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On the BCH-formula in $\mathfrak{so}(3)$

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Abstract

We find a closed-form expression for the Baker–Campbell–Hausdorff formula in the Lie algebra $\mathfrak{so}(3)$, and interpret the formula geometrically in terms of rotation vectors in $\mathbb{R}^3$.

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Key Words: Baker–Campbell–Hausdorff formula, Lie algebra $\mathfrak{so}(3)$, rotations in $\mathbb{R}^3$

1 Introduction

It is known that the expression for the exponential map taking the Lie algebra $\mathfrak{so}(3)$ into the Lie group $\mathrm{SO}(3)$ has a closed-form expression called Rodrigues' formula [2, p. 291].

$$\exp(\hat{x}) = I + \frac{\sin(\theta)}{\theta}\hat{x} + 2\frac{\sin^2(\theta/2)}{\theta^2}\hat{x}^2, \tag{1}$$

where $\theta = \|x\|$. In equation (1) the element of $\mathfrak{so}(3)$ is represented as a skew-symmetric matrix $\hat{x}$, which is obtained from the three-vector x through the hat-map isomorphism:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \longleftrightarrow \quad \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

As shown in Appendix B of [1], similar closed-form expressions for other formulae in the case of $\mathfrak{so}(3)$ and $\mathrm{SO}(3)$ are possible to construct.

We will in this note consider the Baker–Campbell–Hausdorff (BCH) formula in the Lie algebra $\mathfrak{so}(3)$, and find an expression in the spirit of (1), albeit, a bit more involved.

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2 On the BCH formula in $\mathfrak{so}(3)$

2.1 A closed form expression for the BCH formula in $\mathfrak{so}(3)$

We now recall the definition of the BCH formula. See e.g. [3] for more details. In general, define $\mathrm{BCH}(\hat{u},\hat{v})$, for $\hat{u},\hat{v}\in\mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra, as

$$\exp(\mathrm{BCH}(\hat{u},\hat{v})) = \exp(\hat{u})\exp(\hat{v}).$$

For the rest of this note we restrict our attention to the Lie algebra $\mathfrak{so}(3)$, which consists of 3×3 skew-symmetric matrices.

We now state our theorem.

Theorem 2.1 (Expression for BCH in $\mathfrak{so}(3)$)
The BCH formula in $\mathfrak{so}(3)$ has the form

$$\mathrm{BCH}(\hat{u},\hat{v}) = \alpha\hat{u} + \beta\hat{v} + \gamma[\hat{u},\hat{v}],$$

where $[\hat{u},\hat{v}]$ denotes the commutator of $\hat{u}$ and $\hat{v}$, and α,β, and γ are real constants of the form

$$\alpha = \frac{\sin^{-1}(d)}{d}\frac{a}{\theta},\quad \beta = \frac{\sin^{-1}(d)}{d}\frac{b}{\phi},\quad \gamma = \frac{\sin^{-1}(d)}{d}\frac{c}{\theta\,\phi},$$

where a,b,c, and d are defined as

$$\begin{aligned}
a &= \sin(\theta)\cos^2(\phi/2) - \sin(\phi)\sin^2(\theta/2)\cos(\angle(u,v)),\\
b &= \sin(\phi)\cos^2(\theta/2) - \sin(\theta)\sin^2(\phi/2)\cos(\angle(u,v)),\\
c &= \frac{1}{2}\sin(\theta)\sin(\phi) - 2\sin^2(\theta/2)\sin^2(\phi/2)\cos(\angle(u,v)),\\
d &= \sqrt{a^2+b^2+2ab\cos(\angle(u,v)) + c^2\sin^2(\angle(u,v))}.
\end{aligned}$$

In the above formulae $\theta = \|u\|$, $\phi = \|v\|$, and $\angle(u,v)$ is the angle between the two vectors u and v.

The proof of this theorem is by straight forward calculation. Start off by calculating the product of $\exp(\hat{u})\exp(\hat{v})$ using Rodrigues' formula (1), resulting in

$$\begin{aligned}
A &= \exp(\hat{u})\exp(\hat{v})\\
&= I + 2\frac{\sin^2(\theta/2)}{\theta^2}\hat{u}^2 + 2\frac{\sin^2(\phi/2)}{\phi^2}\hat{v}^2 \qquad (2)\\
&\quad + \frac{\sin(\theta)}{\theta}\hat{u} + \frac{\sin(\phi)}{\phi}\hat{v} + \frac{\sin(\theta)}{\theta}\frac{\sin(\phi)}{\phi}\hat{u}\hat{v} + 4\frac{\sin^2(\theta/2)}{\theta^2}\frac{\sin^2(\phi/2)}{\phi^2}\hat{u}^2\hat{v}^2 \qquad (3)\\
&\quad + 2\frac{\sin(\theta)}{\theta}\frac{\sin^2(\phi/2)}{\phi^2}\hat{u}\hat{v}^2 + 2\frac{\sin(\phi)}{\phi}\frac{\sin^2(\theta/2)}{\theta^2}\hat{u}^2\hat{v}. \qquad (4)
\end{aligned}$$

The terms in A are ordered such that (2) is the symmetric part, and (3-4) is the skew-symmetric part. The reason for this ordering of the terms is that we want to take the logarithm of A in order to determine the expression for $\mathrm{BCH}(\hat{u},\hat{v})$, and as a result, the symmetric part drops out.

