A Logic for Set-Valued Functions: Cut-Elimination and Other Improvements

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Abstract

Motivated by the interest in algebraic specification of nondeterminism we sketch the multialgebraic semantics and present an associated sound and complete calculus. Through a series of transformations we arrive at an equivalent calculus not containing the general cut rule. We show also how specific (non-logical) axioms can be replaced by specific cut rules and introduce some other improvements of the original calculus.

1 Introduction

The paper presents a logic introduced in the field of algebraic specifications for the purpose of specifying abstract data types involving nondeterministic operations. The traditional algebraic specifications [EM85] use various variants of equational logic for specfying deterministic data types which are modelled as algebras. However, in many situations, one is faced with the task of specifying a system whose functionality is not uniquely determined by the requirements or, even worse, whose intended behaviour may admit a degree of nondeterminism. In such cases, the use of nondeterminism at the description level either increases the desired level of abstraction (avoiding overspecification) or even, in some cases, proves indispensable. [WM97] surveys the problems and proposals for extending the algebraic framework with nondeterminism. One such proposal is to generalize algebras to *multialgebras* where operations return not necessarily individual elements but, possibly, sets thereof [Hus90, Wal93, WB97]. At the same time, one wants to carefully distinguish the operations which can return a set (i.e., are nondeterministic) from the ones which always return individuals (are deterministic). There are various ways of making such a distinction so here let us, for the sake of example and simplicity, assume that there is a predicate D which, applied to a term t evaluates to true iff t is deterministic. At the language (and logic) level, this poses the following problem. Suppose that we have a nondeterministic operation $f : X \to Y$ and a deterministic one $d : Y \to Z$. In particular, $D(d(y))$ is true for every $y \in Y$. In the traditional setting, this enables us to conclude, by simple substitution, $D(d(f(x)))$. However, since $f(x)$ may denote a set, so $d(f(x))$ may yield a set too, and $D(d(f(x)))$ is not a sound conclusion.[1]

The specific feature of the calculus which we will study is the accommodation of such a restricted substitutivity. In section 2 we introduce a sound and complete calculus NEQ for the multialgebraic semantics of nondeterminism (originally from [Wal93], and whose variants were studied in [WM95a], [WM95b]). Using techniques of [Kan63], [Pli71], [Pli73], and their

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[1]The problem does not, of course, depend on *how* deterministic operations are distinguished from the nondeterminisitc ones – only that they *can* be distinguished.

refinements and modifications, the paper presents a series of transformations, sections 3-5, leading to an equivalent calculus without the general (*cut*) rule. In the context of algebraic specifications one is interested in axiomatizing data types. Typically, a logic is applied to a set of specific (non-logical) axioms for deriving their consequences. We show how the specific (non-logical) axioms and the general cut-rule can be replaced by the *specific-cut* rules (section 3.2). Several other useful specializations of rules are introduced. The main cut-elimination is proved in subsection 4.3 and section 5 gives the final version of the cut-free calculus NEQ_5 with specific-cut rules replacing specific axioms. Then section 6 studies the situation when no specific axioms are present, showing that in this case no cut or cut-like elimination rules are needed, thus yielding quite a simple calculus for proving multialgebraic tautologies. Sections 7-8 contain some concluding remarks and comparison with related works.

The paper consists of a series of transformations of the calculi and lengthy syntactic proofs of their equivalences. The impatient reader may consult directly the main results:

1.	figure 1 on page 3	–	the original version of the calculus NEQ
2.	figure 5 on page 34	–	the final cut-free version NEQ_5 with the specific-cut rules
3.	proposition 4.12, page 26	–	the main cut-elimination result
	lemma 4.10, 5.10, page 22, 35	–	elimination of auxiliary cut-rules
4.	figure 7 on page 39	–	the final calculus NEQ_6 for multialgebraic tautologies not containing cut, specific cut or elimination rules
5.	section 7	–	final comments

2 Specifications of nondeterministic operations

Definition 2.1 (Specifications) *A specification SP is a pair $(\Sigma, \mathcal{A})$, where Σ is a signature, i.e. a pair $(\mathcal{S}, \mathcal{F})$ of sort and operation symbols (the latter with specified arities and rank). $\mathcal{A}$ is a a set of sequents, $\Gamma \mapsto \Delta$, where Γ, Δ are finite sets of atomic formulae.*

The set of terms[2] *built in the standard way from the variables and operation symbols is denoted $\mathcal{T}(\mathcal{F}, \mathcal{V})$. An* atomic formula *is an* equality*, $s = t$, or an* inclusion*, $s \prec t$, of terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ (where t is called the right-hand-side (RHS) and s the left-hand-side (LHS) of inclusion).*

We will write $\mathcal{V}(\sigma)$ to denote the variables occurring in σ (which may be any syntactic entity or a set thereof), and $x \in \mathcal{V}$ to say that x is a variable.

The non-standard aspect of this definition concerns the atomic formulae which are inclusions (typically, one has only equalities). This originates from the fact that the structures to be specified are not standard algebras but *multialgebras*, where operations may return *sets* of individuals rather than only individuals.

Definition 2.2 (Multialgebras) *Given a signature $\Sigma = (\mathcal{S}, \mathcal{F})$, a* Σ-multialgebra *A is given by:*

- *the carrier $|A|$ of A consists of a set S^A for each $S \in \mathcal{S}$*
- *for each operation symbol $f : S_1 \times \ldots \times S_n \to S$, an operation $f^A : S_1^A \times \ldots \times S_n^A \to \mathcal{P}^+(S^A)$, where $\mathcal{P}^+(X)$ denotes the power set of the set X with empty set excluded.*

Given a multialgebra A and an assignment to variables $\beta : X \to |A|$, there is a unique function $\beta[\_]$ which gives, to each term t with $\mathcal{V}(t) \subseteq X$, its interpretation in A defined in the usual way as:

$$\beta[x] = \beta(x) \qquad \beta[c] = c^A \qquad \beta[f(t_1, \ldots, t_n)] = f^A(\beta[t_1] \ldots \beta[t_n])$$

Unlike in classical algebra, this function need not be a homomorphism of multialgebras. An extensive study of multialgebraic homomorphisms can be found in [WB97].

Definition 2.3 (Semantics) *A Σ-multialgebra A satisifes an atomic formula ϕ under assignment $\beta : X \to |A|$, written $A, \beta \models \phi$, iff:*

- *ϕ is $s = t$, and $\beta[s] = \beta[t] = \{a\}$ for some $a \in |A|$*
- *ϕ is $s \prec t$, and $\beta[s] \subseteq \beta[t]$.*

[2] We will not mention sorting explicitly and always assume that terms and formulae are correctly sorted.

*A satisifes a sequent, $A \models \Gamma \mapsto \Delta$ iff, for each assignment β, there exists a $\gamma \in \Gamma : A, \beta \not\models \gamma$, or there exists a $\delta \in \Delta : A, \beta \models \delta$. A Σ-multialgebra A is an SP-*multimodel, *$A \in MMod(SP)$, iff it satisfies all axioms $\mathcal{A}$ of SP.*

The natural deduction-like calculus NEQ consists of the rules given in figure 1 [WM95a]. Each specification contains a set $\mathcal{A}$ of *specific axioms* (2.9).

Axioms :

(2.1) $\mapsto x = x : x \in \mathcal{V}$ (2.2) $s \odot t \mapsto s \odot t \quad \odot \in \{=, \prec\}$

Identity and inclusion rules :

$$(2.3)\ \frac{\Gamma^x_t \mapsto \Delta^x_t \ ; \ \Gamma' \mapsto s = t, \Delta'}{\Gamma^x_s, \Gamma' \mapsto \Delta^x_s, \Delta'} \qquad (2.4)\ \frac{\Gamma \mapsto \Delta, w(t) \preceq q \ ; \ \Gamma' \mapsto s \prec t, \Delta'}{\Gamma, \Gamma' \mapsto \Delta', \Delta, w(s) \preceq q}$$

Elimination rules:

$$(2.5)\ \frac{\Gamma, x \prec t \mapsto \Delta}{\Gamma^x_t \mapsto \Delta^x_t}\ (elim1) \qquad (2.6)\ \frac{x \prec t, y \prec r(x), \Gamma \mapsto \Delta}{y \prec r(t), \Gamma \mapsto \Delta}\ (E_a)$$

- $x \notin \mathcal{V}(t)$;
- x occurs at most once in $\Gamma \mapsto \Delta$
- and not in a RHS of $\prec$ in Γ

- $x \notin \mathcal{V}(t, y, \Gamma, \Delta), y \in \mathcal{V}$
- x occurs at most once in r

Weakening rules (W) :

$$\frac{\Gamma \mapsto \Delta}{\Gamma \mapsto \Delta, A} \qquad (2.7)\ \frac{\Gamma \mapsto \Delta}{\Gamma, A \mapsto \Delta}$$

Cut rule (cut) :

$$(2.8)\ \frac{\Gamma \mapsto \Delta, A \ ; \ A, \Gamma' \mapsto \Delta'}{\Gamma, \Gamma' \mapsto \Delta, \Delta'}$$

Specific axioms :

(2.9) $A_1 ... A_n \mapsto B_1 ... B_m$ $(\mathcal{A}_k)$ where k identifies a particular axiom in the set $\mathcal{A}$.

Figure 1: The rules and axioms of NEQ

A, B stand for atoms, i.e., $s \odot t$ with $\odot \in \{=, \prec\}$. The symbol $\preceq$ abbreviates either $=$ or $\prec$. Γ^x_t indicates replacement of one or more x by t. Sometimes, we may write $\Gamma[x]$, $\phi[x]$ for a (set of) atom(s) containing x, and then $\Gamma[t]$, $\phi[t]$ for the result of replacing these occurrences of x by t.

Remark 2.4 *From the definition of a sequent it follows that NEQ implicitly contains the structural rules "exchange" and "contraction". NEQ also assumes implicitly symmetry of equations, i.e., $s = t$ is treated as an unorded set $\{s, t\}$.*

Two peculiar features of this calculus are the absence of the unrestricted substitutivity, and that $=$ is only a partial equivalence. The equality $t = t$ represents "deterministic equality" – it holds only for terms which denote individual elements of the carrier. Thus, in general, $NEQ \not\vdash \ \mapsto t = t$. But since variables are assigned only individual elements, we do have $NEQ \vdash \ \mapsto x = x$ for any variable x. Consequently, since terms may denote sets *of individuals, unrestricted substitutivity would not be sound wrt. the multialgebraic semantics. For motivations and examples the reader is referred to [WM95a, WM95b, WB95].*

Theorem 2.5 [WM95a] *NEQ is sound and complete with respect to the multialgebraic semantics, i.e., for any sequent S:*

$$MMod(\Sigma, \mathcal{A}) \models S \iff \mathcal{A}, NEQ \vdash S$$

$\mathcal{A}, NEQ \vdash S$, indicating that S is derivable from $\mathcal{A}$ using the rules of NEQ, might be written in a more standard fashion as $\mathcal{A} \vdash^{NEQ} S$. However, we will consider variants of the calculus where the axioms $\mathcal{A}$ are built into the rules, and the latter notation will be used for a different purpose.

Definition 2.6 *Let I, J be arbitrary calculi. We use the following notational convention:*

- $I \Rightarrow J$ *if, for any sequent* $S : \mathcal{A}, I \vdash S \;\Rightarrow\; \mathcal{A}, J \vdash S$;
- $I \equiv J$ *if* $I \Rightarrow J$ *and* $J \Rightarrow I$;
- I^c *denotes* I *extended with the (cut) rule;*
- $I \vdash^D S$ *indicates that* D *is a derivation of* S *using the rules of calculus* I;
- *for derivations* D, D', *we write* $D \Rightarrow D'$ *to indicate that* D' *may be constructed, assuming that* D *is given.*
- *Variable* x *in the applications of any of the elimination rules will be called* eigen-variable.

3 Some intermediary calculi

In this section we transform NEQ to an equivalent calculus NEQ_3 which will provide a basis for the final calculus and (cut)-elimination. Instead of introducing the resulting calculus at once, we proceed step by step to better visualize the crucial transformations. The most significant one occurs in subsection 3.2 where specific axioms are replaced by appropriate inference rules.

3.1 Kanger-like calculus – NEQ_1

We transform NEQ to an equivalent calculus NEQ_1 with Kanger-like rules for equality [Kan63] and inclusion.

Definition 3.1 *The calculus* NEQ_1 *is obtained from* NEQ *by the following transformations: For any* $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ *and* $x \in \mathcal{V}$, *the following axiom is added:*

(3.1) $\quad x \prec t \mapsto x \prec t$

The axiom (2.2), $s \odot t \mapsto s \odot t$ *is replaced by:*

(3.2) $\quad s = t \mapsto s = t$

The inference rules for equality and inclusion (2.3), (2.4) are replaced by:

(3.3) $$\frac{s = t, \Gamma^x_t \mapsto \Delta^x_t}{s = t, \Gamma^x_s \mapsto \Delta^x_s}\ (=_1) \qquad \frac{t = s, \Gamma^x_t \mapsto \Delta^x_t}{t = s, \Gamma^x_s \mapsto \Delta^x_s}\ (=_2)$$

(3.4) $$\frac{\Gamma, s \prec t \mapsto \Delta, w(t) \preceq q}{\Gamma, s \prec t \mapsto \Delta, w(s) \preceq q}\ (\prec_{Ls})$$

The cases for $=_1$ and $=_2$ are entirely symmetric. Since the reader is going to suffer a lot of notational abbreviations, we are allowing ourselves to gloss over this difference in the sequel and treat these two cases uniformly – we write merely one rule $(=)$ which is to be understood as a shortened form for the two rules $(=_1)$ and $(=_2)$.

Lemma 3.2 $NEQ \Rightarrow NEQ_1$

Proof. We show that the axioms (2.2) and inference rules (2.3), (2.4) are derivable in NEQ_1.

1) Let S be $s \prec t \mapsto s \prec t$. The derivation in NEQ_1 is obtained by choosing a fresh variable x:

$$\cfrac{\cfrac{\cfrac{x \prec t \mapsto x \prec t}{\mapsto t \prec t}\ (elim1)}{s \prec t \mapsto t \prec t}\ (W)}{s \prec t \mapsto s \prec t}\ (\prec_{Ls})$$

2) Suppose that NEQ_1 proves the premises of (2.3). Weakening the first premise with the formula $s = t$ allows us to get $s = t, \Gamma^x_s \mapsto \Delta^x_s$ by $(=)$ and (cut) with the second premise yields the conclusion of (2.3) in NEQ_1:

$$\dfrac{\Gamma' \mapsto s = t, \Delta' \quad \dfrac{\dfrac{\Gamma^x_t \mapsto \Delta^x_t}{s = t, \Gamma^x_t \mapsto \Delta^x_t}\,(\mathrm{W})}{s = t, \Gamma^x_s \mapsto \Delta^x_s}\,(=)}{\Gamma^x_s, \Gamma' \mapsto \Delta^x_s, \Delta'}\,(cut)$$

3) An analogous procedure as above allows us to derive (2.4) in NEQ_1

QED.

Lemma 3.3 $NEQ_1 \Rightarrow NEQ$

Proof.

1) Both inference rules (3.3) are derived by a single application of (2.3):

$$\dfrac{s = t, \Gamma^x_t \mapsto \Delta^x_t \quad ; \quad s = t \mapsto s = t}{s = t, \Gamma^x_s \mapsto \Delta^x_s}$$

2) and the rule (3.4) is obtained by an analogous application of (2.4).

QED.

The two lemmas yield:

Proposition 3.4 $NEQ \equiv NEQ_1$.

3.2 NEQ_2 – some restrictions on (cut) and $(elim1)$.

A new equivalent of NEQ is introduced in which the general (cut) rule is replaced by the *specific-cut rules* depending on the specific axioms of the specification. Recall that we have fixed a set of axioms $\mathcal{A}$, of the general form (2.9): $r_i \odot_i s_i \mapsto u_j \odot_j v_j$.

Definition 3.5 *Calculus NEQ_2 is obtained from NEQ_1 by:*

1. *replacing the inference rule (2.5) with:*

$$(3.5) \qquad \dfrac{\Gamma, x \prec t \mapsto \Delta}{\Gamma \mapsto \Delta^x_t} \quad (E_s) \quad x \notin \mathcal{V}(\Gamma, t) \text{ and at most one occurence of } x \text{ in } \Delta.$$

2. *adding the following inference rules for inclusion:*

$$(3.6) \qquad \dfrac{s \prec t, w(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, w(t) \preceq q, \Gamma \mapsto \Delta}\,(\prec_{La}) \qquad \dfrac{s \prec t, \Gamma \mapsto \Delta, w \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, w \prec q(t)}\,(\prec_{Rs})$$

(In the case of equality $w(s) = q$, we implicitly allow $q = w(t)$ in the conclusion.)

3. *removing the cut-rule (2.8);*
4. *replacing each specific axiom $\mathcal{A}_k$ of the form (2.9) by the (Sp.cut$_k$) rule:*

$$(3.7) \qquad \dfrac{\Gamma \mapsto \Delta, A_1 \;;\; ... \;;\; \Gamma \mapsto \Delta, A_n \;;\; B_1, \Gamma \mapsto \Delta \;;\; ... \;;\; B_m, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta} \quad (Sp.cut_k)$$

Formulae A_i and B_j in applications of (Sp.cut) will be called the specific cut formulae.

Before proving the equivalence of NEQ_1 and NEQ_2^c we introduce the following conventions to be used extensively.

Definition 3.6 *In an application of a rule we distinguish the following formulae:*

- active – *the atomic formula explicitly mentioned in the premise(s) and conclusion of the rule but not modified by its application;*
- side – *the formulae in the premise of the rule which are modified (or removed) by its application;*
- modified – *the new formulae obtained in the conclusion of the rule (either as the modified side formulae or as a new formulae introduced in the conclusion).*[3]

[3] This is also called *principal*.

The term which appears in a modified formula as a result of the application of the rule is called the *resulting term*. Sometimes we will refer by "modified" to the term from the side formula which was changed by the application of the rule, by "replaced" to this part of it which was replaced and by "substituted" to this part of the resulting term which appeard for the replaced (sub)term of the modified term.

It is implicitly assumed that each rule modifies at most one term, even if their formulation might indicate that several terms are modified simultaneously (like for instance (2.3) and (2.4) in figure 1). This convention is used throughout the whole paper.

Example 3.7 For instance, each of the weakening rules (2.7) has only one modified formula $(s \odot t)$, and no side formula. The (cut) rule (2.8) has only side formula which is also called the *cut* formula. The atom $s \odot t$ in the rules (3.3), (3.4) is the active formula of these rules. Term t is modified and s substituted in (3.3) and in (3.4). The number of modified and side formulae in these rules depends on the number of atoms in which t is replaced by s. In the following application of $(elim1)$ rule (2.5)

$$\frac{\Gamma, \phi^x_x, x \prec t \mapsto \Delta}{\Gamma, \phi^x_t \mapsto \Delta}$$

ϕ^x_t is the modified formula, $\phi^x_x, x \prec t$ are side formulae, and t is the substituted term.
We apply this notions in the top-down fashion, i.e., the formula above the line will have a side formula, which is modified in the line below. If one thinks of the bottom-up applications of the rules the notions remain the same but what was "modified" in top-down will now be "side" and vice versa.

The rules will be tagged by $(\odot_{Xyz})$ according to the following convention:

- $\odot \in \{=, \prec\}$ reflects the operator of the active formula;
- $X \in \{L, R\}$ identifies the side of the inclusion in which the modified term occured: L for LHS or equality and R for RHS;
- $y \in \{a, s\}$ – a if the side formula is in the antecedent of the sequent, and s if it is in the consequent; the rules will be called "a-rules", respectively, "s-rules";
- z may be an additional annotation, typically, x if the substituted term was a variable, r if the replaced term was a variable, nr if the replaced term was not a variable.

Thus, for instance, $(\prec_{Ls})$ is the rule with active inclusion which modifies a term in either an equality or LHS of an inclusion in the succedent.

We will also use following notions for a derivation D:

- the *height* of D, denoted $h(D)$, with the obvious meaning;
- the *number of applications* of a rule R in D, denoted $\#(R, D)$;

Lemma 3.8 $NEQ_1 \Rightarrow NEQ^c_2$.

Proof. All specific axioms $\mathcal{A}$ are derivable in NEQ_2, using $(Sp.cut)$ rules and derivability of the axiom $s \odot t \mapsto s \odot t$ in NEQ_2.

The rule (E_s) is a restricted version of $(elim1)$ when the modified formula occurs in the succedent. To see that $(elim1)$ is derivable in NEQ_2, first observe that application of $(elim1)$ with the modified formula in the succedent yields the same result as application of (E_s). So, consider an application of $(elim1)$ with the modified formula in the antecedent.

$$\frac{\Gamma, s(x) \preceq p, x \prec t \mapsto \Delta}{\Gamma, s(t) \preceq p \mapsto \Delta}\ (elim1)$$

By the restrictions on $(elim1)$, x occurs either in $=$ or in a LHS of $\prec$, which is indicated by the symbol $\preceq$. We may apply $(\prec_{La})$ and $(elim_s)$ to obtain the same conclusion:

$$\dfrac{\dfrac{\Gamma, s(x) \preceq p, x \prec t \mapsto \Delta}{\Gamma, s(t) \preceq p, x \prec t \mapsto \Delta}\ (\prec_{La})}{\Gamma, s(t) \preceq p \mapsto \Delta}\ (E_s)$$

QED.

Proposition 3.9 $NEQ^c_2 \equiv NEQ_1$.

Proof. $\Leftarrow$ is lemma 3.8. For the opposite implication we have the following situations.

1) The $(Sp.cut)$ rules (3.7) are derivable in NEQ_1. For each $(Sp.cut_k)$, use (cut) to its premises and the corresponding specific axiom $\mathcal{A}_k$.

