2}\hat{u} + \frac{\theta-\sin(\theta)}{\theta^3}\hat{u}^2, && (32)\\ \operatorname{dexp}_u^{-1} &= I - \frac{1}{2}\hat{u} + \frac{2-\theta\cot(\theta/2)}{2\theta^2}\hat{u}^2. && (33) \end{aligned}$$

In a similar way we can calculate a closed form expression of (29), assuming $u, v, x, \in \mathbb{R}^3$:

$$\begin{aligned} \partial\operatorname{dexp}_u(v) &= \frac{1-\cos(\theta)}{\theta^3}\hat{v} + (u^T v)\frac{4\cos(\theta)-4+\theta\sin(\theta)}{2\theta^4}\hat{u}\\ &\quad+\frac{\theta-\sin(\theta)}{\theta^3}(\hat{u}\hat{v}+\hat{v}\hat{u}) + (u^T v)\frac{3\sin(\theta)-\theta\cos(\theta)-2\theta}{\theta^5}\hat{u}^2. && (34) \end{aligned}$$

Now, (32) and (34) together form a closed form expression for (28).

The simplified expression for $\operatorname{dexp}^{-1}_{(u,v)}$ in the $\mathfrak{so}(3)$-case is obtained by using (33) and the following closed form version of formulae (31):

$$\begin{aligned} \partial\operatorname{dexp}_u^{-1}(v)(x) &= -\frac{1}{2}\hat{u} + \frac{2-\theta\cot(\theta/2)}{2\theta^2}(\hat{u}\hat{v}+\hat{v}\hat{u})\\ &\quad-(u^T v)\frac{8-\theta^2-2\theta\cot(\theta/2)-\theta^2\cot^2(\theta/2)}{4\theta^4}\hat{u}^2. \end{aligned}$$

## B The cotangent bundle of a Lie group

Representing the cotangent bundle $\mathrm{T}^*G$ as a direct product is in principle identical to the situation in Section 2.1. As for representing the cotangent bundle as a semidirect product $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$ there are minor differences in the formulaes, but in flavor all the constructions are the same. The results in the case of the semidirect-product representation of $\mathrm{T}^*G$ are collected in Table 4.

We can in a similar way as in the tangent bundle case, lift the binary operation on $G$ up to a binary operation on $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$. Of course, there is a conceptual difference, since this time we treat covectors instead of vectors. Associating a covector to each element in $G$, we see that to be able to add them together the covectors must belong to the same covector space, $\mathrm{T}^*_{ab}G$. Insisting on transforming the covectors directly, instead of first transforming back to the dual Lie algebra $\mathfrak{g}^*$, we obtain the semidirect product as a group structure on $\mathrm{T}\mathfrak{g}^*$. In the literature, one can find 'magnetic extension' as a term used for the trivialized cotangent bundle of a Lie group, when this is represented as a semidirect product, see [2].

The exponential map $\exp_G : \mathfrak{g} \to G$ is lifted up to the trivialized cotangent bundle to yield $\exp :$ $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^* \to G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$ which maps $(u, \alpha) \mapsto (\exp_G(u), \operatorname{dexp}_u(\alpha))$, where $\operatorname{dexp}_u$ is defined in a similar way as the trivialized differential of $\exp_G$ where $\mathrm{ad}$ is substituted with $-\mathrm{ad}^*$ instead.