Lemma 2.2 (Logarithm of $A \in \mathbf{SO}(3)$ [1])
The logarithm of $A \in SO(3)$ is found as follows. Denote by $\hat{z} = (A - A^T)/2$ the skew-symmetric part of A. Then

$$\log(A) = \frac{\sin^{-1}(\|z\|)}{\|z\|}\hat{z}.$$

This lemma is easily proved by staring at the Rodrigues' formula (1), and noticing that I and $\hat{x}^2$ are symmetric. Note that in MATLAB $\sin^{-1}$ is implemented such that in the case of all positive real parts of the eigenvalues of A, $\sin^{-1} = \texttt{asin}$, whereas for the case of two equal negative real parts $\sin^{-1} = \pi - \texttt{asin}$.

We continue the proof of Theorem 2.1 by finding an expression for the skew-symmetric part of A.

$$\begin{aligned}
\hat{z} &= (A - A^T)/2 \\
&= \frac{\sin(\theta)}{\theta}\hat{u} + \frac{\sin(\phi)}{\phi}\hat{v} + \frac{1}{2}\frac{\sin(\theta)}{\theta}\frac{\sin(\phi)}{\phi}[\hat{u}, \hat{v}] \\
&\quad + \frac{\sin(\theta)}{\theta}\frac{\sin^2(\phi/2)}{\phi^2}(\hat{u}\hat{v}^2 + \hat{v}^2\hat{u}) + \frac{\sin(\phi)}{\phi}\frac{\sin^2(\theta/2)}{\theta^2}(\hat{u}^2\hat{v} + \hat{v}\hat{u}^2) \\
&\quad + 2\frac{\sin^2(\theta/2)}{\theta^2}\frac{\sin^2(\phi/2)}{\phi^2}(\hat{u}^2\hat{v}^2 - \hat{v}^2\hat{u}^2).
\end{aligned}$$

Now, using the identities

$$\begin{aligned}
\hat{u}^2\hat{v} + \hat{v}\hat{u}^2 &= -\theta^2\hat{v} - (u^T v)\hat{u}, \\
\hat{u}^2\hat{v}^2 - \hat{v}^2\hat{u}^2 &= -(u^T v)[\hat{u}, \hat{v}], \\
(u^T v) &= \theta\,\phi\cos(\angle(u, v)),
\end{aligned}$$

the expression for $\hat{z}$ simplifies further.

$$\begin{aligned}
\hat{z} &= \left(\sin(\theta)\cos^2(\phi/2) - \sin(\phi)\sin^2(\theta/2)\cos(\angle(u, v))\right)\frac{\hat{u}}{\theta} && (5) \\
&\quad + \left(\sin(\phi)\cos^2(\theta/2) - \sin(\theta)\sin^2(\phi/2)\cos(\angle(u, v))\right)\frac{\hat{v}}{\phi} && (6) \\
&\quad + \left(\frac{1}{2}\sin(\theta)\sin(\phi) - 2\sin^2(\theta/2)\sin^2(\phi/2)\cos(\angle(u, v))\right)\left[\frac{\hat{u}}{\theta}, \frac{\hat{v}}{\phi}\right] && (7) \\
&= a\hat{e}_u + b\hat{e}_v + c\left[\hat{e}_u, \hat{e}_v\right], && (8)
\end{aligned}$$

where we have defined the unit matrices $\hat{e}_x$ as $\frac{\hat{x}}{\|x\|}$, and the constants a, b, and c by (5), (6), and (7), respectively.

Using (8), it is possible to show that the norm of z is given as

$$\|z\|^2 = a^2 + b^2 + 2ab\cos(\angle(u, v)) + c^2\sin^2(\angle(u, v)).$$

Scaling $\hat{z}$ with $\sin^{-1}(\|z\|)/\|z\|$, according to Lemma 2.2, and comparing with the constants in Theorem 2.1, we have completed the proof. □

2.2 Interpreting the BCH formula in $\mathfrak{so}(3)$ as a rotation vector

Interpreting Rodrigues' formula (1) as a rotation around the vector x an angle $\|x\|$, we can interpret $\exp(\hat{u})\exp(\hat{v})$ as the composition of two such rotations. First we rotate around the vector v an angle $\|v\|$, followed by a rotation around the vector u an angle $\|u\|$. Then, $\hat{w} = \mathrm{BCH}(\hat{u}, \hat{v})$ is the vector such that the combined rotation around the vector u and v, is replaced by one rotation around the vector

$$w = \alpha u + \beta v + \gamma u \times v,$$

where the constants are the same as in Theorem 2.1.

3 Conclusion

In this note we have found a closed-form expression for the BCH formula in $\mathfrak{so}(3)$. This adds another member to the family of finite-term expressions for important functions on $\mathfrak{so}(3)$ and SO(3), such as the exponential map, Cayley map, logarithm, etc.

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