2) The rule $(\prec_{La})$ is derivable in NEQ_1. Assume that NEQ_1 proves the premise $s \prec t, w(s) \preceq q, \Gamma \mapsto \Delta$ of the rule. (For simplicity, consider only the case with one occurrence of s in this premise, i.e., when it has the form $s \prec t$.) The following is a derivation of the conclusion of the rule in NEQ_1:

$$
x \text{ not in the RHS of } \prec \text{ in } \phi \quad
\dfrac{\dfrac{\dfrac{w(t) \preceq q \mapsto w(t) \preceq q}{s \prec t, w(t) \preceq q \mapsto w(s) \preceq q}\,(\mathrm{W})}{s \prec t, w(t) \preceq q \mapsto w(s) \preceq q}\,(\prec_{Ls}) \qquad s \prec t, w(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, w(t) \preceq q, \Gamma \mapsto \Delta}\,(cut)
$$

3) In a similar way, we show admissibility of $(\prec_{Rs})$ in NEQ_1. Assume it proves the premise of the rule $s \prec t, \Gamma \mapsto \Delta, w \prec q(s)$. We then construct the following derivation of its conclusion in NEQ_1:

$$
\dfrac{s \prec t, \Gamma \mapsto \Delta, w \prec q(s) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \prec q(t) \mapsto x \prec q(t)}{\mapsto q(t) \prec q(t)}\,(elim1)}{s \prec t, \mapsto q(t) \prec q(t)}\,(W)}{s \prec t, \mapsto q(s) \prec q(t)}\,(\prec_{Ls})}{w \prec q(s), s \prec t, \mapsto q(s) \prec q(t)}\,(W)}{w \prec q(s), s \prec t, \mapsto w \prec q(t)}\,(\prec_{Ls})}{s \prec t, \Gamma \mapsto \Delta, w \prec q(t)}\,(cut)
$$

QED.

3.3 Calculus NEQ_3 – no weakening rules.

Applying the standard technique, we will now eliminate the weakening rules by replacing each axiom of NEQ_2 of the form $a \mapsto s$ by the set of axioms $\Gamma, a \mapsto \Delta, s$ for all Γ, Δ. This simplifies the proofs, nevertheless, it is often easier to understand the arguments if the applications of weakening are made explicit. We will occasionally formulate the proofs using such an explicit mention of weakening which is to be understood as starting the whole derivation with another instance of the same axiom – in particular, it does not increase the height of the derivation. More precisely, letting NEQ_2^* denote NEQ_2 with axioms modified as indicated above and without the weakening rules, we have the following fact:

Lemma 3.10 *$NEQ_2 \vdash^D S \;\Rightarrow\; NEQ_2^* \vdash^{D^*} S$ and $h(D^*) \leq h(D)$.*

Proof. We have to show that weakening rules are admissible in NEQ_2^*. Proceeding by induction on the number of their applications and $h(D)$, consider the last rule R applied above weakening. The cases for (W_a) and (W_s) are exactly the same, so we only formulate the proof for (W_a).

1) If no rules are applied above (W_a) in D, the result is an axiom $s \mapsto s$ (with LHS possibly empty) of NEQ_2 weakened by a formula ϕ. In NEQ_2^* we obtain it directly as an axiom $\phi, s \mapsto s$. The height of this new derivation is less then $h(D)$.

2) R is $(=)$, and D ends as follows

$$
D \left\{ \begin{array}{c} \vdots \\ \dfrac{\dfrac{s = t, f(s) \odot q, \Gamma \mapsto \Delta}{s = t, f(t) \odot q, \Gamma \mapsto \Delta}\,(=)}{\phi, s = t, f(t) \odot q, \Gamma \mapsto \Delta}\,(W) \end{array} \right.
\quad \Rightarrow \quad
\begin{array}{c} \vdots \\ \dfrac{\dfrac{s = t, f(s) \odot q, \Gamma \mapsto \Delta}{\phi, s = t, f(s) \odot q, \Gamma \mapsto \Delta}\,(W)}{\phi, s = t, f(t) \odot q, \Gamma \mapsto \Delta}\,(=) \end{array}
$$

Induction hypothesis on $h(D)$ yields the conclusion.

3) The argument is exactly the same for all the remaining rules, with the exception of the elimination rules.

4) R is (E_s) – If ϕ does not contain the eigen-variable x of (E_s), we can trivially swap the two. If it does, however, we have to rename x in the whole D_1 to a new variable, say y, not occuring there. We then obtain equivalent derivation in which we can swap the applications of (E_s) and (W).

$$D \left\{ \dfrac{D_1 \left\{ \dfrac{\begin{array}{c}\vdots \\ x \prec t, \Gamma \mapsto \Delta\end{array}}{\Gamma \mapsto \Delta^x_t}\ (E_s) \right.}{\phi[x], \Gamma \mapsto \Delta^x_t}\ (W) \right. \quad \Rightarrow \quad D' \left\{ \dfrac{D'_1 \left\{ \dfrac{\begin{array}{c}\vdots \\ y \prec t, \Gamma \mapsto \Delta\end{array}}{\Gamma \mapsto \Delta^y_t}\ (E_s) \right.}{\phi[x], \Gamma \mapsto \Delta^y_t}\ (W) \right.$$

5) The same procedure is applied when the last rule was (E_a).

QED.

Definition 3.11 *Calculus NEQ_3 is obtained from NEQ_2 by:*

- *replacing weakening rules by extended axioms as indicated in lemma 3.10;*
- *replacing rules (3.3) (=) by:*

$$\text{(3.8)} \quad \frac{s = t, \Gamma \mapsto \Delta^x_t}{s = t, \Gamma \mapsto \Delta^x_s}\ (=_s) \quad \textit{and} \quad \frac{s = t, \Gamma^x_t \mapsto \Delta}{s = t, \Gamma^x_s \mapsto \Delta}\ (=_a)$$

adding the inference rules of simple cut

$$\text{(3.9)} \quad \frac{t \prec t, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (cut_t) \qquad \text{(3.10)} \quad \frac{x = x, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (cut_{x=})$$

Axioms :

$$x \prec t, \Gamma \mapsto \Delta, x \prec t \qquad \Gamma \mapsto \Delta, x = x \qquad \Gamma, s = t \mapsto s = t, \Delta$$

Identity rules :

$$\frac{s = t, \Gamma \mapsto \Delta^x_t}{s = t, \Gamma \mapsto \Delta^x_s}\ (=_s) \qquad \frac{s = t, \Gamma^x_t, \mapsto \Delta}{s = t, \Gamma^x_s, \mapsto \Delta}\ (=_a)$$

Inclusion rules :

$$\frac{s \prec t, \Gamma \mapsto \Delta, w(t) \preceq q}{s \prec t, \Gamma \mapsto \Delta, w(s) \preceq q}\ (\prec_{Ls}) \qquad \frac{s \prec t, w(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, w(t) \preceq q, \Gamma \mapsto \Delta}\ (\prec_{La})$$

$$\frac{s \prec t, \Gamma \mapsto \Delta, w \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, w \prec q(t)}\ (\prec_{Rs})$$

Elimination rules :

$$\frac{x \prec t, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta^x_t}\ (E_s) \qquad \frac{x \prec t, y \prec r(x), \Gamma \mapsto \Delta}{y \prec r(t), \Gamma \mapsto \Delta}\ (E_a)$$

- $x \notin \mathcal{V}(\Gamma, t)$;
- at most one x in Δ;

- x is fresh : $x \notin \mathcal{V}(t, \Gamma, \Delta)$;
- at most one x in r

Simple cut rules :

$$\frac{t \prec t, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (cut_t) \qquad \frac{x = x, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (cut_{x=})$$

Specific cut rules :

for each specific axiom $\mathcal{A}_k$: $A_1 ... A_n \mapsto B_1 ... B_m$, a rule:

$$\frac{\Gamma \mapsto \Delta, A_1\ ;...;\ \Gamma \mapsto \Delta, A_n\ ;\ B_1, \Gamma \mapsto \Delta\ ;...;\ B_m, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (Sp.cut_k)$$

Figure 2: The rules of NEQ_3 $(x, y \in \mathcal{V})$.

Any application of the equality rule (3.3) in NEQ_2 can be simulated by repeated applications of the rules (3.8) – recall that each of the rules (3.8) abbreviates two rules: $(=_1)$ substituting the first and $(=_2)$ substituting the second term from the active equality $s = t$. Weakening rules are admissible in NEQ_3 according to lemma 3.10 and, in the presence of the (cut) rule, the simple cut (cut_t), $(cut_{x=})$ do not add anything to the power of the calculus. We thus have the obvious:

Proposition 3.12 $NEQ_2^c \equiv NEQ_3^c$.

4 NEQ_4 and admissibility of (cut).

The appplications of (E_s) and (E_a) substituting a variable will be called *degenerate*:

- (E_{ax}) $\dfrac{\Gamma, x \prec z, y \prec r(x) \mapsto \Delta}{\Gamma, y \prec r(z) \mapsto \Delta}$ $z \in \mathcal{V}$ $\quad (r(x) = x$ is a special case)
- (E_{sx}) $\dfrac{\Gamma, x \prec z \mapsto \Delta}{\Gamma \mapsto \Delta_z^x}$ $z \in \mathcal{V}$

These are excluded from NEQ_4. On the other hand, applications of various rules replacing a non-variable term will be called *non-reduced*. We also divide the applications of the equality rules explicitly into those modifying the LHS or RHS of inclusions. The a-equality rule modifying another equality has been removed. Only reduced applications of the a-rule for modifying an LHS, $(=_{Lar})$, are included in NEQ_4. The simple cut rule of NEQ_3 removing $t \prec t$ is replaced by the elementary cut removing only variable $x \prec x$. The rules of NEQ_4 are shown in figure 3.
The essential facts in the rest of this section are:

1. The possibility of transforming NEQ_4 derivations in two different ways eliminating either some undesirable applications of a-rules or else of some s-rules. These proofs are in subsection 4.1.
2. Admissibility of the rules excluded from NEQ_3 leading to the equivalence of the two calculi. These proofs are partly in subsection 4.1 – the crucial proofs of admissibility of the simple cut rule and equivalence of the two calculi are in subsection 4.2.
3. Admissibility of (cut) in NEQ_4 is shown in subsection 4.3. The proof is based on the possibility of modifying NEQ_4 derivations as in point 1 above.

.

4.1 Some facts about NEQ_4 derivations

Lemma 4.1 *$NEQ_4 \vdash^D S \;\Rightarrow\; NEQ_4 \vdash^{D^*} S$ and D^* does not contain $(=_{Rs})$.*

Proof. By induction on $\langle \#((=_{Rs}), D), h(D) \rangle$. For the axioms, we only have to consider the one with inclusion in the consequent.

$$\frac{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(s)}{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(p)} \; (=_{Rs}) \qquad \Rightarrow \qquad \frac{p = s, x \prec t(p), \Gamma \mapsto \Delta, x \prec t(p)}{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(p)} \; (=_{Ra})$$

Consider the uppermost application of $(=_{Rs})$ in D:

$$\cfrac{\cfrac{\vdots}{r = s, \Gamma \mapsto \Delta, q \prec t(r)} \; R}{r = s, \Gamma \mapsto \Delta, q \prec t(s)} \; (=_{Rs})$$

1) $R = (E_{sr})$ with $q \prec t(r)$ as the modified formula and r as modified term. We have two subcases corresponding to the situation when the term r' substituted by (E_{sr}) for its eigenvariable is a subterm or superterm of (or equal to) r. (When the two are independent, the case is trivial.)

Axioms:

$$x \prec t, \Gamma \mapsto \Delta, x \prec t \qquad s = t, \Gamma \mapsto \Delta, s = t \qquad \Gamma \mapsto \Delta, x = x$$

Identity rules:

$$(=_{Ls})\ \frac{t = s, \Gamma \mapsto \Delta, p(s) \preceq q}{t = s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (=_{Lar})\ \frac{x = t, p(x) \prec q, \Gamma \mapsto \Delta}{x = t, p(t) \prec q, \Gamma \mapsto \Delta}$$

$$(=_{Rs})\ \frac{t = s, \Gamma \mapsto \Delta, p \prec q(s)}{t = s, \Gamma \mapsto \Delta, p \prec q(t)} \qquad (=_{Ra})\ \frac{s = t, p \prec q(s), \Gamma \mapsto \Delta}{s = t, p \prec q(t), \Gamma \mapsto \Delta}$$

Inclusion rules:

$$(\prec_{Ls})\ \frac{s \prec t, \Gamma \mapsto \Delta, p(t) \preceq q}{s \prec t, \Gamma \mapsto \Delta, p(s) \preceq q} \qquad (\prec_{La})\ \frac{s \prec t, p(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, p(t) \preceq q, \Gamma \mapsto \Delta}$$

$$(\prec_{Rs})\ \frac{s \prec t, \Gamma \mapsto \Delta, p \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, p \prec q(t)} \qquad (\prec_{Ra})$$

Elimination rules:

$$(E_{sr})\ \frac{x \prec t, \Gamma \mapsto \Delta, \phi[x]}{\Gamma \mapsto \Delta, \phi[t]} \qquad (E_{ar})\ \frac{x \prec t, y \prec q(x), \Gamma \mapsto \Delta}{y \prec q(t), \Gamma \mapsto \Delta}$$

- $x \notin \mathcal{V}(\Gamma, t)$, $t \notin \mathcal{V}$
- at most one x in Δ;

- $x \notin \mathcal{V}(t, \Gamma, \Delta, y)$, $t \notin \mathcal{V}$
- at most one x in $q(x)$

Elementary cut:

$$(cut_x)\ \frac{x \prec x, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta} \qquad (cut_{x=})\ \frac{x = x, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}$$

Specific cut rules:

for each specific axiom $\mathcal{A}_k$: $A_1, ..., A_n \mapsto B_1, ..., B_m$, a rule:

$$\frac{\Gamma \mapsto \Delta, A_1\ ; ...;\ \Gamma \mapsto \Delta, A_n\ ;\ B_1, \Gamma \mapsto \Delta\ ; ...;\ B_m, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\ (Sp.cut_k)$$

Figure 3: The rules of NEQ_4 $(x, y \in \mathcal{V})$

1.1) r' is a subterm of r,
i.e., $t'(r')$ and $t(r)$ are identical with $r = f(r')$ and $t'(x) = t(f(x))$.

$$D \left\{ \begin{array}{ll} \dfrac{\dfrac{x \prec r', f(r') = s, \Gamma \mapsto \Delta, q \prec t(f(x))}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(f(r'))}\ (E_{sr})}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\ (=_{Rs}) \end{array} \right.$$

$$\Downarrow$$

$$D' \left\{ \begin{array}{ll} \dfrac{\dfrac{\dfrac{x \prec r', f(r') = s, f(x) = s, \Gamma \mapsto \Delta, q \prec t(f(x))}{x \prec r', f(r') = s, f(x) = s, \Gamma \mapsto \Delta, q \prec t(s)}\ (=_{Rs})}{x \prec r', f(r') = s, f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\ (\prec_{La})}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\ (E_{sr}) \end{array} \right.$$

Induction hypothesis allows us to eliminate the application of $(=_{Rs})$ from D' since it occurs higher than in D.

1.2) r' is a superterm of (or equal to) r, i.e., $r' = f(r)$ and $t(x) = t'(f(x))$.
On the right we have the case when $r' = r$ and s is a variable y

$$
\begin{array}{cl}
x \prec f(r), r = s, \Gamma \mapsto \Delta, q \prec t'(x) & \\ \hline
r = s, \Gamma \mapsto \Delta, q \prec t'(f(r)) & (E_{sr}) \\ \hline
r = s, \Gamma \mapsto \Delta, q \prec t'(f(s)) & (=_{Rs}) \\
\Downarrow & \\
x \prec f(r), r = s, \Gamma \mapsto \Delta, q \prec t'(x) & \\ \hline
x \prec f(s), r = s, \Gamma \mapsto \Delta, q \prec t'(x) & (=_{Ra}) \\ \hline
r = s, \Gamma \mapsto \Delta, q \prec t'(f(s)) & (E_{sr})
\end{array}
\qquad
\begin{array}{cl}
x \prec r, r = y, \Gamma \mapsto \Delta, q \prec t'(x) & \\ \hline
r = y, \Gamma \mapsto \Delta, q \prec t'(r) & (E_{sr}) \\ \hline
r = y, \Gamma \mapsto \Delta, q \prec t'(y) & (=_{Rs}) \\
\Downarrow & \\
x \prec r, r = y, \Gamma \mapsto \Delta, q \prec t'(x) & \\ \hline
x \prec y, r = y, \Gamma \mapsto \Delta, q \prec t'(x) & (=_{Ra}) \\ \hline
x \prec y, r = y, \Gamma \mapsto \Delta, q \prec t'(y) & (\prec_{Rs}) \\ \hline
x \prec r, r = y, \Gamma \mapsto \Delta, q \prec t'(y) & (=_{Ra}) \\ \hline
r = y, \Gamma \mapsto \Delta, q \prec t'(y) & (E_{sr})
\end{array}
$$

The case when the modified term is substituted into LHS of $\prec$ allows simple swap since then (E_{sr}) and $(=_{Rs})$ modify different terms.

2) $R = (\prec_{La})$ with the modified formula $r = s(v)$ and v as the modified term:

$$
D \left\{
\begin{array}{cl}
D_1 \left\{ \begin{array}{c} \vdots \\ r = s(u), u \prec v, \Gamma \mapsto \Delta, q \prec t(r) \end{array} \right. & \\ \hline
r = s(v), u \prec v, \Gamma \mapsto \Delta, q \prec t(r) & (\prec_{La}) \\ \hline
r = s(v), u \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) & (=_{Rs})
\end{array}
\right.
$$

$$\Downarrow$$

$$
D^* \left\{
\begin{array}{cl}
D' \left\{
\begin{array}{cl}
D_1 \left\{ \begin{array}{c} \vdots \\ r = s(u), u \prec v, \Gamma \mapsto \Delta, q \prec t(r) \end{array} \right. & \\ \hline
r = s(u), r = s(v), u \prec v, \Gamma \mapsto \Delta, q \prec t(r) & (W_a) \\ \hline
r = s(u), r = s(v), u \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) & (=_{Rs})
\end{array}
\right. & \\ \hline
r = s(v), r = s(v), u \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) & (\prec_{La})
\end{array}
\right.
$$

$h(D') < h(D)$, so D^* can be transformed into a desired derivation without $(=_{Rs})$. (Notice that the result of D^* is the same as the result of D by the implicit application of contraction.)

3) $R = (\prec_{Ls})$:

$$
D \left\{
\begin{array}{cl}
D_1 \left\{ \begin{array}{c} \vdots \\ r = s, q \prec u, \Gamma \mapsto \Delta, u \prec t(r) \end{array} \right. & \\ \hline
r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(r) & (\prec_{Ls}) \\ \hline
r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(s) & (=_{Rs})
\end{array}
\right.
$$

$$\Downarrow$$

$$
D^* \left\{
\begin{array}{cl}
D' \left\{
\begin{array}{cl}
D_1 \left\{ \begin{array}{c} \vdots \\ r = s, q \prec u, \Gamma \mapsto \Delta, u \prec t(r) \end{array} \right. & \\ \hline
r = s, r = s(v), q \prec u, \Gamma \mapsto \Delta, u \prec t(s) & (=_{Rs})
\end{array}
\right. & \\ \hline
r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(s) & (\prec_{Ls})
\end{array}
\right.
$$

Since $h(D') < h(D)$, the induction hypothesis yields a desired derivation.

4) $R = (\prec_{Rs})$

$$
\begin{array}{cl}
f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(f(r)) & \\ \hline
f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(f(p)) & (\prec_{Rs}) \\ \hline
f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(s) & (=_{Rs})
\end{array}
\quad \Rightarrow \quad
\begin{array}{cl}
f(r) = s, f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(f(r)) & \\ \hline
f(r) = s, f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(s) & (=_{Rs}) \\ \hline
f(p) = s, f(p) = s, r \prec p, \Gamma \mapsto \Delta, t \prec q(s) & (\prec_{La})
\end{array}
$$

$$
\begin{array}{cl}
p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(p) & \\ \hline
p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(s)) & (\prec_{Rs}) \\ \hline
p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) & (=_{Rs})
\end{array}
\quad \Rightarrow \quad
\begin{array}{cl}
p \prec f(r), p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(p) & \\ \hline
p \prec f(r), p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) & (\prec_{Rs}) \\ \hline
p \prec f(s), p \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) & (=_{Ra})
\end{array}
$$

5) The other cases when R is (E_{ar}), $(=_{Ls})$, $(=_{Ra})$, $(=_{La})$, $(Sp.cut)$ or (cut_t) can be trivially swapped with the application of $(=_{Rs})$ since they do not affect the active equation nor the modified inclusion of $(=_{Rs})$.

QED.

4.1.1 Some admissible and redundant a-rules

The underlying theme of this subsection are the reduced vs. non-reduced applications of various a-rules. These facts, to be utilized in the proof of the (cut)-elimination, appear in the side-conditions of the lemmas.

Lemma 4.2 *The following rule is admissible in NEQ_4:*

$$\frac{s = t, f(s) = p, \Gamma \mapsto \Delta}{s = t, f(t) = p, \Gamma \mapsto \Delta}\ (=_{a=})$$

Morover, if D^* derives S using this rule while D derives S without it then $\#((\preceq_{Lar}), D^*) \leq \#((\preceq_{Lar}), D)$, and $\#((\preceq_{Lanr}), D^*) \leq \#((\preceq_{Lanr}), D)$.

Proof. By lemma 4.1 we assume no applications of $(=_{Rs})$ in D. By induction on $\#((=_{a=}), D)$ and $h(D)$. For $h(D) = 1$ we have one possible axiom:

$$\frac{s = t, f(s) = p, \Gamma \mapsto \Delta, f(s) = p}{s = t, f(t) = p, \Gamma \mapsto \Delta, f(s) = p}\ (=_{a=}) \quad \Rightarrow \quad \frac{s = t, f(t) = p, \Gamma \mapsto \Delta, f(t) = p}{s = t, f(t) = p, \Gamma \mapsto \Delta, f(s) = p}\ (=_{Ls})$$

Consider the uppermost application of $(=_{a=})$ and the rule R applied just above it. The elimination rules, $(\prec_{Ls})$, $(\prec_{Rs})$, $(Sp.cut)$ and (cut_x) do not affect the active or the principal formula of $(=_{a=})$ so these can be swapped reducing the height at which the application of $(=_{a=})$ occurs. We have then the following cases for R:

1) $(=_{Lar})$:

$$D \left\{ \begin{array}{l} D_1 \left\{ \begin{array}{c} \vdots \\ r = x, x = t, s(x) \prec p, \Gamma \mapsto \Delta \end{array} \right. \\ \hline r = x, x = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (=_{Lar}) \\ \hline r = x, r = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (=_{a=}) \end{array} \right.$$

$$\Downarrow$$

$$D^* \left\{ \begin{array}{l} D_1^* \left\{ \begin{array}{c} \vdots \\ x \prec t, r = x, x = t, s(x) \prec p, \Gamma \mapsto \Delta \end{array} \right. \\ \hline x \prec t, r = x, r = t, s(x) \prec p, \Gamma \mapsto \Delta \quad (=_{a=}) \\ \hline x \prec t, r = x, r = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (\prec_{Lar}) \\ \hline x \prec r, r = x, r = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (=_{Ra}) \\ \hline x \prec x, r = x, r = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (=_{Ra}) \\ \hline r = x, r = t, s(t) \prec p, \Gamma \mapsto \Delta \quad (cut_x) \end{array} \right.$$

$h(D_1^*) < h(D_1) + 1$ yields the conclusion. Exactly the same transformation is made when $(=_{a=})$ in D modifies t (or its subterm). Since the transformation replaces an application of $(=_{Lar})$ by an application of $(\prec_{Lar})$, we have the unchanged total number of applications of $(\preceq_{Lar})$ and $(\preceq_{Lanr})$.