| Domain representation: $G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*$ | Left | Right |
|---|---|---|
| Group operation $(a, \alpha) \cdot (b, \beta)$ | $(ab, \mathrm{Ad}_b^* \alpha + \beta)$ | $(ab, \alpha + \mathrm{Ad}_{a^{-1}}^* \beta)$ |
| Inverse of $(a, \alpha)$ | $(a^{-1}, -\mathrm{Ad}_{a^{-1}}^* \alpha)$ | $(a^{-1}, -\mathrm{Ad}_a^* \alpha)$ |
| $\mathrm{L}_{(a,\alpha)}(b, \beta)$ | $(ab, \mathrm{Ad}_b^* \alpha + \beta)$ | $(ab, \alpha + \mathrm{Ad}_{a^{-1}}^* \beta)$ |
| $\mathrm{R}_{(a,\alpha)}(b, \beta)$ | $(ba, \mathrm{Ad}_a^* \beta + \alpha)$ | $(ba, \beta + \mathrm{Ad}_{b^{-1}}^* \alpha)$ |
| $\mathrm{I}_{(a,\alpha)}(b, \beta)$ | $(\mathrm{I}_a b, \mathrm{Ad}_{ba^{-1}}^* \alpha$ | $(\mathrm{I}_a b, \alpha - \mathrm{Ad}_{\mathrm{I}_a b^{-1}}^* \alpha$ |
| | $+ \mathrm{Ad}_{a^{-1}}^* (\beta - \alpha))$ | $+ \mathrm{Ad}_{a^{-1}}^* \beta)$ |
| Lie algebra | $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ | $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$ |
| $[(u, \delta), (v, \gamma)]$ | $(\mathrm{ad}_u v, \mathrm{ad}_v^* \delta - \mathrm{ad}_u^* \gamma)$ | $(\mathrm{ad}_u v, \mathrm{ad}_v^* \delta - \mathrm{ad}_u^* \gamma)$ |
| $\mathrm{Ad}_{(a,\alpha)}(v, \gamma)$ | $(\mathrm{Ad}_a v, \mathrm{Ad}_{a^{-1}}^* (\mathrm{ad}_v^* \alpha + \gamma))$ | $(\mathrm{Ad}_a v, \mathrm{ad}_{\mathrm{Ad}_a v}^* \alpha + \mathrm{Ad}_{a^{-1}}^* \gamma)$ |
| Exponential map $\exp(u, v)$ | $(\exp_G(u), \mathrm{dexp}_u(v))$ | $(\exp_G(u), \mathrm{dexp}_u(v))$ |
| Logarithm map $\log(g, u)$ | $(\log_G(g), \mathrm{dexp}_{\log(g)}^{-1}(u))$ | $(\log_G(g), \mathrm{dexp}_{\log(g)}^{-1}(u))$ |
| $\mathrm{dexp}_{(u,v)}^{-1}(x, y)$ | $\sum_{j=0}^{\infty} \frac{B_j}{j!} \mathrm{ad}_{(u,v)}^j (x, y)$ | $\sum_{j=0}^{\infty} \frac{B_j}{j!} (-\mathrm{ad})_{(u,v)}^j (x, y)$ |
| $(v, \gamma)_{G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*}^{\mathrm{L}}(b, \beta)$ | $(\mathrm{T}_e \mathrm{R}_b v, \mathrm{Ad}_b^* \gamma)$ | $(\mathrm{T}_e \mathrm{R}_b v, \gamma - \mathrm{ad}_v^* \beta)$ |
| $(v, \gamma)_{G \ltimes_{\mathrm{Ad}^*} \mathfrak{g}^*}^{\mathrm{R}}(b, \beta)$ | $(\mathrm{T}_e \mathrm{L}_b v, \gamma + \mathrm{ad}_v^* \beta)$ | $(\mathrm{T}_e \mathrm{L}_b v, \mathrm{Ad}_{b^{-1}}^* \gamma)$ |

Table 4: The semidirect-product representation of $\mathrm{T}^*G$.

Similar to the trivialized tangent bundle case, we can state the general form of differential equations on the trivialized cotangent bundle. In the left trivialized case we obtain

$$\begin{aligned} b'(t) &= \mathrm{T}_e \mathrm{R}_b \, v(b, \beta, t) \\ \beta'(t) &= \mathrm{Ad}_b^* \, \gamma(b, \beta, t), \end{aligned}$$

for suitable functions $v : G \times \mathfrak{g}^* \times \mathbb{R} \to \mathfrak{g}$ and $\gamma : G \times \mathfrak{g}^* \times \mathbb{R} \to \mathfrak{g}^*$, whereas for the right trivialized case we get

$$\begin{aligned} b'(t) &= \mathrm{T}_e \mathrm{R}_b \, v(b, \beta, t) \\ \beta'(t) &= \gamma(b, \beta, t) - \mathrm{ad}_{v(b,\beta,t)}^* \beta. \end{aligned}$$

# C Trivialized Hamiltonian vector fields

In this section we recall the natural symplectic structure on $\mathrm{T}^*G$ and the Hamiltonian vector field it generates [1]. Since this symplectic structure is defined irrespectively of the group structure on the manifold – in this case $\mathrm{T}^*G$ – this structure is the same for both the direct-product representation and the semidirect-product representation of the trivialized cotangent bundle. The left trivialized symplectic structure at the point $(g, \pi) \in G \times \mathfrak{g}^*$ is

$$\omega(g, \pi) \left( (\mathrm{T}_e \, \mathrm{L}_g \, u_1, \nu_1), (\mathrm{T}_e \, \mathrm{L}_g \, u_2, \nu_2) \right) = \langle \nu_2, u_1 \rangle - \langle \nu_1, u_2 \rangle + \langle \pi, [u_1, u_2] \rangle,$$

where $u_1, u_2 \in \mathfrak{g}$, $\nu_1, \nu_2 \in \mathfrak{g}^*$, and $\langle , \rangle$ denotes a pairing of elements from $\mathfrak{g}^*$ with elements from $\mathfrak{g}$. In the right trivialized case the $'+'$ is changed into a $'-'$.