2) $(=_{Ls})$: The only case when simple swapping of the last two rules is insufficient occurs when the active formula of $(=_{Ls})$ is modified in the last application of $(=_{a=})$. The end of the derivation has then the form:

$$D \left\{ \begin{array}{c} s = t, w(t) = q, \Gamma \mapsto \Delta, p(q) = r \\ \hline s = t, w(t) = q, \Gamma \mapsto \Delta, p(w(t)) = r \quad (=_{Ls}) \\ \hline s = t, w(s) = q, \Gamma \mapsto \Delta, p(w(t)) = r \quad (=_{a=}) \end{array} \right.$$

$$\Downarrow$$

$$D^* \left\{ \begin{array}{l} D_1 \left\{ \begin{array}{c} s = t, w(t) = q, \Gamma \mapsto \Delta, p(q) = r \\ \hline s = t, w(s) = q, \Gamma \mapsto \Delta, p(q) = r \quad (=_{a=}) \end{array} \right. \\ \hline s = t, w(s) = q, \Gamma \mapsto \Delta, p(w(s)) = r \quad (=_{Ls}) \\ \hline s = t, w(s) = q, \Gamma \mapsto \Delta, p(w(t)) = r \quad (=_{Ls}) \end{array} \right.$$

$h(D_1) < h(D)$ so, by the induction hypothesis, there is a NEQ_4 derivation corresponding to D^* deriving the same sequent without $(=_{a=})$. The case when in D,$(=_{Ls})$ replaces $w(t)$ by q is treated analogously.

3) $(\prec_{La})$ – two cases

$$D\left\{\begin{array}{c} D_1\left\{\begin{array}{c}\vdots\\ s=t,w(p)=q,p\prec r(t),\Gamma\mapsto\Delta\end{array}\right. \\ \hline s=t,w(r(t))=q,p\prec r(t),\Gamma\mapsto\Delta \\ \hline s=t,w(r(s))=q,p\prec r(t),\Gamma\mapsto\Delta\end{array}\right. \begin{array}{l}(\prec_{La})\\(=_{a=})\end{array}$$

$$\Downarrow$$

$$D^*\left\{\begin{array}{c} D_1^*\left\{\begin{array}{c}\vdots\\ s=t,w(p)=q,p\prec r(t),p\prec r(s),\Gamma\mapsto\Delta\end{array}\right. \\ \hline s=t,w(r(s))=q,p\prec r(t),p\prec r(s),\Gamma\mapsto\Delta \\ \hline s=t,w(r(s))=q,p\prec r(t),p\prec r(t),\Gamma\mapsto\Delta\end{array}\right. \begin{array}{l}(\prec_{La})\\(=_{Ra})\end{array}$$

where the last sequent is the same as in D by implicit contraction. The last sequent of D_1^* can be obtained as in D_1 by weakening (lemma 3.10) with $h(D_1^*)=h(D_1)$.

$$\begin{array}{c} s=t(r),w(t(p))=q,p\prec r,\Gamma\mapsto\Delta \\ \hline s=t(r),w(t(r))=q,p\prec r,\Gamma\mapsto\Delta \\ \hline s=t(r),w(s)=q,p\prec r,\Gamma\mapsto\Delta \end{array}\begin{array}{l}(\prec_{La})\\(=_{a=})\end{array} \quad\Rightarrow\quad \begin{array}{c} s=t(p),s=t(r),w(t(p))=q,p\prec r,\Gamma\mapsto\Delta \\ \hline s=t(p),s=t(r),w(s)=q,p\prec r,\Gamma\mapsto\Delta \\ \hline s=t(r),s=t(r),w(s)=q,p\prec r,\Gamma\mapsto\Delta \end{array}\begin{array}{l}(=_{a=})\\(\prec_{La})\end{array}$$

4) $(=_{Ra})$:

$$\begin{array}{c} \vdots \\ r=u,w(u)=t,p\prec f(w(u)),\Gamma\mapsto\Delta \\ \hline r=u,w(u)=t,p\prec f(t),\Gamma\mapsto\Delta \\ \hline r=u,w(r)=t,p\prec f(t),\Gamma\mapsto\Delta \end{array}\begin{array}{l}(=_{Ra})\\(=_{a=})\end{array} \quad\Rightarrow\quad \begin{array}{c} \vdots \\ r=u,w(u)=t,p\prec f(w(u)),\Gamma\mapsto\Delta \\ \hline r=u,w(r)=t,p\prec f(w(u)),\Gamma\mapsto\Delta \\ \hline r=u,w(r)=t,p\prec f(w(r)),\Gamma\mapsto\Delta \\ \hline r=u,w(r)=t,p\prec f(t),\Gamma\mapsto\Delta \end{array}\begin{array}{l}(=_{a=})\\(=_{Ra})\\(=_{Ra})\end{array}$$

Analogous transformation is applied when $(=_{a=})$ modifies t rather than $w(u)$.

5) All other cases involve only $\prec$ in the antecedent so they admit trivial swap.

In all cases no new applications of $(\preceq_{La})$ appear in the transformed derivations D^*, and the ones which occur there are the same as in the original derivations D. QED.

Lemma 4.3 *$NEQ_4 \vdash^D S \;\Rightarrow\; NEQ_4 \vdash^{D^*} S$ and D^* contains no applications of $(\prec_{Lanr})$ nor $(=_{Ranr})$. Moreover, $\#((\preceq_{Lar}),D^*) \leq \#((\preceq_{Lar}),D)$ and $\#((\preceq_{Lanr}),D^*) \leq \#((\preceq_{Lanr}),D)$.*

Proof. By induction on $\langle(\#(\prec_{Lanr})+\#(=_{Ranr}),D),h(D)\rangle$. By lemma 4.1, we assume that the derivation D does not contain any $(=_{Rs})$. We first consider the case where the uppermost application of any of the two rules is $(\prec_{Lanr})$.

The basis case of $h(D)=1$ is as follows:

$$\frac{s\prec t,f(s)=p,\Gamma\mapsto\Delta,f(s)=p}{s\prec t,f(t)=p,\Gamma\mapsto\Delta,f(s)=p}\,(\prec_{Lanr}) \quad\Rightarrow\quad \frac{s\prec t,f(t)=p,\Gamma\mapsto\Delta,f(t)=p}{s\prec t,f(t)=p,\Gamma\mapsto\Delta,f(s)=p}\,(\prec_{Ls})$$

If we were to use $(=_{Ranr})$ the axiom must be of the form $x\prec t,\Gamma\mapsto\Delta,x\prec t$, so the application would be reduced.

Consider the uppermost application of $(\prec_{Lanr})$ and the rule R applied immediately above it:

$$q\notin\mathcal{V}\quad \begin{array}{c} \vdots \\ \hline q\prec t,s(q)\preceq p,\Gamma\mapsto\Delta \\ \hline q\prec t,s(t)\preceq p,\Gamma\mapsto\Delta \end{array}\begin{array}{l}R\\(\prec_{Lanr})\end{array}$$

1) R is $(\prec_{Ls})$ with the active formula $s(q)\prec p$.

$$D\left\{\begin{array}{c} q\prec t,s(q)\prec p,\Gamma\mapsto\Delta,s_1(p)\preceq q_1 \\ \hline q\prec t,s(q)\prec p,\Gamma\mapsto\Delta,s_1(s(q))\preceq q_1 \\ \hline q\prec t,s(t)\prec p,\Gamma\mapsto\Delta,s_1(s(q))\preceq q_1 \end{array}\right.\begin{array}{l}(\prec_{Ls})\\(\prec_{Lanr})\end{array}$$

If the active formula of $(\prec_{Ls})$ does not contain q in the way indicated above, the two applications can be trivially swapped and the induction hypothesis allows to eliminate $(\prec_{Lanr})$.

From the above D, we may deduce the existence of the following derivation:

$$\dfrac{\dfrac{D_1 \left\{ \dfrac{q \prec t, s(q) \prec p, \Gamma \mapsto \Delta, s_1(p) \preceq q_1}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, s_1(p) \preceq q_1} (\prec_{Lanr}) \right.}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, s_1(s(t)) \preceq q_1} (\prec_{Ls})}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, s_1(s(q)) \preceq q_1} (\prec_{Ls})$$

Since $h(D_1) < h(D)$, the induction hypothesis allows us to eliminate $(\prec_{Lanr})$.

2) R is (E_{ar}). Since $q \notin \mathcal{V}$ also $s(q) \notin \mathcal{V}$. Therefore none of $q \prec t$ and $s(q) \preceq p$ can be the modified formula of an application of (E_{ar}). Hence the two applications can be swapped and induction hypothesis applied.

3) R is $(=_{Lar})$:

$$\dfrac{\dfrac{q(x) \prec t, s(q(r)) \preceq p, x = r, \Gamma \mapsto \Delta}{q(r) \prec t, s(q(r)) \preceq p, x = r, \Gamma \mapsto \Delta} (=_{Lar})}{q(r) \prec t, s(t) \preceq p, x = r, \Gamma \mapsto \Delta} (\prec_{Lanr})$$

$$\Downarrow$$

$$\dfrac{\dfrac{q(r) \prec t, q(x) \prec t, s(q(r)) \preceq p, x = r, \Gamma \mapsto \Delta}{q(r) \prec t, q(x) \prec t, s(t) \preceq p, x = r, \Gamma \mapsto \Delta} (\prec_{Lanr})}{q(r) \prec t, q(r) \prec t, s(t) \preceq p, x = r, \Gamma \mapsto \Delta} (=_{Lar})$$

Analogous weakening is used when the modified formula of $(=_{La})$ is modified also in the subsequent application of $(\prec_{Lanr})$.

4) R is $(=_{Rar})$ If the active formula of $(\prec_{Lanr})$ is not modified by $(=_{Rar})$ we obtain the required reduction in height by swapping analogous to the previous case. So, assume it was modified. The derivation ends then as follows:

$$\dfrac{\dfrac{t(y) \prec q, y \prec p(k), r = k, \Gamma \mapsto \Delta}{t(y) \prec q, y \prec p(r), r = k, \Gamma \mapsto \Delta} (=_{Rar})}{t(p(r)) \prec q, y \prec p(r), r = k, \Gamma \mapsto \Delta} (\prec_{Lar})$$

Since we have $(=_{Rar})$, the LHS in $y \prec p(k)$ must be a variable, which means that the final application is $(\prec_{Lar})$.

5) R is $(\prec_{Lar})$: This case is treated exactly as case 3) with the applications of $(\prec_{Lar})$ instead:

$$\dfrac{\dfrac{q(x) \prec t, s(q(r)) \preceq p, x \prec r, \Gamma \mapsto \Delta}{q(r) \prec t, s(q(r)) \preceq p, x \prec r, \Gamma \mapsto \Delta} (\prec_{Lar})}{q(r) \prec t, s(t) \preceq p, x \prec r, \Gamma \mapsto \Delta} (\prec_{Lanr}) \quad \Rightarrow \quad \dfrac{\dfrac{q(r) \prec t, q(x) \prec t, s(q(r)) \preceq p, x \prec r, \Gamma \mapsto \Delta}{q(r) \prec t, q(x) \prec t, s(t) \preceq p, x \prec r, \Gamma \mapsto \Delta} (\prec_{Lanr})}{q(r) \prec t, q(r) \prec t, s(t) \preceq p, x \prec r, \Gamma \mapsto \Delta} (\prec_{Lar})$$

Analogous procedure yields the result if the modified formula of the application of $(\prec_{Lar})$ is not active but modified in the subsequent $(\prec_{Lanr})$.

6) R is $(\prec_{Rs})$

$$\dfrac{\dfrac{q \prec t, s(q) \prec p, \Gamma \mapsto \Delta, r \prec f(s(q))}{q \prec t, s(q) \prec p, \Gamma \mapsto \Delta, r \prec f(p)} (\prec_{Rs})}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, r \prec f(p)} (\prec_{Lanr}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{q \prec t, s(q) \prec p, \Gamma \mapsto \Delta, r \prec f(s(q))}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, r \prec f(s(q))} (\prec_{Lanr})}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, r \prec f(s(t))} (\prec_{Rs})}{q \prec t, s(t) \prec p, \Gamma \mapsto \Delta, r \prec f(p)} (\prec_{Rs})$$

7) (E_{sr}), $(Sp.cut)$ and (cut_t) can be swapped with $(\prec_{Lanr})$.

Now the case when the uppermost application is of $(=_{Ranr})$

1) R is $(=_{Lar})$:

$$\dfrac{\dfrac{x = u, s = t, q(x) \prec p(t), \Gamma \mapsto \Delta}{x = u, s = t, q(u) \prec p(t), \Gamma \mapsto \Delta} (=_{Lar})}{x = u, s = t, q(u) \prec p(s), \Gamma \mapsto \Delta} (=_{Ranr}) \quad \Rightarrow \quad \dfrac{\dfrac{x = u, s = t, q(x) \prec p(t), \Gamma \mapsto \Delta}{x = u, s = t, q(x) \prec p(s), \Gamma \mapsto \Delta} (=_{Ranr})}{x = u, s = t, q(u) \prec p(s), \Gamma \mapsto \Delta} (=_{Lar})$$

2) R is $(\prec_{Lar})$: The side formula of $(=_{Ranr})$ cannot be active here, since we are using reduced version $(\prec_{Lar})$:

$$\dfrac{\dfrac{x \prec u, s=t, q(x) \prec p(t), \Gamma \mapsto \Delta}{x \prec u, s=t, q(u) \prec p(t), \Gamma \mapsto \Delta}\ (\prec_{Lar})}{x \prec u, s=t, q(u) \prec p(s), \Gamma \mapsto \Delta}\ (=_{Ranr}) \quad \Rightarrow \quad \dfrac{\dfrac{x \prec u, s=t, q(x) \prec p(t), \Gamma \mapsto \Delta}{x \prec u, s=t, q(x) \prec p(s), \Gamma \mapsto \Delta}\ (=_{Ranr})}{x \prec u, s=t, q(u) \prec p(s), \Gamma \mapsto \Delta}\ (\prec_{Lar})$$

3) R is $(\prec_{Ls})$:

$$\dfrac{\dfrac{s=t, q \prec p(t), \Gamma \mapsto \Delta, w(p(t)) \preceq r}{s=t, q \prec p(t), \Gamma \mapsto \Delta, w(q) \preceq r}\ (\prec_{Ls})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, w(q) \preceq r}\ (=_{Ranr}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{s=t, q \prec p(t), \Gamma \mapsto \Delta, w(p(t)) \preceq r}{s=t, q \prec p(s), \Gamma \mapsto \Delta, w(p(t)) \preceq r}\ (=_{Ranr})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, w(p(s)) \preceq r}\ (=_{Ls})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, w(q) \preceq r}\ (\prec_{Ls})$$

4) R is $(\prec_{Rs})$:

$$\dfrac{\dfrac{s=t, q \prec p(t), \Gamma \mapsto \Delta, r \prec f(q)}{s=t, q \prec p(t), \Gamma \mapsto \Delta, r \prec f(p(t))}\ (\prec_{Rs})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, r \prec f(p(t))}\ (=_{Ranr}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{s=t, q \prec p(t), \Gamma \mapsto \Delta, r \prec f(q)}{s=t, q \prec p(s), \Gamma \mapsto \Delta, r \prec f(q)}\ (=_{Ranr})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, r \prec f(p(s))}\ (\prec_{Rs})}{s=t, q \prec p(s), \Gamma \mapsto \Delta, r \prec f(p(t))}\ (=_{Rs})$$

5) Other cases are trivial.
If R is $(=_{Rar})$ or (E_{ar}) then they do not affect the active or side formula of $(=_{Ranr})$ since, otherwise, the LHS of the latter would have to be a variable.
$(=_{Ls})$, (E_{sr}), $(Sp.cut)$ and (cut_x) can be trivially swapped.

As we can see, the applications of $(\preceq_{La})$ occurring in the transformed derivations are the same as the original ones. Hence $\#((\preceq_{La}), D^*) \leq \#((\preceq_{La}), D)$ and $\#((\preceq_{Lar}), D^*) \leq \#((\preceq_{Lar}), D^*)$. QED.

Lemma 4.4 *The following rule is admissible in NEQ_4:*

$$(=_{Lanr})\ \dfrac{s=t, p(s) \prec q, \Gamma \mapsto \Delta}{s=t, p(t) \prec q, \Gamma \mapsto \Delta}\ s \notin \mathcal{V}$$

Moreover, if $NEQ_4 \vdash^D S$ and $NEQ_4 + (=_{Lanr}) \vdash^{D'} S$, we can construct D' so that $\#((\preceq_{Lar}), D') \leq \#((\preceq_{Lar}), D)$, and $\#((\preceq_{Lanr}), D') \leq \#((\preceq_{Lanr}), D)$.

Proof. By lemma 4.3, we assume that D does not contain any applications of $(\prec_{Lanr})$. By induction on $\langle \#(=_{Lanr}, D), h(D) \rangle$. If $h(D) = 1$, the only relevant possibility is the axiom with an inclusion in the antecednet, but it has a variable in the LHS, so an application of $(=_{La})$ has to be reduced.

Consider the last rule above the uppermost application of $(=_{Lanr})$.

1) $(\prec_{La})$, i.e, $(\prec_{Lar})$.
The first case is when both $(\prec_{Lar})$ and $(=_{Lanr})$ modify the same formula:

$$\dfrac{\dfrac{f(x) \prec p, r=u, x \prec s(u), \Gamma \mapsto \Delta}{f(s(u)) \prec p, r=u, x \prec s(u), \Gamma \mapsto \Delta}\ (\prec_{Lar})}{f(s(r)) \prec p, r=u, x \prec s(u), \Gamma \mapsto \Delta}\ (=_{Lanr})$$

$$\Downarrow$$

$$\dfrac{\dfrac{f(x) \prec p, r=u, x \prec s(u), x \prec s(r), \Gamma \mapsto \Delta}{f(s(r)) \prec p, r=u, x \prec s(u), x \prec s(r), \Gamma \mapsto \Delta}\ (\prec_{Lar})}{f(s(r)) \prec p, r=u, x \prec s(u), x \prec s(u), \Gamma \mapsto \Delta}\ (=_{Rar})$$

Another case is the following:

$$\dfrac{\dfrac{f(g(x)) \prec p, x \prec t, q=g(t), \Gamma \mapsto \Delta}{f(g(t)) \prec p, x \prec t, q=g(t), \Gamma \mapsto \Delta}\ (\prec_{Lar})}{f(q) \prec p, x \prec t, q=g(t), \Gamma \mapsto \Delta}\ (=_{Lanr})$$

$$\Downarrow$$

$$\cfrac{\cfrac{f(g(x)) \prec p, x \prec t, q = g(t), q = g(x), \Gamma \mapsto \Delta}{f(q) \prec p, x \prec t, q = g(t), q = g(x), \Gamma \mapsto \Delta}\ (=_{Lanr})}{f(q) \prec p, x \prec t, q = g(t), q = g(t), \Gamma \mapsto \Delta}\ (\prec_{Lar})$$

The last case is when the formula ϕ modified by $(=_{Lanr})$ in D was active in the preceding application of $(\prec_{Lar})$. But in this case, since we only have reduced applications $(\prec_{Lar})$, ϕ's LHS must be a variable, so the application of $(=_{La})$ would not be $(=_{Lanr})$ but $(=_{Lar})$.

2) $(\prec_{Ls})$: The two applications $(\prec_{Ls})$-$(=_{Lanr})$ are easily replaced by $(=_{Lanr})$-$(=_{Ls})$.

3) $(\prec_{Rs})$:

$$\cfrac{\cfrac{f(s) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(f(s))}{f(s) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(t)}\ (\prec_{Rs})}{f(r) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(t)}\ (=_{Lanr}) \quad \Rightarrow \quad \cfrac{\cfrac{\cfrac{f(s) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(f(s))}{f(r) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(f(s))}\ (=_{Lanr})}{f(r) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(f(r))}\ (=_{Rs})}{f(r) \prec t, s = r, \Gamma \mapsto \Delta, p \prec q(t)}\ (\prec_{Rs})$$

4) $(=_{Lar})$:

$$D \left\{ \cfrac{\cfrac{f(x) \prec p, r = u, x = s(u), \Gamma \mapsto \Delta}{f(s(u)) \prec p, r = u, x = s(u), \Gamma \mapsto \Delta}\ (=_{Lar})}{f(s(r)) \prec p, r = u, x = s(u), \Gamma \mapsto \Delta}\ (=_{Lanr}) \right.$$

$$\Downarrow$$

$$D' \left\{ \cfrac{\cfrac{f(x) \prec p, r = u, x = s(u), x = s(r), \Gamma \mapsto \Delta}{f(s(r)) \prec p, r = u, x = s(u), x = s(r), \Gamma \mapsto \Delta}\ (=_{Lanr})}{f(s(r)) \prec p, r = u, x = s(u), x = s(u), \Gamma \mapsto \Delta}\ (=_{a=}) \right.$$

By induction hypothesis $(=_{Lanr})$ can be eliminated from D', and by lemma 4.2 D' can be transformed so that it does not contain $(=_{a=})$ and this transformation does not introduce any additional applications of $(\preceq_{La})$. The same applies in the other case:

$$\cfrac{\cfrac{f(s(x)) \prec p, x = u, s(u) = r, \Gamma \mapsto \Delta}{f(s(u)) \prec p, x = u, s(u) = r, \Gamma \mapsto \Delta}\ (=_{Lar})}{f(r) \prec p, x = u, s(u) = r, \Gamma \mapsto \Delta}\ (=_{Lanr})$$

$$\Downarrow$$

$$\cfrac{\cfrac{f(s(x)) \prec p, x = u, s(u) = r, s(x) = r, \Gamma \mapsto \Delta}{f(r) \prec p, x = u, s(u) = r, s(x) = r, \Gamma \mapsto \Delta}\ (=_{Lanr})}{f(r) \prec p, x = u, s(u) = r, s(u) = r, \Gamma \mapsto \Delta}\ (=_{a=})$$

5) Elimination rules are not relevant, neither is $(Sp.cut)$ and (cut_x), and $(=_{Ra})$ and $(=_{Rs})$ are trivially swapped since their active formula (equality) is not affected by $(=_{Lanr})$.

We see that no new applications of $(\preceq_{Lar})$ or $(\preceq_{Lanr})$ emerge. QED.

Lemma 4.5 *The following rule is admissible in NEQ_4:*

$$\frac{s \prec t, q \prec t, \Gamma \mapsto \Delta}{s \prec t, q \prec s, \Gamma \mapsto \Delta}\ (\prec_{Ra})$$

Moreover, if D^* is a derivation using the rule $(\prec_{Ra})$ and D is the corresponding derivation not using this rule, then $\#((\preceq_{Lar}), D) \leq \#((\preceq_{Lar}), D^*)$, and $\#((\preceq_{Lanr}), D) \leq \#((\preceq_{Lanr}), D^*)$.

Proof. By induction on $\langle \#((\prec_{Ra}), D^*), h(D^*) \rangle$. By lemma 4.3 we assume that the derivation does not contain any $(\prec_{Lanr})$ or $(=_{Ranr})$. The possible application to an axiom is:

$$\frac{x \prec t, s \prec t, \Gamma \mapsto \Delta, x \prec t}{x \prec s, s \prec t, \Gamma \mapsto \Delta, x \prec t}\ (\prec_{Ra}) \quad \Rightarrow \quad \frac{x \prec s, s \prec t, \Gamma \mapsto \Delta, x \prec s}{x \prec s, s \prec t, \Gamma \mapsto \Delta, x \prec t}\ (\prec_{Rs})$$

Consider the rule applied just above $(\prec_{Ra})$.

1) $(\prec_{Rs})$

$$\dfrac{\dfrac{q \prec t, s \prec t, \Gamma \mapsto \Delta, p \prec f(q)}{q \prec t, s \prec t, \Gamma \mapsto \Delta, p \prec f(t)}\,(\prec_{Rs})}{q \prec s, s \prec t, \Gamma \mapsto \Delta, p \prec f(t)}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{q \prec t, s \prec t, \Gamma \mapsto \Delta, p \prec f(q)}{q \prec s, s \prec t, \Gamma \mapsto \Delta, p \prec f(q)}\,(\prec_{Ra})}{q \prec s, s \prec t, \Gamma \mapsto \Delta, p \prec f(s)}\,(\prec_{Rs})}{q \prec s, s \prec t, \Gamma \mapsto \Delta, p \prec f(t)}\,(\prec_{Rs})$$

2) $(\prec_{Lar})$:

$$\dfrac{\dfrac{s(y) \prec t, q \prec t, y \prec p, \Gamma \mapsto \Delta}{s(p) \prec t, q \prec t, y \prec p, \Gamma \mapsto \Delta}\,(\prec_{Lar})}{s(p) \prec t, q \prec s(p), y \prec p, \Gamma \mapsto \Delta}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{s(p) \prec t, s(y) \prec t, q \prec t, y \prec p, \Gamma \mapsto \Delta}{s(p) \prec t, s(y) \prec t, q \prec s(p), y \prec p, \Gamma \mapsto \Delta}\,(\prec_{Ra})}{s(p) \prec t, s(p) \prec t, q \prec s(p), y \prec p, \Gamma \mapsto \Delta}\,(\prec_{Lar})$$

3) $(\prec_{Ls})$

$$\dfrac{\dfrac{s \prec t, q \prec t, \Gamma \mapsto \Delta, w(t) \preceq p}{s \prec t, q \prec t, \Gamma \mapsto \Delta, w(q) \preceq p}\,(\prec_{Ls})}{s \prec t, q \prec s, \Gamma \mapsto \Delta, w(q) \preceq p}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{s \prec t, q \prec t, \Gamma \mapsto \Delta, w(t) \preceq p}{s \prec t, q \prec s, \Gamma \mapsto \Delta, w(t) \preceq p}\,(\prec_{Ra})}{s \prec t, q \prec s, \Gamma \mapsto \Delta, w(s) \preceq p}\,(\prec_{Ls})}{s \prec t, q \prec s, \Gamma \mapsto \Delta, w(q) \preceq p}\,(\prec_{Ls})$$

4) (E_{ar}):

$$\dfrac{\dfrac{x \prec t, y \prec r(x), s \prec r(t), \Gamma \mapsto \Delta}{y \prec r(t), s \prec r(t), \Gamma \mapsto \Delta}\,(E_{ar})}{y \prec s, s \prec r(t), \Gamma \mapsto \Delta}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{x \prec t, y \prec r(x), s \prec r(t), s \prec r(x), \Gamma \mapsto \Delta}{x \prec t, y \prec s, s \prec r(t), s \prec r(x), \Gamma \mapsto \Delta}\,(\prec_{Ra})}{y \prec s, s \prec r(t), s \prec r(t), \Gamma \mapsto \Delta}\,(E_{ar})$$

We do exactly the same if (E_{ar}) modifies the active (rather than the side) formula of the application of $(\prec_{Ra})$.

5) $(=_{Lar})$:

$$\dfrac{\dfrac{s(x) \prec t, q \prec t, x = p, \Gamma \mapsto \Delta}{s(p) \prec t, q \prec t, x = p, \Gamma \mapsto \Delta}\,(=_{Lar})}{s(p) \prec t, q \prec s(p), x = p, \Gamma \mapsto \Delta}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{s(p) \prec t, s(x) \prec t, q \prec t, x = p, \Gamma \mapsto \Delta}{s(p) \prec t, s(x) \prec t, q \prec s(p), x = p, \Gamma \mapsto \Delta}\,(\prec_{Ra})}{s(p) \prec t, s(p) \prec t, q \prec s(p), x = p, \Gamma \mapsto \Delta}\,(=_{Lar})$$

6) $(=_{Rar})$:

$$\dfrac{\dfrac{s \prec t(p), x \prec t(r), r = p, \Gamma \mapsto \Delta}{s \prec t(p), x \prec t(p), r = p, \Gamma \mapsto \Delta}\,(=_{Rar})}{s \prec t(p), x \prec s, r = p, \Gamma \mapsto \Delta}\,(\prec_{Ra}) \quad \Rightarrow \quad \dfrac{\dfrac{s \prec t(r), s \prec t(p), x \prec t(r), r = p, \Gamma \mapsto \Delta}{s \prec t(r), s \prec t(p), x \prec s, r = p, \Gamma \mapsto \Delta}\,(\prec_{Ra})}{s \prec t(p), s \prec t(p), x \prec s, r = p, \Gamma \mapsto \Delta}\,(=_{Ra})$$

Analogous transformation is applied if $(\prec_{Ra})$ in the original derivation yields $s \prec x, x \prec t(p) \ldots$.

7) The rules $(=_{Ls})$, $(=_{Rs})$, (E_{sr}), $(Sp.cut)$, (cut_x) and $(cut_{x=})$ can be trivially swapped, since they do not affect the antecedent – the active formula, which is either equality or disappears, cannot be involved in the final application of $(\prec_{Ra})$.

We see that no new applications of $(\preceq_{La})$ appear in the transformed derivations D, and the ones which occur there are the same as in the original derivations D^*. [Also, no new applications of $(cut_{x=})$ appear in the transformed derivations.] QED.

4.1.2 Some redundant s-rules

First lemma shows the possibility of eliminating most of the s-rules from the NEQ_4 derivations.

Lemma 4.6 *$NEQ_4 \vdash^D S \Rightarrow NEQ_4 \vdash^{D^*} S$ and D^* uses no applications of $(=_{Rs})$, $(\prec_{Ls})$ or $(=_{Ls})$.*

Proof. We proceed by simultaneous induction on the number of applications of all the listed rules and by subinduction on the height of the derivation. By lemma 4.1, we may assume that derivation contains no applications of $(=_{Rs})$.

Writing $(\preceq_{Ls})$, we mean either $(\prec_{Ls})$ or $(=_{Ls})$ and use this notation when the two cases are treated analogously.

1) Each form of the axioms gives rise to several cases:

1.1) $x \prec t, \Gamma \mapsto \Delta, x \prec t$

1.1.1) If $s \notin \mathcal{V}$, we choose a fresh variable y :

$$\frac{s \preceq x, x \prec t, \Gamma \mapsto \Delta, x \prec t}{s \preceq x, x \prec t, \Gamma \mapsto \Delta, s \prec t}\,(\preceq_{Ls}) \quad\Rightarrow\quad \cfrac{\cfrac{y \prec s, s \preceq x, y \prec t, \Gamma \mapsto \Delta, y \prec t}{y \prec s, s \preceq x, s \prec t, \Gamma \mapsto \Delta, y \prec t}\,(\prec_{Lar})}{s \preceq x, s \prec t, \Gamma \mapsto \Delta, s \prec t}\,(E_{sr})$$

1.1.2) If s is a variable, say y:

$$\frac{y \preceq x, x \prec t, \Gamma \mapsto \Delta, x \prec t}{y \preceq x, x \prec t, \Gamma \mapsto \Delta, y \prec t}\,(\preceq_{Ls}) \quad\Rightarrow\quad \frac{y \preceq x, y \prec t, \Gamma \mapsto \Delta, y \prec t}{y \preceq x, x \prec t, \Gamma \mapsto \Delta, y \prec t}\,(\preceq_{Lar})$$

1.2) $s = t, \Gamma \mapsto \Delta, s = t$ gives rise to four cases

1.2.1) The substituted term is a variable x

$$\frac{x \prec p, s(p) = t, \Gamma \mapsto \Delta, s(p) = t}{x \prec p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(\prec_{Ls}) \quad\Rightarrow\quad \frac{x \prec p, s(x) = t, \Gamma \mapsto \Delta, s(x) = t}{x \prec p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(\prec_{Lar})$$

1.2.2) and variable substituted by $(=_{Ls})$:

$$\frac{x = p, s(p) = t, \Gamma \mapsto \Delta, s(p) = t}{x = p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(=_{Ls}) \quad\Rightarrow\quad \cfrac{\cfrac{\cfrac{x \prec p, x = p, s(x) = t, \Gamma \mapsto \Delta, s(x) = t}{x \prec p, x = p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(\prec_{Lar})}{x \prec x, x = p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(=_{Rar})}{x = p, s(p) = t, \Gamma \mapsto \Delta, s(x) = t}\,(cut_x)$$

1.2.3) The substituted term $r \notin \mathcal{V}$ – we choose a fresh variable y :

$$\frac{r = p, s(p) = t, \Gamma \mapsto \Delta, s(p) = t}{r = p, s(p) = t, \Gamma \mapsto \Delta, s(r) = t}\,(=_{Ls}) \quad\Rightarrow\quad \cfrac{\cfrac{\cfrac{y \prec p, r = p, s(y) = t, \Gamma \mapsto \Delta, s(y) = t}{y \prec p, r = p, s(p) = t, \Gamma \mapsto \Delta, s(y) = t}\,(\prec_{Lar})}{y \prec r, r = p, s(p) = t, \Gamma \mapsto \Delta, s(y) = t}\,(=_{Rar})}{r = p, s(p) = t, \Gamma \mapsto \Delta, s(r) = t}\,(E_{sr})$$

1.2.4) and non-variable substituted by $(\prec_{Ls})$

$$\frac{r \prec p, s(p) = t, \Gamma \mapsto \Delta, s(p) = t}{r \prec p, s(p) = t, \Gamma \mapsto \Delta, s(r) = t}\,(\prec_{Ls}) \quad\Rightarrow\quad \frac{r \prec p, s(r) = t, \Gamma \mapsto \Delta, s(r) = t}{r \prec p, s(p) = t, \Gamma \mapsto \Delta, s(r) = t}\,(\prec_{La})$$

1.3) $\Gamma \mapsto \Delta, x = x$. The situation

$$\frac{s = x, \Gamma \mapsto \Delta, x = x}{s = x, \Gamma \mapsto \Delta, s = x}\,(=_{Ls})$$

gives an instance of an axiom. The case of $(\prec_{Ls})$ is treated as follows:

$$\frac{t \prec x, \Gamma \mapsto \Delta, x = x}{t \prec x, \Gamma \mapsto \Delta, t = x}\,(\prec_{Ls}) \quad\Rightarrow\quad \cfrac{\cfrac{t = x, t \prec x, \Gamma \mapsto \Delta, t = x}{x = x, t \prec x, \Gamma \mapsto \Delta, t = x}\,(\prec_{La})}{t \prec x, \Gamma \mapsto \Delta, t = x}\,(cut_{x=})$$

2) $(\prec_{La})$

$$\cfrac{\cfrac{q(t) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(r) \preceq t}{q(s) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(r) \preceq t}\,(\prec_{La})}{q(s) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(q(s)) \preceq t}\,(\preceq_{Ls}) \quad\Rightarrow\quad \cfrac{\cfrac{q(s) \preceq r, q(t) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(r) \preceq t}{q(s) \preceq r, q(t) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(q(s)) \preceq t}\,(\preceq_{Ls})}{q(s) \preceq r, q(s) \preceq r, t \prec s, \Gamma \mapsto \Delta, p(q(s)) \preceq t}\,(\prec_{La})$$

3) (E_{ar})

$$\dfrac{\dfrac{y \prec t, x \prec q(y), \Gamma \mapsto \Delta, f(q(t)) \preceq p}{x \prec q(t), \Gamma \mapsto \Delta, f(q(t)) \preceq p}\,(E_{ar})}{x \prec q(t), \Gamma \mapsto \Delta, f(x) \preceq p}\,(\prec_{Ls}) \quad \Rightarrow \quad \dfrac{\dfrac{x \prec q(t), y \prec t, x \prec q(y), \Gamma \mapsto \Delta, f(q(t)) \preceq p}{x \prec q(t), y \prec t, x \prec q(y), \Gamma \mapsto \Delta, f(x) \preceq p}\,(\prec_{Ls})}{x \prec q(t), x \prec q(t), \Gamma \mapsto \Delta, f(x) \preceq p}\,(E_{ar})$$

4) $(=_{Lar})$

$$\dfrac{\dfrac{s(y) \prec q, t = y, \Gamma \mapsto \Delta, f(q) \preceq t}{s(t) \prec q, t = y, \Gamma \mapsto \Delta, f(q) \preceq t}\,(=_{Lar})}{s(t) \prec q, t = y, \Gamma \mapsto \Delta, f(s(t)) \preceq t}\,(\prec_{Ls}) \quad \Rightarrow \quad \dfrac{\dfrac{s(t) \prec q, s(y) \prec q, t = y, \Gamma \mapsto \Delta, f(q) \preceq t}{s(t) \prec q, s(y) \prec q, t = y, \Gamma \mapsto \Delta, f(s(t)) \preceq t}\,(\prec_{Ls})}{s(t) \prec q, s(t) \prec q, t = y, \Gamma \mapsto \Delta, f(s(t)) \preceq t}\,(=_{Lar})$$

5) $(=_{Ra})$

$$\dfrac{\dfrac{x \prec p(t), t = s, \Gamma \mapsto \Delta, f(p(s)) \preceq q}{x \prec p(s), t = s, \Gamma \mapsto \Delta, f(p(s)) \preceq q}\,(=_{Ra})}{x \prec p(s), t = s, \Gamma \mapsto \Delta, f(x) \preceq q}\,(\prec_{Ls}) \quad \Rightarrow \quad \dfrac{\dfrac{x \prec p(s), x \prec p(t), t = s, \Gamma \mapsto \Delta, f(p(s)) \preceq q}{x \prec p(s), x \prec p(t), t = s, \Gamma \mapsto \Delta, f(x) \preceq q}\,(\prec_{Ls})}{x \prec p(s), x \prec p(s), t = s, \Gamma \mapsto \Delta, f(x) \preceq q}\,(=_{Ra})$$

6) $(\prec_{Rs})$, $(=_{Rs})$, (E_{sr}), (cut_x), $(cut_{x=})$ and $(Sp.cut)$ can be trivially swapped.

As we see, no new applications of s-rules appear in the resulting derivations. QED.

In lemma 4.1 we have shown that rule $(=_{Rs})$ can be eliminated from the derivations. However, new applications of a-rules appeared in the transformed derivations. Moreover, in lemma 4.3, eliminating non-reduced applications of $(=_{Ranr})$ we used an application of $(=_{Rs})$ in the transformed derivation (case 4). The final lemma shows now that, as a matter of fact, we can eliminate $(=_{Rs})$, as well as a non-reduced version of $(\prec_{Rsnr})$, from derivations without increasing the number of $(\preceq_{Lanr})$ and $(=_{Ranr})$.

Lemma 4.7 *$NEQ_4 \vdash^D S \;\Rightarrow\; NEQ_4 \vdash^{D^*} S$ and D^* contains no applications of $(=_{Rs})$ nor*

$$(\prec_{Rsnr}) \quad \dfrac{s \prec t, \Gamma \mapsto \Delta, p \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, p \prec q(t)} \quad s \notin \mathcal{V}$$

Moreover $\#((\preceq_{Lanr}), D^*) \leq \#((\preceq_{Lanr}), D)$ and $\#((=_{Ranr}), D^*) \leq \#((=_{Ranr}), D)$.

Proof. By lemma 4.3 we assume that no non-reduced applications $(\prec_{Lanr})$ or $(=_{Ranr})$ appear in D. Proceeding by simultaneous induction on the number of applications of $(=_{Rs})$ and $(\prec_{Rsnr})$ and subinduction on the height of D, we consider first the case when the uppermost application of one of these two rules is $(=_{Rs})$. (This part of the proof is essentially revisting the proof of lemma 4.1 and checking that the current side-conditions are satisfied.)

For the axioms, we only have to consider the one with inclusion in the consequent.

$$\dfrac{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(s)}{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(p)}\,(=_{Rs}) \quad \Rightarrow \quad \dfrac{p = s, x \prec t(p), \Gamma \mapsto \Delta, x \prec t(p)}{p = s, x \prec t(s), \Gamma \mapsto \Delta, x \prec t(p)}\,(=_{Rar})$$

The application is reduced $(=_{Rar})$. Consider the uppermost application of $(=_{Rs})$ in D:

1) $R = (E_{sr})$ with $q \prec t(r)$ as the modified formula and r as modified term. We have two subcases corresponding to the situation when the term r' substituted by (E_{sr}) for its eigenvariable is a subterm or superterm of (or equal to) r. (When the two are independent, the case is trivial.)

   1.1) r' is a subterm of r,
   i.e., $t'(r')$ and $t(r)$ are identical with $r = f(r')$ and $t'(x) = t(f(x))$.

$$D \left\{ \dfrac{\dfrac{x \prec r', f(r') = s, \Gamma \mapsto \Delta, q \prec t(f(x))}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(f(r'))}\,(E_{sr})}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\,(=_{Rs}) \right.$$

$$\Downarrow$$

$$D' \left\{ \dfrac{\dfrac{\dfrac{x \prec r', f(r') = s, f(x) = s, \Gamma \mapsto \Delta, q \prec t(f(x))}{x \prec r', f(r') = s, f(x) = s, \Gamma \mapsto \Delta, q \prec t(s)}\,(=_{Rs})}{x \prec r', f(r') = s, f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\,(\prec_{Lar})}{f(r') = s, \Gamma \mapsto \Delta, q \prec t(s)}\,(E_{sr}) \right.$$

The obtained application of $(\prec_{Lar})$ is reduced.

1.2) r' is a superterm of (or equal to) r,
i.e., $r' = f(r)$ and $t(x) = t'(f(x))$.

$$D\left\{\begin{array}{c} x \prec f(r), r = s, \Gamma \mapsto \Delta, q \prec t'(x) \\ \hline r = s, \Gamma \mapsto \Delta, q \prec t'(f(r)) \\ \hline r = s, \Gamma \mapsto \Delta, q \prec t'(f(s)) \end{array}\right. \begin{array}{l} (E_{sr}) \\ (=_{Rs}) \end{array}$$

$$\Downarrow$$

$$D\left\{\begin{array}{c} x \prec f(r), r = s, \Gamma \mapsto \Delta, q \prec t'(x) \\ \hline x \prec f(s), r = s, \Gamma \mapsto \Delta, q \prec t'(x) \\ \hline r = s, \Gamma \mapsto \Delta, q \prec t'(f(s)) \end{array}\right. \begin{array}{l} (=_{Rar}) \\ (E_{sr}) \end{array}$$

Again, the resulting application of $(=_{Rar})$ is reduced. The case when the modified term is substituted into LHS of $\prec$ is treated analogously.

2) $R = (\prec_{Lar})$ with the modified formula $r = s(v)$ and v as the modified term:

$$D\left\{\begin{array}{c} D_1\left\{\begin{array}{c} \vdots \\ r = s(x), x \prec v, \Gamma \mapsto \Delta, q \prec t(r) \end{array}\right. \\ \hline r = s(v), x \prec v, \Gamma \mapsto \Delta, q \prec t(r) \\ \hline r = s(v), x \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) \end{array}\right. \begin{array}{l} (\prec_{La}) \\ (=_{Rs}) \end{array}$$

$$\Downarrow$$

$$D^*\left\{\begin{array}{c} D'\left\{\begin{array}{c} D_1\left\{\begin{array}{c} \vdots \\ r = s(x), x \prec v, \Gamma \mapsto \Delta, q \prec t(r) \end{array}\right. \\ \hline r = s(x), r = s(v), x \prec v, \Gamma \mapsto \Delta, q \prec t(r) \\ \hline r = s(x), r = s(v), x \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) \end{array}\right. \begin{array}{l} (W_a) \\ (=_{Rs}) \end{array} \\ \hline r = s(v), r = s(v), x \prec v, \Gamma \mapsto \Delta, q \prec t(s(v)) \end{array}\right. (\prec_{Lar})$$

The resulting application of $(\prec_{La})$ is the same as the original one – hence reduced. The same transformation is applied if $(=_{Rs})$ substitutes r for $s(v)$.

3) $R = (\prec_{Ls})$:

$$D\left\{\begin{array}{c} D_1\left\{\begin{array}{c} \vdots \\ r = s, q \prec u, \Gamma \mapsto \Delta, u \prec t(r) \end{array}\right. \\ \hline r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(r) \\ \hline r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(s) \end{array}\right. \begin{array}{l} (\prec_{Ls}) \\ (=_{Rs}) \end{array}$$

$$\Downarrow$$

$$D^*\left\{\begin{array}{c} D'\left\{\begin{array}{c} D_1\left\{\begin{array}{c} \vdots \\ r = s, q \prec u, \Gamma \mapsto \Delta, u \prec t(r) \end{array}\right. \\ \hline r = s, r = s(v), q \prec u, \Gamma \mapsto \Delta, u \prec t(s) \end{array}\right. (=_{Rs}) \\ \hline r = s, q \prec u, \Gamma \mapsto \Delta, q \prec t(s) \end{array}\right. (\prec_{Ls})$$

4) $R = (\prec_{Rsr})$

$$\begin{array}{c} f(p) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(f(x)) \\ \hline f(p) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(f(p)) \\ \hline f(p) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(s) \end{array} \begin{array}{l} (\prec_{Rsr}) \\ (=_{Rs}) \end{array} \quad \Rightarrow \quad \begin{array}{c} f(p) = s, f(x) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(f(x)) \\ \hline f(p) = s, f(x) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(s) \\ \hline f(p) = s, f(p) = s, x \prec p, \Gamma \mapsto \Delta, t \prec q(s) \end{array} \begin{array}{l} (=_{Rs}) \\ (\prec_{Lar}) \end{array}$$

$$\begin{array}{c} x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(x) \\ \hline x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(s)) \\ \hline x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) \end{array} \begin{array}{l} (\prec_{Rsr}) \\ (=_{Rs}) \end{array} \quad \Rightarrow \quad \begin{array}{c} x \prec f(r), x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(x) \\ \hline x \prec f(r), x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) \\ \hline x \prec f(s), x \prec f(s), s = r, \Gamma \mapsto \Delta, t \prec q(f(r)) \end{array} \begin{array}{l} (\prec_{Rs}) \\ (=_{Rar}) \end{array}$$

In both cases, since the original application of $(\prec_{Rsr})$ is reduced, so are the resulting applications of $(\prec_{Lar})$ and $(=_{Rar})$.