The above symplectic structure generates the following Hamiltonian vector fields in the left and right trivialized cases.

$$\begin{aligned} X_H^{\mathrm{L}}(g,\Pi) &= (\mathrm{T}_e\mathrm{L}_g \frac{\delta H}{\delta \Pi}, -\frac{\delta H}{\delta g} + \mathrm{ad}^*_{\frac{\delta H}{\delta \Pi}}\Pi) &(35)\\ X_H^{\mathrm{R}}(g,\pi) &= (\mathrm{T}_e\mathrm{R}_g \frac{\delta H}{\delta \pi}, -\frac{\delta H}{\delta g} - \mathrm{ad}^*_{\frac{\delta H}{\delta \pi}}\pi). &(36)\end{aligned}$$

The functional derivatives of the Hamiltonian are defined as

$$dH(g,\Pi)(\mathrm{T}_e\,\mathrm{L}_g\,u,\alpha) = \langle \frac{\delta H}{\delta g}, u\rangle + \langle \alpha, \frac{\delta H}{\delta \Pi}\rangle.$$

Comparing (35) and (36) with the infinitesimal generators of the left and right multiplicative actions in Table 4, it is interesting to note that (35) is equal to $(v,\gamma)^{\mathrm{R}}_{G\ltimes_{\mathrm{Ad}^*}\mathfrak{g}^*}(b,\beta)$ and (36) is equal to $(v,\gamma)^{\mathrm{L}}_{G\ltimes_{\mathrm{Ad}^*}\mathfrak{g}^*}(b,\beta)$, provided that $v = \frac{\delta H}{\delta \pi}$ and $\gamma = -\frac{\delta H}{\delta g}$.

**The unreduced rigid body equations:** The rigid body equations are usually stated in their reduced form on $\mathfrak{g}^*$, where they are known as the Euler equations. The Euler equations are in fact Hamiltonian on $\mathfrak{g}^*$ with Hamiltonian $H(\Pi) = \frac{1}{2}\Pi \cdot I^{-1}\Pi$. Now we want to consider the unreduced equations on the full cotangent bundle $\mathrm{T}^*G$.

We start with the equations in body coordinates (left trivialization). The Hamiltonian is in this case $H(g,\Pi) = \frac{1}{2}\Pi \cdot I^{-1}\Pi$, where $g$ is the current configuration, and $\Pi$ is the body angular momentum. The Hamiltonian vector field is given by (35), and the functional derivatives are easily checked to equal

$$\begin{aligned}\frac{\delta H}{\delta g} &= 0\\ \frac{\delta H}{\delta \Pi} &= \Omega,\end{aligned}$$

where $\Omega = I^{-1}\Pi$ is the body angular velocity. Putting this into (35), *the unreduced rigid body equations in body coordinates* are

$$\begin{aligned} g'(t) &= \mathrm{T}_e\,\mathrm{L}_g\,\Omega\\ \Pi'(t) &= \mathrm{ad}^*_\Omega\,\Pi.\end{aligned}$$

Specifying $G = \mathrm{SO}(3)$, the equations reduce to

$$\begin{aligned} g'(t) &= g\Omega\\ \Pi'(t) &= \Pi \times \Omega.\end{aligned}$$

Next, we consider the equations in space coordinates (right trivialization). Let $g$ still denote the configuration of the body and let $\pi$ denote the space angular momentum. The Hamiltonian is now expressed as $H(g,\pi) = \frac{1}{2}(g^T\pi)\cdot I^{-1}(g^T\pi)$ in the SO(3)-case. The functional derivatives are

$$\begin{aligned}\frac{\delta H}{\delta g} &= (gI^{-1}g^T)\pi \times \pi\\ \frac{\delta H}{\delta \pi} &= (gI^{-1}g^T)\pi,\end{aligned}$$

which inserted in (36) yields *the unreduced rigid body equations in space coordinates*

$$
\begin{aligned}
g'(t) &= (gI^{-1}g^T)\pi g \\
&= \omega g \\
\pi'(t) &= -(gI^{-1}g^T)\pi \times \pi - \pi \times (gI^{-1}g^T)\pi \\
&= 0,
\end{aligned}
$$

where $\omega = (gI^{-1}g^T)\pi$ is the space angular velocity.

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