5) The other cases are easily swapped as in lemma 4.1.

So the case when the uppermost application is $(\prec_{Rsnr})$. For the axiom we have the following case (y is a fresh variable):

$$\dfrac{s \prec t, x \prec q(s), \Gamma \mapsto \Delta, x \prec q(s)}{s \prec t, x \prec q(s), \Gamma \mapsto \Delta, x \prec q(t)}\,(\prec_{Rsnr}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{y \prec s, y \prec t, x \prec q(y), \Gamma \mapsto \Delta, x \prec q(y)}{y \prec s, y \prec t, x \prec q(y), \Gamma \mapsto \Delta, x \prec q(t)}\,(\prec_{Rsr})}{y \prec s, s \prec t, x \prec q(y), \Gamma \mapsto \Delta, x \prec q(t)}\,(\prec_{Lar})}{s \prec t, x \prec q(y), \Gamma \mapsto \Delta, x \prec q(t)}\,(E_{ar})$$

Consider the rule applied just above $(\prec_{Rsnr})$.

1) $(\prec_{Rsr})$ – we choose a fresh variable y

$$\dfrac{\dfrac{s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(x)}{s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(s))}\,(\prec_{Rsr})}{s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Rsnr})$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{y \prec s, y \prec t, x \prec f(y), s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(x)}{y \prec s, y \prec t, x \prec f(y), s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(y))}\,(\prec_{Rsr})}{y \prec s, y \prec t, x \prec f(y), s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Rsr})}{y \prec s, s \prec t, x \prec f(y), s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Lar})}{s \prec t, x \prec f(s), s \prec t, x \prec f(s), \Gamma \mapsto \Delta, p \prec q(f(t))}\,(E_{ar})$$

Another case is as follows:

$$\dfrac{\dfrac{f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(x))}{f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(s))}\,(\prec_{Rsr})}{f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Rsnr})$$

$$\Downarrow$$

$$\dfrac{\dfrac{f(x) \prec f(t), f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(x))}{f(x) \prec f(t), f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Rsnr})}{f(s) \prec f(t), f(s) \prec f(t), x \prec s, \Gamma \mapsto \Delta, p \prec q(f(t))}\,(\prec_{Lar})$$

The new application of $(\prec_{Lar})$ is reduced.

2) $(\prec_{Lar})$ or $(=_{Lar})$ are treated analogously

$$\dfrac{\dfrac{p(x) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(p(t))}{p(t) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(p(t))}\,(\preceq_{Lar})}{p(t) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(q)}\,(\prec_{Rsnr}) \quad\Rightarrow\quad \dfrac{\dfrac{p(t) \prec q, p(x) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(p(t))}{p(t) \prec q, p(x) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(q)}\,(\prec_{Rsnr})}{p(t) \prec q, p(t) \prec q, x \preceq t, \Gamma \mapsto \Delta, r \prec f(q)}\,(\preceq_{Lar})$$

3) $(=_{Rar})$ – since this requires a variable in the LHS of the modified formula, the subsequent application will be reduced $(\prec_{Rsr})$:

$$\dfrac{\dfrac{x \prec t(p), p = r, \Gamma \mapsto \Delta, w \prec q(x)}{x \prec t(r), p = r, \Gamma \mapsto \Delta, w \prec q(x)}\,(=_{Ra})}{x \prec t(r), p = r, \Gamma \mapsto \Delta, w \prec q(t(r))}\,(\prec_{Rsr})$$

4) (E_{sr})

$$\dfrac{\dfrac{x \prec t, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(q(x))}{x \prec t, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(q(t))}\,(E_{sr})}{x \prec t, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(p)}\,(\prec_{Rsnr}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{x \prec t, q(x) \prec p, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(q(x))}{x \prec t, q(x) \prec p, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(p))}\,(\prec_{Rsnr})}{x \prec t, q(t) \prec p, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(p))}\,(\prec_{Lar})}{x \prec t, q(t) \prec p, q(t) \prec p, \Gamma \mapsto \Delta, r \prec w(p))}\,(E_{sr})$$

5) The rules $(=_{Ls})$ and $(\prec_{Ls})$ can be trivially swapped since they affect only the LHS of the inclusion in the consequent. The rule (E_{ar}) requires a variable in the LHS of the modified formula, so if this formula is active in a subsequent application of $(\prec_{Rs})$ this must be $(\prec_{Rsr})$. By the assumption, we do not have applications of $(=_{Rs})$. (cut_x) and $(Sp.cut)$ can be trivially swapped.

The new applications of $(\preceq_{La})$ are all reduced and in case 2) are of the same kind as in the original derivation. Also, no new applications of $(=_{Ra})$ appear. Hence $\#(\preceq_{Lanr})$ and $\#(=_{Ranr})$ are not increased. QED.

Remark 4.8 *Lemma 4.6 allows us to conclude that the applications of $(=_{Rs})$, $(=_{Ls})$ and $(\prec_{Ls})$ can be eliminated from any derivation in NEQ_4.*

Similarly, though alternatively, combining lemmas 4.3 and 4.7 – in this order, and with their side-conditions – allows us to conclude that the applications of $(\prec_{Lanr})$, $(=_{Ranr})$, $(=_{Rs})$ and $(\prec_{Rsnr})$ can be eliminated.

These two facts will be of crucial importance in the proof of cut-elimination in 4.3.

4.2 Equivalence $NEQ_4 \equiv NEQ_3$

The rules of NEQ_4 are merely restricted versions of the rules of NEQ_3, so we trivially have $NEQ_4 \Rightarrow NEQ_3$. For the opposite implication, we need to show admissibility in NEQ_4 of the unrestricted rules. Lemma 4.2 showed admissibility of $(=_{a=})$ and lemma 4.4 of $(=_{Lanr})$. We now show admissibility in NEQ_4 of the remaining rules (E_{ax}), (E_{sx}) and (cut_t).

Lemma 4.9 *The rules (E_{ax}) and (E_{sx}) are admissible in NEQ_4.*

Proof. Consider first the uppermost application of (E_{ax}). Renaming the eigen-variable z in the whole D_1 to x, and applying (cut_x) to the result of D_1', we get the desired result. (Remember that we may have $r(z) = z$.)

$$D_1 \left\{ \begin{array}{c} \vdots \\ z \prec x, y \prec r(z), \Gamma \mapsto \Delta \end{array} \right. \over y \prec r(x), \Gamma \mapsto \Delta \quad (E_{ax}) \qquad \Rightarrow \qquad D_1' \left\{ \begin{array}{c} \vdots \\ x \prec x, y \prec r(x), \Gamma \mapsto \Delta \end{array} \right. \over y \prec r(x), \Gamma \mapsto \Delta \quad (cut_x)$$

In a similar way, we consider the uppermost application of (E_{sx}):

$$\dfrac{D \left\{ \begin{array}{c} \vdots \\ \Gamma, x \prec y \mapsto \Delta \end{array} \right.}{\Gamma \mapsto \Delta^x_y} \ (E_{sx})$$

In all applications of (E_{sr}) and (E_{ar}) in D, replace the eigen-variables so that they differ from the eigen-variable x and from y. In the obtained derivation, replace all the occurrences of x by y. In this way, we obtain a derivation of $\Gamma, y \prec y \mapsto \Delta^x_y$. Application of (cut_x) gives the desired derivation D^* of the sequent $\Gamma \mapsto \Delta^x_y$ without using degenerated applications of (E_{sx}). QED.

Lemma 4.10 $NEQ_4 \vdash^D t \prec t, \Gamma \mapsto \Delta \quad \Rightarrow \quad NEQ_4 \vdash^{D^*} \Gamma \mapsto \Delta$.

Proof. By lemma 4.3 we assume that D contains no applications of $(\prec_{Lanr})$, $(=_{Ranr})$

By induction on $\langle \gamma[t], h(D) \rangle$, where $\gamma[t]$ is the complexity of the term t (defined in the usual way). If $\gamma[t] = 0$, i.e., t is a variable x, application of (cut_x) yields the conclusion. This is all that may happen if $h(D) = 1$, i.e. we merely modify once some axiom. (Superposition of the antecedent in $x \prec t, \Gamma \mapsto \Delta, x \prec t$ leads to $t \prec t, x \prec t, \Gamma \mapsto \Delta, x \prec t$ – removal of $t \prec t$ yields the same axiom.)

So let $\gamma[t] > 0$ and R be the last rule applied. The only rules which may generate $t \prec t$ in the antecedent are $(\prec_{Lar})$, (E_{ar}) and $(=_{Rar})$ or $(=_{Lar})$. We consider these cases by looking at the rule R_1 applied just above

1) R is $(\prec_{Lar})$ and D ends as shown below, where t is $t(s)$:

$$D \left\{ \begin{array}{c} \vdots \\ \hline t(y) \prec t(s), y \prec s, \Gamma \mapsto \Delta \ \ R_1 \\ \hline t(s) \prec t(s), y \prec s, \Gamma \mapsto \Delta \ \ (\prec_{Lar}) \end{array} \right.$$

1.1) R_1 is $(\prec_{Lar})$: The two possibilities are that $t(y) \prec t(s)$ was 1.1.1) active or 1.1.2) modified in R_1. The former requires $t(y)$ to be the variable y and so $t(s)$ to be s.

1.1.1) $y \prec t$ was active:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ y \prec t, y \prec t, q(y) \preceq p, \Gamma \mapsto \Delta\end{array}}{y \prec t, y \prec t, q(t) \preceq p, \Gamma \mapsto \Delta}\,(\prec_{Lar})}{t \prec t, y \prec t, q(t) \preceq p, \Gamma \mapsto \Delta}\,(\prec_{Lar})$$

Swapping the two applications gives a derivation with an earlier appearance of $t \prec t$.

1.1.2) $y \prec t$ was modified (we write it here in full as $t(y) \prec t(p(q))$):

$$D\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\\ y \prec p(x), t(y) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta\end{array}}{y \prec p(x), t(p(x)) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta}\,(\prec_{Lar})}{y \prec p(x), t(p(q)) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta}\,(\prec_{Lar})\right.$$

We first use a weakened derivation with $y \prec p(q)$

$$\dfrac{\dfrac{D_1\left\{\dfrac{\begin{array}{c}\vdots\\ y \prec p(q), y \prec p(x), t(y) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta\end{array}}{y \prec p(q), y \prec p(x), t(p(q)) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta}\,(\prec_{Lar})\right.}{p(x) \prec p(q), y \prec p(x), t(p(q)) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta}\,(\prec_{Lar})}{p(q) \prec p(q), y \prec p(x), t(p(q)) \prec t(p(q)), x \prec q, \Gamma \mapsto \Delta}\,(\prec_{Lar})$$

By induction on $h(D_1) < h(D)$ we can obtain the sequent concluding D_1 without $t(p(q)) \prec t(p(q))$. Then, the final $p(q) \prec p(q)$, can be eliminated by induction hypothesis on complexity of terms, since $\gamma[p(q)] < \gamma[t(p(q))]$.

1.2) R_1 is (E_{ar}):

$$\dfrac{\dfrac{x \prec p, y \prec t(x), y \prec t(p), \Gamma \mapsto \Delta}{y \prec t(p), y \prec t(p), \Gamma \mapsto \Delta}\,(E_{ar})}{t(p) \prec t(p), y \prec t(p), \Gamma \mapsto \Delta}\,(\prec_{Lar})$$

$$\Downarrow$$

$$\dfrac{\dfrac{x \prec p, y \prec t(x), y \prec t(p), y \prec t(x), \Gamma \mapsto \Delta}{x \prec p, t(x) \prec t(x), y \prec t(p), y \prec t(x), \Gamma \mapsto \Delta}\,(\prec_{Lar})}{t(x) \prec t(x), y \prec t(p), y \prec t(p), \Gamma \mapsto \Delta}\,(E_{ar})$$

Induction on complexity (or height) gives the conclusion (having eliminated $t(x) \prec t(x)$ the application of (E_{ar}) becomes correct).

1.3) R is $(=_{Rar})$:

$$\dfrac{\dfrac{y \prec t(s), y \prec t(p), p = s, \Gamma \mapsto \Delta}{y \prec t(p), y \prec t(p), p = s, \Gamma \mapsto \Delta}\,(=_{Rar})}{t(p) \prec t(p), y \prec t(p), p = s, \Gamma \mapsto \Delta}\,(\prec_{Lar})$$

Weakening the derivation with $y \prec t(s)$ (if $\gamma[t(s)] < \gamma[t(p)]$, and with $y \prec t(p)$ otherwise), we obtain:

$$\dfrac{\dfrac{y \prec t(s), y \prec t(s), y \prec t(p), p = s, \Gamma \mapsto \Delta}{t(s) \prec t(s), y \prec t(s), y \prec t(p), p = s, \Gamma \mapsto \Delta}\,(\prec_{Lar})}{t(s) \prec t(s), y \prec t(p), y \prec t(p), p = s, \Gamma \mapsto \Delta}\,(=_{Rar})$$

Induction on height gives the conclusion.

1.4) R is $(=_{Lar})$:

$$D\left\{\dfrac{\dfrac{t(x) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}{t(s(y)) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}\,(=_{Lar})}{t(s(p)) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}\,(\prec_{Lar})\right.$$

$$\Downarrow$$

$$\cfrac{\cfrac{D_1\left\{\cfrac{x \prec s(p), t(x) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}{x \prec s(p), t(s(p)) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}\ (\prec_{Lar})\right.}{s(y) \prec s(p), t(s(p)) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}\ (=_{Lar})}{s(p) \prec s(p), t(s(p)) \prec t(s(p)), x = s(y), y \prec p, \Gamma \mapsto \Delta}\ (\prec_{Lar})$$

Occurence of $t(s(p)) \prec t(s(p))$ in D_1 can be eliminated by induction hypothesis $h(D_1) < h(D)$, and of $s(p) \prec s(p)$ by $\gamma[s(p)] < \gamma[t(s(p))]$.

1.5) All the remaining cases can be easily swapped yielding an earlier occurrence of $t \prec t$.
(E_{sr}) : Since the active and the side formula of the subsequent $(\prec_{Lar})$ have to share a variable in the LHS, neither might be active in this application of (E_{sr}).
$(\prec_{Ls})$, $(\prec_{Rs})$: Activity of $t(y) \prec t(s)$ in this rule, can be simulated by the activity of $y \prec s$ which also must be present.
$(=_{Ls})$, $(=_{Rs})$, $(Sp.cut)$, (cut_x) and $(cut_{x=})$ are trivial.

2) R is $(=_{Lar})$ and D ends as shown below :

$$D\left\{\cfrac{\cfrac{\vdots}{t(y) \prec t(s), y = s, \Gamma \mapsto \Delta}\ R_1}{t(s) \prec t(s), y = s, \Gamma \mapsto \Delta}\ (=_{Lar})\right.$$

2.1) R_1 is $(\prec_{Lar})$ The two possibilities are that $t'(y) \prec t$ was 2.1.1) active or 2.1.2) modified in R_1. The former requires $t'(y)$ to be the variable y.

2.1.1) $y \prec t$ active:

$$D\left\{\cfrac{\cfrac{\begin{array}{c}\vdots\\ y \prec t, y = t, q(y) \preceq p, \Gamma \mapsto \Delta\end{array}}{y \prec t, y = t, q(t) \preceq p, \Gamma \mapsto \Delta}\ (\prec_{Lar})}{t \prec t, y = t, q(t) \preceq p, \Gamma \mapsto \Delta}\ (=_{Lar})\right.$$

$$\Downarrow$$

$$\cfrac{D_1\left\{\cfrac{\begin{array}{c}\vdots\\ y \prec t, y = t, q(y) \preceq p, \Gamma \mapsto \Delta\end{array}}{t \prec t, y = t, q(y) \preceq p, \Gamma \mapsto \Delta}\ (=_{Lar})\right.}{t \prec t, y = t, q(t) \preceq p, \Gamma \mapsto \Delta}\ (=_{Lar})$$

and induction hypothesis, $h(D_1) < h(D)$, gives the conclusion.

2.1.2) $y \prec t$ was modified (we write it as $q(y) \prec q(p(t))$)

$$D\left\{\cfrac{\cfrac{\begin{array}{c}\vdots\\ y \prec p(x), q(y) \prec q(p(t)), x = t, \Gamma \mapsto \Delta\end{array}}{y \prec p(x), q(p(x)) \prec q(p(t)), x = t, \Gamma \mapsto \Delta}\ (\prec_{Lar})}{y \prec p(x), q(p(t)) \prec q(p(t)), x = t, \Gamma \mapsto \Delta}\ (=_{Lar})\right.$$

$$\Downarrow$$

$$D'\left\{\cfrac{D_1\left\{\cfrac{\begin{array}{c}\vdots\\ y \prec p(t), y \prec p(x), q(y) \prec q(p(t)), x = t, \Gamma \mapsto \Delta\end{array}}{y \prec p(t), y \prec p(x), q(p(t)) \prec q(p(t)), x = t, \Gamma \mapsto \Delta}\ (\prec_{Lar})\right.}{y \prec p(x), y \prec p(x), q(p(t)) \prec q(p(t)), x = t, \Gamma \mapsto \Delta}\ (=_{Rar})\right.$$

2.2) R_1 is (E_{ar})

$$D\left\{\cfrac{\cfrac{x \prec p, y \prec t(x), t(p) = y, \Gamma \mapsto \Delta}{y \prec t(p), t(p) = y, \Gamma \mapsto \Delta}\ (E_{ar})}{t(p) \prec t(p), t(p) = y, \Gamma \mapsto \Delta}\ (=_{Lar})\right.$$

$$\Downarrow$$

$$D'\left\{\begin{array}{c}\underline{x \prec p, y \prec t(x), t(p) = y, \Gamma \mapsto \Delta} \;(E_{ar})\\ \underline{y \prec t(p), t(p) = y, \Gamma \mapsto \Delta}\;(=_{Rar})\\ y \prec y, t(p) = y, \Gamma \mapsto \Delta\end{array}\right.$$

and $y \prec y$ can be eliminated by induction hypothesis on $\gamma[t]$.

2.3) R_1 is $(=_{Rar})$ Then LHS of the side inclusion must be a variable, i.e.,

$$D\left\{\begin{array}{c}\underline{y \prec t(p), p = q, y = t(q), \Gamma \mapsto \Delta}\;(=_{Rar})\\ \underline{y \prec t(q), p = q, y = t(q), \Gamma \mapsto \Delta}\;(=_{Lar})\\ t(q) \prec t(q), p = q, y = t(q), \Gamma \mapsto \Delta\end{array}\right.$$

$$\Downarrow$$

$$D'\left\{\begin{array}{c}\underline{D_1\left\{\begin{array}{c}\underline{y \prec t(p), p = q, y = t(q), y = t(p), \Gamma \mapsto \Delta}\;(=_{Lar})\\ t(p) \prec t(p), p = q, y = t(q), y = t(p), \Gamma \mapsto \Delta\end{array}\right.}\;(=_{a=})\\ t(p) \prec t(p), p = q, y = t(q), y = t(q), \Gamma \mapsto \Delta\end{array}\right.$$

$h(D_1) < h(D)$ yields the conclusion.

2.4) R_1 is $(=_{Lar})$

$$D\left\{\begin{array}{c}\underline{t(x) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta}\;(=_{Lar})\\ \underline{t(s(y)) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta}\;(=_{Lar})\\ t(s(p)) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta\end{array}\right.$$

$$\Downarrow$$

$$\begin{array}{c}\underline{D_1\left\{\begin{array}{c}\underline{x = s(p), t(x) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta}\;(=_{Lar})\\ x = s(p), t(s(p)) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta\end{array}\right.}\;(=_{a=})\\ x = s(y), t(s(p)) \prec t(s(p)), y = p, x = s(y), \Gamma \mapsto \Delta\end{array}$$

$h(D_1) < h(D)$ gives the conclusion.
In the last two cases we have used lemma 4.2 which admits $(=_{a=})$.

2.5) All other cases of R_1 can be easily swapped leading to an earlier occurrence of $t \prec t$ (cf. the end of case 1. for $R = (\prec_{Lar})$)

3) R is (E_{ar}) This would require a degenerate application of this rule which does not belong to NEQ_4 (cf. lemma 4.9).

4) R is $(=_{Rar})$ Since this requires a variable in the LHS, we are back in the ground case for $\gamma[t] = 0$.

QED.

The remark at the very begining of this section 4.2, together with lemmas 4.2, 4.4, 4.9 and 4.10 (showing the admissibility of the missing NEQ_3 rules in NEQ_4), give us the first part of the following fact from which the second follows trivially.

Proposition 4.11 *$NEQ_3 \equiv NEQ_4$, and thus, also $NEQ_3^c \equiv NEQ_4^c$.*

4.3 Admissibility of (cut).

Referring to the remark 4.8, in the following proof of (cut)-elimination, we may introduce additional

Assumptions: If $NEQ_4 \vdash^D S$ then $NEQ_4 \vdash^{D^*} S$ where D^* contains

1.	either no $(=_{Rs})$, $(=_{Ls})$, $(\prec_{Ls})$	lemma 4.6 (4.1)
2.	or no $(\prec_{Lanr})$, $(=_{Ranr})$, $(=_{Rs})$, $(\prec_{Rsnr})$	lemma 4.3, 4.7

Also, NEQ_4 admits the rules

3.	$(=_{a=})$, $(\prec_{Lanr})$, $(=_{Ranr})$, $(=_{Lanr})$ and $(\prec_{Ra})$ without increasing the total number of $(\prec_{Lar})$ and $(=_{Lar})$	lemma 4.2, 4.3, 4.4, 4.5
4.	(cut_t)	lemma 4.10

Proposition 4.12 *If D is a NEQ_4^c and D_1, D_2 are NEQ_4 derivations as follows:*

$$D\left\{\dfrac{D_1\left\{\begin{array}{c}\vdots\\ \Gamma\mapsto\Delta,\phi\end{array}\right. \quad D_2\left\{\begin{array}{c}\vdots\\ \phi,\Gamma'\mapsto\Delta'\end{array}\right.}{\Gamma,\Gamma'\mapsto\Delta,\Delta'}\right.\ (cut)$$

then $NEQ_4 \vdash \Gamma, \Gamma' \mapsto \Delta, \Delta'$.

Proof. By the assumption 1. we let D_1 contain no $(=_{Rs})$, $(=_{Ls})$ or $(\prec_{Ls})$. By the assumption 2. we let D_2 contain only reduced applications $(\prec_{Lar})$, $(=_{Rar})$, $(\prec_{Rsr})$ – $(=_{Lanr})$ does not belong to NEQ_4 – and no $(=_{Rs})$.

The induction is then on $\langle\#((\preceq_{Lar}), D_2), h(D_1), h(D_2)\rangle$, where the first parameter is the total number of applications of $(\prec_{Lar})$ *or* $(=_{Lar})$ *which modify the cut formula* ϕ in D_2 (this qualification is essential in case 2.3) on page 28).

If $h(D_1) = 0$, then the resulting sequent of D_1 is an axiom. If the cut formula is not the one mentioned explicitly in the axioms (Fig. 3) then we obtain a cut-free derivation directly by choosing another instance of the same axiom. In the other case, if the cut formula is $t \prec t$ or $x = x$, the assumption 4. allows us to conclude the existence of a derivation of the conclusion of D_2 without this formula in the antecedent. Finally, if the axiom in D_1 is $s = t, \Gamma \mapsto \Delta, s = t$ or $x \prec t, \Gamma \mapsto \Delta, x \prec t$ and the cut formula is $s = t$, resp. $x \prec t$, then it will also appear in the antecedent of the conclusion of (cut). Then this conclusion can be obtained without (cut) directly from D_2 by starting it with the instance of the axiom extended with Γ and Δ.

For $h(D_1) > 0$, we consider the last rule R applied in D_1.

I. $(\prec_{La})$, (E_{ar}), $(=_{Lar})$, $(=_{Ra})$, $(Sp.cut)$, $(cut_{x=})$ or (cut_x): Since none of these rules modifies the succedent, their application as R in D_1 means that we can apply (cut) with D_1^* – before R – instead. The induction on $h(D_1)$ and a subsequent application of R gives the conclusion.

II. The rules $(=_{Rs})$, $(=_{Ls})$ and $(\prec_{Ls})$ are not used in D_1 by assumption 1.

III. $(\prec_{Rs})$: and the derivations end as follows:

$$D\left\{\dfrac{D_1\left\{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ s\prec t,\Gamma\mapsto\Delta,w\prec q(s)\end{array}\right.}{s\prec t,\Gamma\mapsto\Delta,w\prec q(t)}\,(\prec_{Rs})\right. \quad D_2\left\{\begin{array}{c}\vdots\\ w\prec q(t),\Gamma'\mapsto\Delta'\end{array}\right.}{s\prec t,\Gamma,\Gamma'\mapsto\Delta,\Delta'}\right.\ (cut)$$

We drop this last application of $(\prec_{Rs})$ in D_1 and extend weakened D_2 as follows:

$$D'\left\{\dfrac{\dfrac{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ \vdots\\ s\prec t,\Gamma\mapsto\Delta,w\prec q(s)\end{array}\right. \quad D_2'\left\{\dfrac{D_2\left\{\begin{array}{c}\vdots\\ w\prec q(t),q(s)\prec q(t),\Gamma'\mapsto\Delta'\end{array}\right.}{w\prec q(s),q(s)\prec q(t),\Gamma'\mapsto\Delta'}\,(\prec_{Ra})\right.}{s\prec t,q(s)\prec q(t)\Gamma,\Gamma'\mapsto\Delta,\Delta'}\,(cut)}{s\prec t,q(t)\prec q(t),\Gamma,\Gamma'\mapsto\Delta,\Delta'}\,(\prec_{La})}{s\prec t,\Gamma,\Gamma'\mapsto\Delta,\Delta'}\right.\ (cut_t)$$

By assumption 3. (lemma 4.5) the application of $(\prec_{Ra})$ is admissible without increasing the number of $(\preceq_{Lar})$, so (cut) can be eliminated by the second argument of induction. The result will be a NEQ_4 derivation which admits (cut_t) by assumption 4 (lemma 4.10).

IV. (E_{sr}) :

$$\dfrac{D_1\left\{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x\prec t,\Gamma\mapsto\Delta,\phi[x]\end{array}\right.}{\Gamma\mapsto\Delta,\phi[t]}\,(E_{sr})\right. \quad D_2\left\{\dfrac{\vdots}{\phi[t],\Gamma'\mapsto\Delta'}\,R'\right.}{\Gamma,\Gamma'\mapsto\Delta,\Delta'}\ (cut)$$

By restrictions on (E_{sr}), $t \notin \mathcal{V}$ and, furthermore, $x \notin \mathcal{V}(\Gamma, \Delta, t)$ so we can choose x so that $x \notin \mathcal{V}(\Gamma', \Delta')$. We consider two cases: 1) when in D_1, x in ϕ is in the RHS of $\prec$, and 2) when x occurs in a LHS of $\prec$ or when ϕ is an equality.

1) Let ϕ be $p \prec s(t)$

– t indicating the only occurrence which has been substituted for x. We can then extend D_2 and construct the following derivation:

$$
\dfrac{\dfrac{\dfrac{D_1^* \left\{ \begin{array}{c} \vdots \\ x \prec t, \Gamma \mapsto \Delta, p \prec s(x) \end{array} \right. \qquad D_2' \left\{ \dfrac{\dfrac{D_2 \left\{ \begin{array}{c} \vdots \\ p \prec s(t), \Gamma' \mapsto \Delta' \end{array} \right.}{s(x) \prec s(t), p \prec s(t), \Gamma' \mapsto \Delta'}\,(\mathrm{W})}{s(x) \prec s(t), p \prec s(x), \Gamma' \mapsto \Delta'}\,(\prec_{Ra}) \right.}{x \prec t, s(x) \prec s(t), \Gamma', \Gamma \mapsto \Delta, \Delta'}\,(cut)}{x \prec t, s(t) \prec s(t), \Gamma', \Gamma \mapsto \Delta, \Delta'}\,(\prec_{Lar})}{\dfrac{s(t) \prec s(t), \Gamma', \Gamma \mapsto \Delta, \Delta'}{\Gamma', \Gamma \mapsto \Delta, \Delta'}\,(cut_t)}\,(E_{sr})
$$

The rule $(\prec_{Ra})$ is admissible (assumption 3, lemma 4.5) and does not increase the number of applications of $(\preceq_{Lar})$. So (cut) can be eliminated by induction on $h(D_1^*) < h(D_1)$. The resulting derivation is in NEQ_4 so, by assumption 4. (lemma 4.10) (cut_t) is admissible.

2) Let ϕ be $s(t) \preceq p$

Here we have three subcases, depending on whether ϕ in the application of the last rule R' in D_2 was 2.1) neither modified nor active, 2.2) modified, or 2.3) active.

2.1) The cut formula ϕ is neither modified nor active.
We may then swap the applications of R' and (cut), and the induction on the height of D_2 yields the required elimination of (cut).

2.2) ϕ is modified by R'.
The two possibilities of ϕ being an inclusion and equality are entirely analogous. This case depends only on the rule R' – if ϕ is an equality, R' cannot be $(=_{a=})$, but except for that the two cases are identical.

2.2.1) R' is (E_{ar}) or $(=_{Rar})$:
This would require ϕ to be an inclusion with a single variable in the LHS, i.e, $y \prec p$. Since in the current case 2), (E_{sr}) in D_1 replaces x in the LHS of $\prec$, this would mean that this application is actually degenerate (E_{sx}), what is excluded by lemma 4.9.

2.2.2) R' is $(\prec_{Lar})$ or $(=_{Lar})$:
These two cases are identical so we treat them jointly and write $(\preceq_{Lar})$. In each subcase, all occurrences of $(\preceq_{Lar})$ must be replaced consistently either by $(\prec_{Lar})$ or by $(=_{Lar})$, unless we mention the latter rules explicitly. The cut formula ϕ has the form $\phi[x]$ and $\phi[t]$ after the application of (E_{sr}) in D_1, while in D_2 we write it as $\phi'[y]$, and as $\phi'[t']$ after the final application of $(\preceq_{Lar})$, i.e. $\phi[t] = \phi = \phi'[t']$. This is to indicate that the term t introduced in D_1 and t' introduced in D_2 into ϕ may be different. The relation between the two does not matter, however – what makes it possible to treat all the (sub)cases in the same way, is the fact that if $\phi'[y]$ is an inclusion the modified term is introduced into its LHS.
D_2 ends as follows:

$$
D_2 \left\{ \dfrac{D_2^* \left\{ \begin{array}{c} \vdots \\ \phi'[y], y \preceq t', \Gamma' \mapsto \Delta' \end{array} \right.}{\phi'[t'], y \preceq t', \Gamma' \mapsto \Delta'}\,(\preceq_{Lar}) \right.
$$

We drop this last application of $(\preceq_{Lar})$ and, instead, extend D_1 weakened with $y \preceq t'$.

$$
\dfrac{D_1' \left\{ \dfrac{D_1 \left\{ \dfrac{\begin{array}{c} \vdots \\ y \preceq t', x \prec t, \Gamma \mapsto \Delta, \phi[x] \end{array}}{y \preceq t', \Gamma \mapsto \Delta, \phi[t]}\,(E_{sr}) \right.}{y \preceq t', \Gamma \mapsto \Delta, \phi'[y]}\,(\preceq_{Ls}) \right. \qquad D_2^* \left\{ \begin{array}{c} \vdots \\ \vdots \\ \phi'[y], y \preceq t', \Gamma' \mapsto \Delta' \end{array} \right.}{y \preceq t', \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut)
$$

The obtained application of $(\preceq_{Ls})$ is admissible and here harmless. (cut) can be eliminated by induction hypothesis on $\#((\preceq_{La}), D_2)$.

2.3) ϕ is active in R'

2.3.1) ϕ is inclusion $s(t) \prec p$ (active in R'), that is R' is either $(\prec_{Ls})$, $(\prec_{Rsr})$ or $(\prec_{Lar})$. The last two cases are excluded because they would require $s(t)$ to be a variable. Then we would have a degenerate application of (E_{sx}) in D_1, which does not belong to NEQ_4. So let R' be $(\prec_{Ls})$. We have the following derivation:

$$\dfrac{D_1\left\{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x \prec t, \Gamma \mapsto \Delta, s(x) \prec p\end{array}\right.}{\Gamma \mapsto \Delta, s(t) \prec p}\,(E_{sr})\right. \qquad D_2\left\{\dfrac{D_2^*\left\{\begin{array}{c}\vdots\\ s(t) \prec p, \Gamma' \mapsto \Delta', w(p) \preceq q\end{array}\right.}{s(t) \prec p, \Gamma' \mapsto \Delta', w(s(t)) \preceq q}\,(\prec_{Ls})\right.}{\Gamma, \Gamma' \mapsto \Delta, \Delta', w(s(t)) \preceq q}\,(cut)$$

First, construct the derivation M' by cutting $s(t) \prec p$ after D_1 and D_2^*. Since $h(D_2^*) < h(D_2)$ this (cut) can be eliminated using the third argument of induction. Then extend M' to M as follows:

$$M\left\{\dfrac{\dfrac{M'\left\{\dfrac{D_1\left\{\begin{array}{c}\vdots\\ \Gamma \mapsto \Delta, s(t) \prec p\end{array}\right. \qquad D_2^*\left\{\begin{array}{c}\vdots\\ s(t) \prec p, \Gamma' \mapsto \Delta', w(p) \preceq q\end{array}\right.}{\Gamma, \Gamma' \mapsto \Delta, \Delta', w(p) \preceq q}\,(cut)\right.}{s(x) \prec p, \Gamma, \Gamma' \mapsto \Delta, \Delta', w(p) \preceq q}\,(W_a)}{s(x) \prec p, \Gamma, \Gamma' \mapsto \Delta, \Delta', w(s(x)) \preceq q}\,(\prec_{Ls})\right.$$

The application of (cut) to M with D_1^* can be eliminated by induction hypothesis $h(D_1^*) < h(D_1)$ – notice that we are using here the fact that $\#((\preceq_{La}), M)$ *modifying the cut formula* is not greater than $\#((\preceq_{La}), D_2^*)$, even if the total number of arbitrary applications of $(\preceq_{La})$ in M may be far greater than in D_2^* (due to applications in D_1). It yields the following sequent leading to the desired conclusion by an application of (E_{sr}) – x may be chosen so that $x \notin \mathcal{V}(\Gamma', \Delta', w, q)$:

$$\dfrac{x \prec t, \Gamma, \Gamma' \mapsto \Delta, \Delta', w(s(x)) \preceq q}{\Gamma, \Gamma' \mapsto \Delta, \Delta', w(s(t)) \preceq q}\,(E_{sr})$$

2.3.2) ϕ is equality $s(t) = p$ (active in R').
We have only three cases for R' which are all treated analogously to the previous case 2.3.1). By assumption 2., the case of $(=_{Rs})$ for R' is excluded.

2.3.2.1) R' is $(=_{Ls})$
This is treated exactly as 2.3.1) with applications of $(=_{Ls})$ instead of $(\prec_{Ls})$.

2.3.2.2) R' is $(=_{Lar})$ or $(=_{Rar})$
We proceed as above with the construction of M', M and (cut). The differences occur only in the last step, so we make the following generic description where ψ is the side and ψ' the modified formula of R':

$$\dfrac{D_1\left\{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x \prec t, \Gamma \mapsto \Delta, s(x) = p\end{array}\right.}{\Gamma \mapsto \Delta, s(t) = p}\,(E_{sr})\right. \qquad D_2\left\{\dfrac{D_2^*\left\{\begin{array}{c}\vdots\\ s(t) = p, \psi, \Gamma' \mapsto \Delta'\end{array}\right.}{s(t) = p, \psi', \Gamma' \mapsto \Delta',}\,(R')\right.}{\psi', \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut)$$

First, construct the derivation M' by cutting $s(t) = p$ after D_1 and D_2^* which is weakened with $s(x) = p$. $h(D_2^*) < h(D_2)$ means that this (cut) can be eliminated. This yields the following sequent

$$s(x) = p, \psi, \Gamma, \Gamma' \mapsto \Delta, \Delta'$$

leading to the desired conclusion by the procedure depending on R'. (Remember that x may be chosen so that $x \notin \mathcal{V}(\Gamma', \Delta', \psi)$, and $\#(\preceq_a, M')$ is not greater than in D_2^*):

2.3.2.2.1) R' is $(=_{Lar})$
$s(t)$ cannot be a variable (since then (E_s) in D_1 would be degenerate), so p must be a

variable y and ψ is $f(y) \prec q$, and ψ' is $f(s(t)) \prec q$.

$$\dfrac{\dfrac{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x \prec t, \Gamma \mapsto \Delta, s(x)=y\end{array}\right. \qquad \dfrac{s(x)=y, f(y)\prec q, \Gamma, \Gamma' \mapsto \Delta, \Delta'}{s(x)=y, f(s(x))\prec q, \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(=_{Lar})}{x \prec t, f(s(x)) \prec q, \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut)}{x \prec t, f(s(t)) \prec q, \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(\prec_{Lar})}{f(s(t)) \prec q, \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(E_{sr})$$

This (cut) can be eliminated since $h(D_1^*) < h(D_1)$.

2.3.2.2.2) R' is $(=_{Rar})$
substituting $s(t)$ for p, i.e., ψ is $y \prec f(p)$, and ψ' is $y \prec f(s(t))$ for a $y \in \mathcal{V}$:

$$\dfrac{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x \prec t, \Gamma \mapsto \Delta, s(x)=p\end{array}\right. \qquad \dfrac{s(x)=p, y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}{s(x)=p, y \prec f(s(x)), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(=_{Rar})}{x \prec t, y \prec f(s(x)), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut)}{y \prec f(s(t)), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(E_{ar})$$

This (cut) can be eliminated since $h(D_1^*) < h(D_1)$.

2.3.2.2.3) R' is $(=_{Rar})$
substituting p for $s(t)$, i.e., ψ is $y \prec f(s(t))$, and ψ' is $y \prec f(p)$ for a $y \in \mathcal{V}$. Here we have to weaken D_2^* also with $f(s(x)) \prec f(s(t))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{D_1^*\left\{\begin{array}{c}\vdots\\ x \prec t, \Gamma \mapsto \Delta, s(x)=p\end{array}\right. \qquad \dfrac{\dfrac{f(s(x)) \prec f(s(t)), s(x)=p, y \prec f(s(t)), \Gamma, \Gamma' \mapsto \Delta, \Delta'}{f(s(x)) \prec f(s(t)), s(x)=p, y \prec f(s(x)), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(\prec_{Ra})}{f(s(x)) \prec f(s(t)), s(x)=p, y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(=_{Rar})}{x \prec t, f(s(x)) \prec f(s(t)), y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut)}{x \prec t, f(s(t)) \prec f(s(t)), y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(\prec_{Lar})}{x \prec t, y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(cut_t)}{y \prec f(p), \Gamma, \Gamma' \mapsto \Delta, \Delta'}\,(E_{sr})$$

Again, (cut) can be eliminated since $h(D_1^*) < h(D_1)$.

QED.

The proposition yields the problematic implication of the

Theorem 4.13 $NEQ_4^c \equiv NEQ_4$

Combined with proposition 4.11 (and the equivalences from propositions 3.4, 3.9 and 3.12) we have also:

Corollary 4.14 $NEQ \equiv NEQ_3^c \equiv NEQ_4^c \equiv NEQ_4 \equiv NEQ_3$.

5 The final cut-free calculus NEQ_5

Although the elementary cut rules are relatively innocent, we make the final step showing that they are redundant. We do it in two steps. First, in 5.1, we remove the rule (cut_x) obtaining NEQ_5' which is equivalent to NEQ_4. This step isn't really problematic. The rule $(cut_{x=})$, however, expresses the specific character of variables as compared to arbitrary other terms. In 5.2 we show that it can be replaced by two other rules which capture this specificity of variables, thus obtaining the final cut-free version of the calculus NEQ_5.

5.1 NEQ_5' – the calculus with one elementary cut

Definition 5.1 *NEQ_5' is obtained from NEQ_4 by*

- *replacing the axiom $x \prec t, \Gamma \mapsto \Delta, x \prec t$ by $\Gamma \mapsto \Delta, t \prec t$;*
- *admitting non-reduced applications of $(=_{Lanr})$ but removing the degenerate applications $(=_{Rax})$ and $(=_{Lax})$ which result in a single variable;*
- *removing the elementary cut rule (cut_x) and $(=_{Rs})$;*

The rules of NEQ_5' are given in figure 4.

Axioms:

$$\Gamma \mapsto \Delta, t \prec t \qquad \Gamma \mapsto \Delta, x = x \qquad s = t, \Gamma \mapsto \Delta, s = t$$

Identity rules:

$$(=_{Ls})\ \frac{t = s, \Gamma \mapsto \Delta, p(s) \preceq q}{t = s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (=_{Lanx})\ \frac{s = t, p(s) \prec q, \Gamma \mapsto \Delta}{s = t, p(t) \prec q, \Gamma \mapsto \Delta}\quad p(t) \notin \mathcal{V}$$

$$(=_{Ranx})\ \frac{s = t, p \prec q(s), \Gamma \mapsto \Delta}{s = t, p \prec q(t), \Gamma \mapsto \Delta}\quad q(t) \notin \mathcal{V}$$

Inclusion rules:

$$(\prec_{Ls})\ \frac{t \prec s, \Gamma \mapsto \Delta, p(s) \preceq q}{t \prec s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (\prec_{La})\ \frac{s \prec t, p(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, p(t) \preceq q, \Gamma \mapsto \Delta}$$

$$(\prec_{Rs})\ \frac{s \prec t, \Gamma \mapsto \Delta, p \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, p \prec q(t)}$$

Elimination rules:

$$(E_{sr})\ \frac{\Gamma, x \prec t \mapsto \Delta, \phi[x]}{\Gamma \mapsto \Delta, \phi[t]} \qquad (E_{ar})\ \frac{x \prec t, y \prec r(x), \Gamma \mapsto \Delta}{y \prec r(t), \Gamma \mapsto \Delta}$$

- $x \notin \mathcal{V}(\Gamma, \Delta, t),\ t \notin \mathcal{V}$
- at most one x in ϕ;

- $x \notin \mathcal{V}(t, \Gamma, \Delta, y),\ t \notin \mathcal{V}$
- at most one occurrence of x in r

Elementary cut:

$$(cut_{x=})\ \frac{x = x, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}$$

Specific cut rules:

for each specific axiom $\mathcal{A}_k$: $a_1, ..., a_n \mapsto s_1, ..., s_m$, a rule:

$$\frac{\Gamma \mapsto \Delta, a_1 \ ;...;\ \Gamma \mapsto \Delta, a_n \ ;\ s_1, \Gamma \mapsto \Delta \ ;...;\ s_m, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\quad (Sp.cut_k)$$

Figure 4: The calculus NEQ'_5 $(x, y \in \mathcal{V})$

Lemma 5.2 $NEQ'_5 \Rightarrow NEQ_4$.

Proof. Obviously, the axiom $\Gamma \mapsto \Delta, t \prec t$ is derivable in NEQ_4 by choosing a fresh variable in the axiom $x \prec t, \Gamma \mapsto \Delta, x \prec t$ and applying (E_{sr}). If t is a variable $x \in \mathcal{V}$, then one starts with $x \prec x, \Gamma \mapsto \Delta, x \prec x$ and applies (cut_x). The non-reduced applications $(=_{Lanr})$ are admissible in NEQ_4 by lemma 4.4. QED.

For the other implication $NEQ_4 \Rightarrow NEQ'_5$ we show admissibility in NEQ'_5 of the missing rules: (cut_x) and unrestricted $(=_{Lar})$ and $(=_{Ra})$.

Lemma 5.3 *The rule* (cut_x) *is admissible in* NEQ'_5.

Proof. Instead of any axiom with $x \prec x$ in the antecedent, we may take the one without it. The only rules which might possibly generate this formula in the antecedent are (E_{ar}), $(=_{Lanx})$, $(=_{Ranx})$ or $(\prec_{La})$. But since the first three of these rules are non-degenerate, they cannot introduce a single variable, i.e. they cannot result in a formula $x \prec x$. Thus the only possibility is the following:

$$\frac{t \prec x, t \prec x, \Gamma \mapsto \Delta}{x \prec x, t \prec x, \Gamma \mapsto \Delta}\ (\prec_{La})$$

But the resulting sequent is the same as the premise weakened with $x \prec x$. QED.

Lemma 5.4 NEQ'_5 *admits degenerate applications of* $(=_{Lar})$*, namely,*

$$(=_{Larx})\ \frac{y = x, y \prec t, \Gamma \mapsto \Delta}{y = x, x \prec t, \Gamma \mapsto \Delta} \qquad y, x \in \mathcal{V}$$

Proof. By induction on the number of applications of the rule and the height of derivation. The rules is not applicable to any axiomatic formula, so consider the rule applied above.

1) $(=_{Lanx})$ – the two applications cannot interact and can be swapped since the subsequent application $(=_{Larx})$ is reduced, i.e., it replaces a variable which couldn't have been introduced just above it by a non-degenerate application of $(=_{Lanx})$.
2) $(\prec_{Lanx})$ – two cases

$$\dfrac{\dfrac{x=y, s\prec y, s\prec q, \Gamma \mapsto \Delta}{x=y, s\prec y, y\prec q, \Gamma \mapsto \Delta}\,(\prec_{La})}{x=y, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(=_{Larx}) \quad \Rightarrow \quad \dfrac{\dfrac{x=s, x=y, s\prec y, s\prec q, \Gamma \mapsto \Delta}{x=s, x=y, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(=_{Larx})}{x=y, x=y, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(\prec_{La})$$

Analogous weakening when $(\prec_{La})$ modifies the formula active in the application of $(=_{Larx}$);

$$\dfrac{\dfrac{x=s, s\prec y, y\prec q, \Gamma \mapsto \Delta}{x=y, s\prec y, y\prec q, \Gamma \mapsto \Delta}\,(\prec_{La})}{x=y, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(=_{Larx}) \quad \Rightarrow \quad \dfrac{\dfrac{x=y, x=s, s\prec y, y\prec q, \Gamma \mapsto \Delta}{x=y, x=s, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(=_{Larx})}{x=y, x=y, s\prec y, x\prec q, \Gamma \mapsto \Delta}\,(\prec_{La})$$

3) (E_{ar}) – swapping is possible since $x \in \mathcal{V}$

$$\dfrac{\dfrac{z\prec p, x=y, y\prec r(z), \Gamma \mapsto \Delta}{x=y, y\prec r(p), \Gamma \mapsto \Delta}\,(E_{ar})}{x=y, x\prec r(p), \Gamma \mapsto \Delta}\,(=_{Larx}) \quad \Rightarrow \quad \dfrac{\dfrac{z\prec p, x=y, y\prec r(z), \Gamma \mapsto \Delta}{z\prec p, x=y, x\prec r(z), \Gamma \mapsto \Delta}\,(=_{Larx})}{x=y, x\prec r(p), \Gamma \mapsto \Delta}\,(E_{ar})$$

4) $(\prec_{Ls})$

$$\dfrac{\dfrac{y\prec t, y=x, \Gamma \mapsto \Delta, p(t)\preceq q}{y\prec t, y=x, \Gamma \mapsto \Delta, p(y)\preceq q}\,(\prec_{Ls})}{x\prec t, y=x, \Gamma \mapsto \Delta, p(y)\preceq q}\,(=_{Larx}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{y\prec t, y=x, \Gamma \mapsto \Delta, p(t)\preceq q}{x\prec t, y=x, \Gamma \mapsto \Delta, p(t)\preceq q}\,(=_{Larx})}{x\prec t, y=x, \Gamma \mapsto \Delta, p(x)\preceq q}\,(\prec_{Ls})}{x\prec t, y=x, \Gamma \mapsto \Delta, p(y)\preceq q}\,(=_{Ls})$$

5) $(\prec_{Rs})$ – is treated analogously to $(\prec_{Ls})$.
6) Neither active nor modified formulae of $(=_{Ls})$ or (E_{sr}) can be affected by $(=_{Larx})$, so these, as well as $(Sp.cut)$, allow trivial swap with $(=_{Larx})$. $(=_{Ra})$ modifies at most the RHS of inclusion modified then by $(=_{Larx})$, so these two can be swapped, too.

QED.

To show admissibility of the unrestricted $(=_{Ra})$ we need the following lemma.

Lemma 5.5 *The rule* $(=_{a=})$ *is admissible in* NEQ'_5

$$(=_{a=}) \quad \dfrac{s=t, p(s)=q, \Gamma \mapsto \Delta}{s=t, p(t)=q, \Gamma \mapsto \Delta} \quad p(t) \notin \mathcal{V}$$

Proof. Application to an axiom is transformed as follows:

$$\dfrac{s=t, p(s)=q, \Gamma \mapsto \Delta, p(s)=q}{s=t, p(t)=q, \Gamma \mapsto \Delta, p(s)=q}\,(=_{a=}) \quad \Rightarrow \quad \dfrac{s=t, p(t)=q, \Gamma \mapsto \Delta, p(t)=q}{s=t, p(t)=q, \Gamma \mapsto \Delta, p(s)=q}\,(=_{Ls})$$

The trivial differences from the proof of lemma 4.2 concern the presence of non-reduced version of $(=_{La})$. In addition this case used there (cut_x). Here it will be treated as follows:

1) $(=_{Lanx})$ – $p(f(s)), f(t) \notin \mathcal{V}$

$$\dfrac{\dfrac{s=t, f(s)=r, p(r)\prec q, \Gamma \mapsto \Delta}{s=t, f(s)=r, p(f(s))\prec q, \Gamma \mapsto \Delta}\,(=_{Lanx})}{s=t, f(t)=r, p(f(s))\prec q, \Gamma \mapsto \Delta}\,(=_{a=}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{s=t, f(s)=r, p(r)\prec q, \Gamma \mapsto \Delta}{s=t, f(t)=r, p(r)\prec q, \Gamma \mapsto \Delta}\,(=_{a=})}{s=t, f(t)=r, p(f(t))\prec q, \Gamma \mapsto \Delta}\,(=_{Lanx})}{s=t, f(t)=r, p(f(s))\prec q, \Gamma \mapsto \Delta}\,(=_{Lanx})$$

The first application is $(=_{Lanx})$ because $f(t) \notin \mathcal{V} \Rightarrow p(f(t)) \notin \mathcal{V}$.

And the other possibility – $p(r), f(t) \notin \mathcal{V}$

$$\begin{array}{c} s = t, f(s) = r, p(f(s)) \prec q, \Gamma \mapsto \Delta \\ \hline s = t, f(s) = r, p(r) \prec q, \Gamma \mapsto \Delta \\ \hline s = t, f(t) = r, p(r) \prec q, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{Lanx}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} s = t, f(s) = r, p(f(s)) \prec q, \Gamma \mapsto \Delta \\ \hline s = t, f(t) = r, p(f(s)) \prec q, \Gamma \mapsto \Delta \\ \hline s = t, f(t) = r, p(f(t)) \prec q, \Gamma \mapsto \Delta \\ \hline s = t, f(t) = r, p(r) \prec q, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{a=}) \\ (=_{Lanx}) \\ (=_{Lanx}) \end{array}$$

Again, the first application is $(=_{Lanx})$ because $f(t) \notin \mathcal{V} \Rightarrow p(f(t)) \notin \mathcal{V}$.

2) $(\prec_{La})$ – $p(t) \notin \mathcal{V}$

$$\begin{array}{c} s \prec t, p(s) = q, p(t) = r, \Gamma \mapsto \Delta \\ \hline s \prec t, p(t) = q, p(t) = r, \Gamma \mapsto \Delta \\ \hline s \prec t, r = q, p(t) = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (\prec_{La}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} p(s) = r, s \prec t, p(s) = q, p(t) = r, \Gamma \mapsto \Delta \\ \hline p(s) = r, s \prec t, p(s) = q, p(t) = r, \Gamma \mapsto \Delta \\ \hline p(t) = r, s \prec t, p(s) = q, p(t) = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{a=}) \\ (\prec_{La}) \end{array}$$

Furthermore, we have the case when $t(k) \notin \mathcal{V}$ which needs $(=_{Ranx})$

$$\begin{array}{c} s \prec t(k), p(s) = q, k = r, \Gamma \mapsto \Delta \\ \hline s \prec t(k), p(t(k)) = q, k = r, \Gamma \mapsto \Delta \\ \hline s \prec t(k), p(t(r)) = q, k = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (\prec_{La}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} s \prec t(r), s \prec t(k), p(s) = q, k = r, \Gamma \mapsto \Delta \\ \hline s \prec t(r), s \prec t(k), p(t(r)) = q, k = r, \Gamma \mapsto \Delta \\ \hline s \prec t(k), s \prec t(k), p(t(r)) = q, k = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{a=}) \\ (=_{Ranx}) \end{array}$$

If $t(k) \in \mathcal{V}$, i.e. it is an x we have the following situation with $p(r) \notin \mathcal{V}$:

$$\begin{array}{c} s \prec x, p(s) = q, x = r, \Gamma \mapsto \Delta \\ \hline s \prec x, p(x) = q, x = r, \Gamma \mapsto \Delta \\ \hline s \prec x, p(r) = q, x = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (\prec_{La}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} s \prec x, p(s) = q, x = r, s = r, \Gamma \mapsto \Delta \\ \hline s \prec x, p(r) = q, x = r, s = r, \Gamma \mapsto \Delta \\ \hline s \prec x, p(r) = q, x = r, x = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{a=}) \\ (\prec_{La}) \end{array}$$

3) $(=_{Ranx})$ – $q(t), f(r) \notin \mathcal{V}$

$$\begin{array}{c} t = s, p \prec f(q(s)), q(s) = r, \Gamma \mapsto \Delta \\ \hline t = s, p \prec f(r), q(s) = r, \Gamma \mapsto \Delta \\ \hline t = s, p \prec f(r), q(t) = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{Ranx}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} t = s, p \prec f(q(s)), q(s) = r, \Gamma \mapsto \Delta \\ \hline t = s, p \prec f(q(s)), q(t) = r, \Gamma \mapsto \Delta \\ \hline t = s, p \prec f(q(t)), q(t) = r, \Gamma \mapsto \Delta \\ \hline t = s, p \prec f(r), q(t) = r, \Gamma \mapsto \Delta \end{array} \begin{array}{l} (=_{a=}) \\ (=_{Ranx}) \\ (=_{Ranx}) \end{array}$$

The first resulting application is $(=_{Ranx})$ since $q(t) \notin \mathcal{V} \Rightarrow f(q(t)) \notin \mathcal{V}$. Analogous transformation is applied when $(=_{a=})$ modifies r rather than $q(s)$.

4) $(=_{Ls})$

$$\begin{array}{c} s = t, q(s) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \\ \hline s = t, q(s) = r, \Gamma \mapsto \Delta, f(r) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(r) \prec p \end{array} \begin{array}{l} (=_{Ls}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} s = t, q(s) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(q(t)) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(r) \prec p \end{array} \begin{array}{l} (=_{a=}) \\ (=_{Ls}) \\ (=_{Ls}) \end{array}$$

Another case is analogous

$$\begin{array}{c} s = t, q(s) = r, \Gamma \mapsto \Delta, f(r) \prec p \\ \hline s = t, q(s) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \end{array} \begin{array}{l} (=_{Ls}) \\ (=_{a=}) \end{array} \quad \Rightarrow \quad \begin{array}{c} s = t, q(s) = r, \Gamma \mapsto \Delta, f(r) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(r) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(q(t)) \prec p \\ \hline s = t, q(t) = r, \Gamma \mapsto \Delta, f(q(s)) \prec p \end{array} \begin{array}{l} (=_{a=}) \\ (=_{Ls}) \\ (=_{Ls}) \end{array}$$

5) Other cases are trivial. The active formula of $(\prec_{Ls})$ and $(\prec_{Rs})$ is an inclusion and, similarly, elimination rules involve only inclusions. $(Sp.cut)$ can be trivially swapped.

QED.

Lemma 5.6 *NEQ'_5 admits degenerate applications of $(=_{Ra})$, namely,*

$$(=_{Rax}) \quad \frac{t = x, s \prec t, \Gamma \mapsto \Delta}{t = x, s \prec x, \Gamma \mapsto \Delta} \quad x \in \mathcal{V}$$

Proof. The rule isn't applicable to any axiomatic formula, so we consider the rule applied just above.

1) $(=_{Ranx})$ – $q(t) \notin \mathcal{V}$

$$\dfrac{\dfrac{s = t, q(t) = x, p \prec q(s), \Gamma \mapsto \Delta}{s = t, q(t) = x, p \prec q(t), \Gamma \mapsto \Delta}\,(=_{Ranx})}{s = t, q(t) = x, p \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{q(s) = x, s = t, q(t) = x, p \prec q(s), \Gamma \mapsto \Delta}{q(s) = x, s = t, q(t) = x, p \prec x, \Gamma \mapsto \Delta}\,(=_{Rax})}{q(t) = x, s = t, q(t) = x, p \prec x, \Gamma \mapsto \Delta}\,(=_{a=})$$

Since $q(t) \notin \mathcal{V}$, the rule $(=_{a=})$ is admissible by lemma 5.5.

2) $(\prec_{La})$

$$\dfrac{\dfrac{x = p(s), s \prec t, q \prec p(t), \Gamma \mapsto \Delta}{x = p(t), s \prec t, q \prec p(t), \Gamma \mapsto \Delta}\,(\prec_{La})}{x = p(t), s \prec t, q \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{x = p(t), x = p(s), s \prec t, q \prec p(t), \Gamma \mapsto \Delta}{x = p(t), x = p(s), s \prec t, q \prec x, \Gamma \mapsto \Delta}\,(=_{Rax})}{x = p(t), x = p(t), s \prec t, q \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})$$

Similar weakening when $(\prec_{Lanx})$ introduces the subsequently substituted variable:

$$\dfrac{\dfrac{r = p, r \prec x, q \prec p, \Gamma \mapsto \Delta}{x = p, r \prec x, q \prec p, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = p, r \prec x, q \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{x = p, r = p, r \prec x, q \prec p, \Gamma \mapsto \Delta}{x = p, r = p, r \prec x, q \prec x, \Gamma \mapsto \Delta}\,(=_{Rax})}{r = p, r = p, r \prec x, q \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})$$

Interaction of both formulae gives two cases; the first with $t \notin \mathcal{V}$

$$\dfrac{\dfrac{x = p, p \prec t, \Gamma \mapsto \Delta}{x = t, p \prec t, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = t, p \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{x = t, x = p, p \prec t, \Gamma \mapsto \Delta}{x = t, x = p, p \prec x, \Gamma \mapsto \Delta}\,(=_{Rax})}{x = t, x = x, p \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = t, x = t, p \prec x, \Gamma \mapsto \Delta}\,(=_{a=})$$

where $(=_{a=})$ is admissible by lemma 5.5 since $t \notin \mathcal{V}$. If t is an $y \in \mathcal{V}$

$$\dfrac{\dfrac{x = p, p \prec y, \Gamma \mapsto \Delta}{x = y, p \prec y, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = y, p \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{p \prec y, x = p, p \prec x, \Gamma \mapsto \Delta}{p \prec y, x = y, p \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x \prec y, x = y, p \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{y \prec y, x = y, p \prec x, \Gamma \mapsto \Delta}\,(=_{Lax})}{x = y, p \prec x, \Gamma \mapsto \Delta}\,(cut_x)$$

The rule $(=_{Lax})$ is admissible by lemma 5.4 and (cut_x) by 5.3.

3) $(\prec_{Ls})$

$$\dfrac{\dfrac{x = q, p \prec q, \Gamma \mapsto \Delta, r(q)) \preceq s}{x = q, p \prec q, \Gamma \mapsto \Delta, r(p) \preceq s}\,(\prec_{Ls})}{x = q, p \prec x, \Gamma \mapsto \Delta, r(p) \preceq s}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{x = q, p \prec q, \Gamma \mapsto \Delta, r(q) \preceq s}{x = q, p \prec x, \Gamma \mapsto \Delta, r(q) \preceq s}\,(=_{Rax})}{x = q, p \prec x, \Gamma \mapsto \Delta, r(x) \preceq s}\,(=_{Ls})}{x = q, p \prec x, \Gamma \mapsto \Delta, r(p) \preceq s}\,(\prec_{Ls})$$

4) $(\prec_{Rs})$ is treated analogously with applications of $(\prec_{Rs})$ and $(=_{Rs})$.

5) (E_{ar}) – $t \notin \mathcal{V}$

$$\dfrac{\dfrac{z \prec t, r(t) = x, y \prec r(z), \Gamma \mapsto \Delta}{r(t) = x, y \prec r(t), \Gamma \mapsto \Delta}\,(E_{ar})}{r(t) = x, y \prec x, \Gamma \mapsto \Delta}\,(=_{Rax}) \quad\Rightarrow\quad \dfrac{\dfrac{\dfrac{z \prec t, r(z) = x, r(t) = x, y \prec r(z), \Gamma \mapsto \Delta}{z \prec t, r(z) = x, r(t) = x, y \prec x, \Gamma \mapsto \Delta}\,(=_{Rax})}{z \prec t, r(t) = x, r(t) = x, y \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{r(t) = x, r(t) = x, y \prec x, \Gamma \mapsto \Delta}\,(E_{ar})$$

6) All other rules allow trivial swap. $(=_{Lanx})$ modifies LHS of an inclusion so it does not interact with $(=_{Rax})$. The latter cannot affect the active equalities of $(=_{Ls})$.

QED.

We have thus obtained:

Proposition 5.7 $NEQ'_5 \equiv NEQ_4$.

Proof. $\Rightarrow$ is lemma 5.2. For the opposite implication we use lemma 4.1, i.e. the fact that $NEQ_4 \equiv NEQ_4 - (=_{Rs})$. The NEQ_4 axiom $x \prec t, \Gamma \mapsto \Delta, x \prec t$ is derivable in NEQ'_5 from the axiom $x \prec t, \Gamma \mapsto \Delta, t \prec t$ by a single application of $(\prec_{Ls})$. Admissibility of all the missing rules was shown above: (cut_x) in lemma 5.3, $(=_{Larx})$ in 5.4 and $(=_{Rax})$ in 5.6. (By lemma 4.1 wee don't need to show admissibility of $(=_{Rs})$ in NEQ'_5.) QED.

5.2 NEQ_5 – the calculus with no cut

We now remove the rule $(cut_{x=})$ and introduce two new rules $(\prec_{Lsix})$ and $(\prec_{Laix})$ – see fig. 5.

Axioms:

$$\Gamma \mapsto \Delta, t \prec t \qquad \Gamma \mapsto \Delta, x = x \qquad s = t, \Gamma \mapsto \Delta, s = t$$

Identity rules:

$$(=_{Ls})\ \frac{t = s, \Gamma \mapsto \Delta, p(s) \preceq q}{t = s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (=_{Lanx})\ \frac{s = t, p(s) \prec q, \Gamma \mapsto \Delta}{s = t, p(t) \prec q, \Gamma \mapsto \Delta}\quad p(t) \notin \mathcal{V}$$

$$(=_{Ranx})\ \frac{s = t, p \prec q(s), \Gamma \mapsto \Delta}{s = t, p \prec q(t), \Gamma \mapsto \Delta}\quad q(t) \notin \mathcal{V}$$

Inclusion rules:

$$(\prec_{Ls})\ \frac{t \prec s, \Gamma \mapsto \Delta, p(s) \preceq q}{t \prec s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (\prec_{La})\ \frac{s \prec t, p(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, p(t) \preceq q, \Gamma \mapsto \Delta}$$

$$(\prec_{Rs})\ \frac{s \prec t, \Gamma \mapsto \Delta, p \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, p \prec q(t)}$$

$$(\prec_{Lsix})\ \frac{t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}{t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q} \qquad (\prec_{Laix})\ \frac{t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\quad p(t) \notin \mathcal{V}$$

Elimination rules:

$$(E_{sr})\ \frac{\Gamma, x \prec t \mapsto \Delta, \phi[x]}{\Gamma \mapsto \Delta, \phi[t]} \qquad (E_{ar})\ \frac{x \prec t, y \prec r(x), \Gamma \mapsto \Delta}{y \prec r(t), \Gamma \mapsto \Delta}$$

- $x \notin \mathcal{V}(\Gamma, \Delta, t),\ t \notin \mathcal{V}$
- at most one x in ϕ;

- $x \notin \mathcal{V}(t, \Gamma, \Delta, y),\ t \notin \mathcal{V}$
- at most one occurrence of x in r

Specific cut rules:

for each specific axiom $\mathcal{A}_k$: $a_1, ..., a_n \mapsto s_1, ..., s_m$, a rule:

$$\frac{\Gamma \mapsto \Delta, a_1\ ; ...;\ \Gamma \mapsto \Delta, a_n\ ;\ s_1, \Gamma \mapsto \Delta\ ; ...;\ s_m, \Gamma \mapsto \Delta}{\Gamma \mapsto \Delta}\quad (Sp.cut_k)$$

Figure 5: The calculus NEQ_5 $(x, y \in \mathcal{V})$

Lemma 5.8 *Rules $(\prec_{Lsix})$ and $(\prec_{Laix})$ are admissible in NEQ_5', i.e., $NEQ_5 \Rightarrow NEQ_5'$.*

Proof. The activity of these rules can be simulated by the activity of $t = x$ which, then, can be reduced to $x = x$ and eliminated by $(cut_{x=})$. The cases for both rules are entirely analogous.

$$\dfrac{\begin{array}{c}\vdots\\ t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q\end{array}}{t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (\prec_{Lsix}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ t = x, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q\end{array}}{t = x, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (=_{Ls})}{x = x, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (\prec_{La})}{t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (cut_{x=})$$

and analogous case for $(\prec_{Laix})$ – here $p(t) \notin \mathcal{V}$:

$$\dfrac{\begin{array}{c}\vdots\\ t \prec x, p(x) \prec q, \Gamma \mapsto \Delta\end{array}}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (\prec_{Laix}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ t = x, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta\end{array}}{t = x, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (=_{Lanx})}{x = x, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La})}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (cut_{x=})$$

QED.

To show admissibility of $(cut_{x=})$ in NEQ_5, we will need

Lemma 5.9 *If* $NEQ_5 \vdash^D x \prec x, \Gamma \mapsto \Delta$ *then* $NEQ_5 \vdash^{D^*} \Gamma \mapsto \Delta$ *and* $h(D^*) \leq h(D)$.

Proof. The argument is exactly the same as in the proof of lemma 5.3. The new rule $(\prec_{Laix})$ cannot generate $x \prec x$ because of the restriction $p(t) \notin \mathcal{V}$. QED.

Lemma 5.10 *Admissibility of* $(cut_{x=})$: *If* $NEQ_5 \vdash^D x = x, \Gamma \mapsto \Delta$ *then* $NEQ_5 \vdash \Gamma \mapsto \Delta$.

Proof. By induction on $h(D)$. Instead of any axiom with $x = x$ in the antecedent, we may take the one without it. So, consider the last rule applied before the appearance of $x = x$, i.e., when it was the modified formula. The only possibility is $(\prec_{La})$. First we have the case when this rule was applied to an axiom, and the only problematic case is the following:

$$\frac{x = t, t \prec x, \Gamma \mapsto \Delta, x = t}{x = x, t \prec x, \Gamma \mapsto \Delta, x = t}\,(\prec_{La}) \quad \Rightarrow \quad \frac{t \prec x, \Gamma \mapsto \Delta, x = x}{t \prec x, \Gamma \mapsto \Delta, x = t}\,(\prec_{Ls})$$

(Notice that this case shows that $(cut_{x=})$ and $(\prec_{Ls})$ cannot be removed simultaneously – the case 1.3) of the proof of lemma 4.6 needed $(cut_{x=})$ to eliminate $(\prec_{Ls})$.) Consider the rule applied just above $(\prec_{La})$.

1) $(\prec_{La})$ – first there are two similar cases

$$\dfrac{\dfrac{x = p(t), p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}{x = p(s), p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = p(t), p(t) \prec x, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}{x = x, p(t) \prec x, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p(s) \prec x, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})$$

$$\dfrac{\dfrac{x = p(s), p(t) \prec x, t \prec s, \Gamma \mapsto \Delta}{x = p(s), p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = p(s), p(s) \prec t, p(t) \prec x, t \prec s, \Gamma \mapsto \Delta}{x = x, p(s) \prec t, p(t) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p(s) \prec t, p(s) \prec x, t \prec s, \Gamma \mapsto \Delta}\,(\prec_{La})$$

Then we have the following case:

$$\dfrac{\dfrac{\dfrac{\vdots}{t = t, t \prec x, \Gamma \mapsto \Delta}\,R}{x = t, t \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, t \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})$$

We consider the rule R – the only relevant cases are when it modified a formula resulting in $t = t$ or $t \prec x$.

1.1) The first case is the axiom:

$$\dfrac{\dfrac{t = t, t \prec x, \Gamma \mapsto \Delta, t = t}{x = t, t \prec x, \Gamma \mapsto \Delta, t = t}\,(\prec_{La})}{x = x, t \prec x, \Gamma \mapsto \Delta, t = t}\,(\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{t \prec x, \Gamma \mapsto \Delta, x = x}{t \prec x, \Gamma \mapsto \Delta, x = t}\,(\prec_{Ls})}{t \prec x, \Gamma \mapsto \Delta, t = t}\,(\prec_{Ls})$$

1.2) $(\prec_{La})$ – two analogously treated cases:

1.2.1) modified $t = t$

$$\dfrac{\dfrac{\dfrac{t(s) = t(p), p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}{t(s) = t(s), p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = t(s), p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{t(s) = t(p), p \prec s, t(s) \prec x, t(p) \prec x, \Gamma \mapsto \Delta}{x = t(p), p \prec s, t(s) \prec x, t(p) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(s) \prec x, t(p) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(s) \prec x, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})$$

1.2.2) modified $t \prec x$

$$\dfrac{\dfrac{\dfrac{t(s) = t(s), p \prec s, t(p) \prec x, \Gamma \mapsto \Delta}{t(s) = t(s), p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = t(s), p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{t(s) = t(s), p \prec s, t(p) \prec x, t(s) \prec x, \Gamma \mapsto \Delta}{x = t(s), p \prec s, t(p) \prec x, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(p) \prec x, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})}{x = x, p \prec s, t(s) \prec x, t(s) \prec x, \Gamma \mapsto \Delta}\,(\prec_{La})$$

1.3) $(=_{Lanx})$ – this case is entirely analogous to the case 1.2.2) of $(\prec_{La})$ and is treated by the same form of weakening.

1.4) $(\prec_{Laix})$ – modifying $t \prec x$, is again entirely analogous to the case 1.2.2).

1.5) All other rules allow trivial swaps.

2) $(=_{Lanx})$ – we have following cases

2.1) – the modified formula was $t \prec x$

$$\dfrac{\dfrac{x = p(t), p(y) \prec x, y = t, \Gamma \mapsto \Delta}{x = p(t), p(t) \prec x, y = t, \Gamma \mapsto \Delta}\ (=_{Lanx})}{x = x, p(t) \prec x, y = t, \Gamma \mapsto \Delta}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = p(t), p(t) \prec x, p(y) \prec x, y = t, \Gamma \mapsto \Delta}{x = x, p(t) \prec x, p(y) \prec x, y = t, \Gamma \mapsto \Delta}\ (\prec_{La})}{x = x, p(t) \prec x, p(t) \prec x, y = t, \Gamma \mapsto \Delta}\ (=_{Lanx})$$

2.2) – again with modified $t \prec x$

$$\dfrac{\dfrac{x = t, x \prec x, \Gamma \mapsto \Delta}{x = t, t \prec x, \Gamma \mapsto \Delta}\ (=_{Lanx})}{x = x, t \prec x, \Gamma \mapsto \Delta}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{x = t, t \prec x, \Gamma \mapsto \Delta}{x = x, t \prec x, \Gamma \mapsto \Delta}\ (\prec_{La})$$

The conversion is possible by weakening and admissibility of (cut_x), lemma 5.9, without increasing the length of the derivation.

2.3) – another modfied formula

$$\dfrac{\dfrac{x = t, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{x = t, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (=_{Lanx})}{x = x, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = t, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{x = x, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La})}{x = x, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (\prec_{Laix})$$

The application of $(\prec_{Laix})$ is legal since $p(t) \notin \mathcal{V}$.

2.4) – the case symmetric to the above

$$\dfrac{\dfrac{x = t, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}{x = t, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}\ (=_{Lanx})}{x = x, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = t, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}{x = x, t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La})}{x = x, t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}\ (\prec_{La})$$

3) $(\prec_{Laix})$ – this is similar to the case 2.1):

$$\dfrac{\dfrac{x = t(s), t(y) \prec x, s \prec y, \Gamma \mapsto \Delta}{x = t(s), t(s) \prec x, s \prec y, \Gamma \mapsto \Delta}\ (\prec_{Laix})}{x = x, t(s) \prec x, s \prec y, \Gamma \mapsto \Delta}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = t(s), t(y) \prec x, s \prec y, t(s) \prec y, \Gamma \mapsto \Delta}{x = x, t(y) \prec x, s \prec y, t(s) \prec y, \Gamma \mapsto \Delta}\ (\prec_{La})}{x = x, t(s) \prec x, s \prec y, t(s) \prec y, \Gamma \mapsto \Delta}\ (\prec_{Laix})$$

4) $(=_{Ls})$ – replacing x for t

$$\dfrac{\dfrac{x = t, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}{x = t, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (=_{Ls})}{x = x, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (\prec_{La}) \quad \Rightarrow \quad \dfrac{\dfrac{x = t, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}{x = x, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}\ (\prec_{La})}{x = x, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (\prec_{Lsix})$$

The dual case is even simpler not requiring the rule $(\prec_{Lsix})$

$$\dfrac{\dfrac{x = t, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}{x = t, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}\ (=_{Ls})}{x = x, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}\ (=_{Ls}) \quad \Rightarrow \quad \dfrac{\dfrac{x = t, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}{x = x, t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\ (\prec_{La})}{x = x, t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}\ (\prec_{Ls})$$

5) Other rules allow trivial swaps: $(=_{Ranx})$ and (E_{ar}) cannot result in $t \prec x$, $(\prec_{Ls})$ and $(\prec_{Rs})$ may involve active $t \prec x$ which remains unchanged and hence can be swapped; (E_{sr}) and $(Sp.cut)$ can be trivially swapped.

QED.

Proposition 5.11 $NEQ_5 \equiv NEQ_5'$.

Proof. $\Rightarrow$ is lemma 5.8. $\Leftarrow$ follows from lemma 5.10 – admissibility of $(cut_{x=})$ QED.

This is the end of the cut-elimination story. Combining the chain of equivalences of the intermediary calculi, summarised in theorem 4.13, and the propositions 5.7, 5.11 from this section, we conclude that NEQ_5 is a cut-free equivalent of NEQ.

In the following section, we study the special case of the calculus NEQ_5 without any specific axioms, showing further possibility of simplifications.

6 No specific axioms – no elimination rules

We have shown that calculus NEQ with specific axioms can be replaced by an equivalent calculus with specific cut rules. In the presence of the specific cut rules, the general (cut) can be eliminated. Obviously, derivations of tautologies in the absence of any specific axioms will not require the specific cut rules. Below we show a much less obvious and stronger result – when no specific axioms are present, one can even dispense with the elimination rules! Observe that these rules do have a cut-like flavor. Thus, in spite of the complications in the world of multialgebras, one can obtain the multialgebraic tautologies with quite a simple means of the identity and inclusion rules. Of course, when specific axioms are present, the elimination rules are indispensable since they play the role of substitution rules.

Letting NEQ_5^- be NEQ_5 without the $(Sp.cut)$ rules, we have the obvious:

Lemma 6.1 *If there are no specific axioms then $NEQ_5^- \equiv NEQ_5$.*

Redundancy of elimination rules will be shown using the following fact.

Lemma 6.2 *Assume that $NEQ_5^- \vdash^D \Gamma \mapsto \Delta$ and for an $x \in \mathcal{V}$*

1. *if an equation $p(x) = q \in \Delta$ then there is only this one occurrence of x in Δ,*
2. *if $x \prec r \in \Gamma$ then x is different from the LHS of any inclusion modified by an application of (E_{ar}) in D,*
3. *if $s \prec x \in \Gamma$ then x is different from any variable modified by $(\prec_{Lsix})$ in D, and*
4. *if $s \prec x \in \Gamma$ then x is different from any variable modified by $(\prec_{Laix})$ in D*

then $NEQ_5^- \vdash^{D^*} (\Gamma \mapsto \Delta)^x_t$ (with all *x's replaced by t)*

Proof. The intention of these conditions is to enable exactly the same derivations with t occurring instead of x. The case 1 prevents us from concluding derivability of $\Gamma \mapsto \Delta, t = t$ from the fact that $\Gamma \mapsto \Delta, x = x$, being an axiom, is derivable. In all other cases, we can start derivation with an axiom where t occurs instead of x. All the rules applied in D and involving x can be applied in the same order in D^* with t instead of x. The possible exceptions are:

$$D\left\{\begin{array}{c}\vdots\\ \dfrac{y \prec q, x \prec r(y), \Gamma \mapsto \Delta}{x \prec r(q), \Gamma \mapsto \Delta}\end{array}\right. \qquad D\left\{\begin{array}{c}\vdots\\ \dfrac{s \prec x, \Gamma \mapsto \Delta, p(s) \preceq q}{s \prec x, \Gamma \mapsto \Delta, p(x) \preceq q}\end{array}\right. \qquad D\left\{\begin{array}{c}\vdots\\ \dfrac{s \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{s \prec x, p(s) \prec q, \Gamma \mapsto \Delta}\end{array}\right.$$

$$(E_{ar}) \qquad\qquad (\prec_{Lsix}) \qquad\qquad (\prec_{Laix})$$

The sequent resulting from (E_{ar}) cannot be obtained with t substituted for x by the application of (E_{ar}) unless t is a variable. This case, however, is excluded by the condition 2. Similarly, sequents resulting from $(\prec_{Lsix})$, resp. $(\prec_{Laix})$, cannot be obtained with t instead of x if t isn't a variable. Conditions 3, resp. 4, exclude these cases. QED.

Definition 6.3 *NEQ_6' is NEQ_5^- but without the elimination rules (E_{sr}) and (E_{ar}).*

The rules of the calculus NEQ_6' for the case without specific axioms are given in figure 6.

Remark 6.4 *Revisiting the proof of lemma 4.10, we see that the transformations used there did not introduce any new applications of the elimination rules (neither did lemma 4.2 used in this proof). The same holds (trivially) for the proof of lemma 5.9 (cf. lemma 5.3). Hence:*

1. *(cut_x) is admissible in NEQ_6' (cf. lemma 5.9, 5.3);*
2. *(cut_t) is admissible in NEQ_6' (cf. lemma 4.10);*
3. *lemma 6.2 holds for NEQ_6'.*

Using this observation, we get:

Proposition 6.5 *In the absence of specific axioms $NEQ_5^- \equiv NEQ_6'$*

Proof. $\Leftarrow$ is obvious, and for $\Rightarrow$ we have to show admissibility of the elimination rules in NEQ_6'. Consider the uppermost application of an elimination rule E:

Axioms:

$$\Gamma \mapsto \Delta, t \prec t; \qquad \Gamma \mapsto \Delta, x = x; \qquad s = t, \Gamma \mapsto \Delta, s = t$$

Identity rules:

$$(=_{Ls})\ \frac{t = s, \Gamma \mapsto \Delta, p(s) \preceq q}{t = s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (=_{Lanx})\ \frac{s = t, p(s) \prec q, \Gamma \mapsto \Delta}{s = t, p(t) \prec q, \Gamma \mapsto \Delta}\ p(t) \notin \mathcal{V}$$

$$(=_{Ranx})\ \frac{s = t, p \prec q(s), \Gamma \mapsto \Delta}{s = t, p \prec q(t), \Gamma \mapsto \Delta}\ q(t) \notin \mathcal{V}$$

Inclusion rules:

$$(\prec_{Ls})\ \frac{t \prec s, \Gamma \mapsto \Delta, p(s) \preceq q}{t \prec s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (\prec_{La})\ \frac{s \prec t, p(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, p(t) \preceq q, \Gamma \mapsto \Delta}$$

$$(\prec_{Rs})\ \frac{s \prec t, \Gamma \mapsto \Delta, p \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, p \prec q(t)}$$

$$(\prec_{Lsix})\ \frac{t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}{t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q} \qquad (\prec_{Laix})\ \frac{t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\ p(t) \notin \mathcal{V}$$

Figure 6: The rules of NEQ'_6 $(x \in \mathcal{V})$ – no specific axioms

1) E is (E_{ar}):

$$\frac{D_1 \left\{ \begin{array}{c} \vdots \\ x \prec t, y \prec r(x), \Gamma \mapsto \Delta \end{array} \right.}{y \prec r(t), \Gamma \mapsto \Delta}\ (E_{ar}) \quad \Rightarrow \quad \frac{D'_1 \left\{ \begin{array}{c} \vdots \\ t \prec t, y \prec r(t), \Gamma \mapsto \Delta \end{array} \right.}{y \prec r(t), \Gamma \mapsto \Delta}\ (cut_t)$$

The application of (E_{ar}) implies $x \notin \mathcal{V}(t, y, \Gamma, \Delta)$, and hence the conditions of lemma 6.2 are satisfied. Applying it to D_1, we obtain D'_1 and (cut_t) is admissible by remark 6.4.

2) E is (E_{sr}):

$$\frac{D_1 \left\{ \begin{array}{c} \vdots \\ x \prec t, \Gamma \mapsto \Delta, \phi[x] \end{array} \right.}{\Gamma \mapsto \Delta, \phi[t]}\ (E_{sr}) \quad \Rightarrow \quad \frac{D_1 \left\{ \begin{array}{c} \vdots \\ t \prec t, \Gamma \mapsto \Delta, \phi[t] \end{array} \right.}{\Gamma \mapsto \Delta, \phi[t]}\ (cut_t)$$

By the restrictions on (E_{sr}), there is at most one occurrence of x in $\Gamma, \Delta, \phi[x]$ so, again, the conditions of lemma 6.2 hold and we can apply it to D_1 obtaining D'_1, which leads to the desired conclusion by admissibility of (cut_t).

QED.

Lemma 6.6 *The rule* $(\prec_{Rs})$ *is redundant.*

Proof.

1) Axiom

$$\frac{s \prec t, \Gamma \mapsto \Delta, q(s) \prec q(s)}{s \prec t, \Gamma \mapsto \Delta, q(s) \prec q(t)}\ (\prec_{Rs}) \quad \Rightarrow \quad \frac{s \prec t, \Gamma \mapsto \Delta, q(t) \prec q(t)}{s \prec t, \Gamma \mapsto \Delta, q(s) \prec q(t)}\ (\prec_{Ls})$$

2) $(\prec_{Laix})$

$$\cfrac{\cfrac{t \prec x, p(x) \prec q, \Gamma \mapsto \Delta, r \prec f(p(t))}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta, r \prec f(p(t))}\ (\prec_{Laix})}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta, r \prec f(q)}\ (\prec_{Rs}) \quad \Rightarrow \quad \cfrac{\cfrac{t \prec x, p(x) \prec q, p(t) \prec q, \Gamma \mapsto \Delta, r \prec f(p(t))}{t \prec x, p(x) \prec q, p(t) \prec q, \Gamma \mapsto \Delta, r \prec f(q)}\ (\prec_{Rs})}{t \prec x, p(t) \prec q, p(t) \prec q, \Gamma \mapsto \Delta, r \prec f(q)}\ (\prec_{Laix})$$

3) $(\prec_{La})$, $(=_{Lanx})$, $(=_{Ranx})$ – these are treated by the same kind of weakening as the above case of $(\prec_{Laix})$

4) The remaining rules modify at most LHS of a formula in the consequent and thus allow trivial swaps.

QED.

Thus all multialgebraic tautologies are derivable with the calculus NEQ_6 given in figure 7.

Axioms:

$$\Gamma \mapsto \Delta, t \prec t; \qquad \Gamma \mapsto \Delta, x = x; \qquad s = t, \Gamma \mapsto \Delta, s = t$$

Identity rules:

$$(=_{Ls})\ \frac{t = s, \Gamma \mapsto \Delta, p(s) \preceq q}{t = s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (=_{Lanx})\ \frac{s = t, p(s) \prec q, \Gamma \mapsto \Delta}{s = t, p(t) \prec q, \Gamma \mapsto \Delta}\quad p(t) \notin \mathcal{V}$$

$$(=_{Ranx})\ \frac{s = t, p \prec q(s), \Gamma \mapsto \Delta}{s = t, p \prec q(t), \Gamma \mapsto \Delta}\quad q(t) \notin \mathcal{V}$$

Inclusion rules:

$$(\prec_{Ls})\ \frac{t \prec s, \Gamma \mapsto \Delta, p(s) \preceq q}{t \prec s, \Gamma \mapsto \Delta, p(t) \preceq q} \qquad (\prec_{La})\ \frac{s \prec t, p(s) \preceq q, \Gamma \mapsto \Delta}{s \prec t, p(t) \preceq q, \Gamma \mapsto \Delta}$$

$$(\prec_{Lsix})\ \frac{t \prec x, \Gamma \mapsto \Delta, p(t) \preceq q}{t \prec x, \Gamma \mapsto \Delta, p(x) \preceq q} \qquad (\prec_{Laix})\ \frac{t \prec x, p(x) \prec q, \Gamma \mapsto \Delta}{t \prec x, p(t) \prec q, \Gamma \mapsto \Delta}\quad p(t) \notin \mathcal{V}$$

Figure 7: The rules of NEQ_6 ($x \in \mathcal{V}$) – no specific axioms

7 Other connectives

Since satisfaction of a sequent is defined in the usual way, soundness/completeness of NEQ and its equivalence with NEQ_5 imply that the entailment relation in NEQ_5 coincides with the semantic consequence. If we restrict our formulae ϕ to the atomic ones (i.e., $s \prec t$ or $s = t$), this means that, given a specification with specific axioms $\mathcal{A} = \{\phi_1, ..., \phi_n\}$, $MMod(\mathcal{A}) \models \phi \Leftrightarrow NEQ_5 \vdash \mathcal{A} \mapsto \phi$. Since the specific axioms are thus incorporated into the actual sequents to be proved, we need no (*Sp.cut*) rules and can conclude that

$$MMod(\mathcal{A}) \models \phi \iff NEQ_6 \vdash \mathcal{A} \mapsto \phi$$

Notice that this formulation, however simple and convincing, requires us to "guess" the variable names which have to be used in ϕ in order to "match" the appropriate variable names in the axioms.

Furthermore, we may extend the language introducing other logical operators in the usual way. We obtain a sound and complete system NEQ_i^+ (for $i \leq 6$) by extending NEQ_i with the standard rules

$$(\neg \mapsto)\ \frac{\Gamma \mapsto \Delta, \phi}{\Gamma, \neg\phi \mapsto \Delta} \qquad (\mapsto \neg)\ \frac{\Gamma, \phi \mapsto \Delta}{\Gamma \mapsto \Delta, \neg\phi}$$

$$(\vee \mapsto)\ \frac{\Gamma, \phi_1 \mapsto \Delta \;\; ; \;\; \Gamma, \phi_2 \mapsto \Delta}{\Gamma, \phi_1 \vee \phi_2 \mapsto \Delta} \qquad (\mapsto \vee)\ \frac{\Gamma \mapsto \Delta, \phi_1, \phi_2}{\Gamma \mapsto \Delta, \phi_1 \vee \phi_2}$$

$$(\wedge \mapsto)\ \frac{\Gamma, \phi_1, \phi_2 \mapsto \Delta}{\Gamma, \phi_1 \wedge \phi_2 \mapsto \Delta} \qquad (\mapsto \wedge)\ \frac{\Gamma \mapsto \Delta, \phi_1 \;\; ; \;\; \Gamma \mapsto \Delta, \phi_2}{\Gamma \mapsto \Delta, \phi_1 \wedge \phi_2}$$

$$(\Rightarrow \mapsto)\ \frac{\Gamma, \phi_1 \mapsto \Delta \;\; ; \;\; \Gamma \mapsto \Delta, \phi_2}{\Gamma, \phi_2 \Rightarrow \phi_1 \mapsto \Delta} \qquad (\mapsto \Rightarrow)\ \frac{\Gamma, \phi_1 \mapsto \Delta, \phi_2}{\Gamma \mapsto \Delta, \phi_1 \Rightarrow \phi_2}$$

We then have that, for a set of specific axioms $\mathcal{A} = \{\Phi_1, ..., \Phi_n\}$ and a formula Φ, all built from the atomic equalities and inclusions with the logical operators, $MMod(\mathcal{A}) \models \Phi \Leftrightarrow NEQ_5^+ \vdash \mathcal{A} \mapsto \Phi$. A proof in NEQ_5^+ will typically consist of a series of applications of the above eight rules leading to a sequent $\Gamma \mapsto \Delta$, to which one then applies the rules of NEQ_5. Here again axioms are incorporated into the formulae, so we can use NEQ_6 instead:

$$MMod(\mathcal{A}) \models \Phi \iff NEQ_6^+ \vdash \mathcal{A} \mapsto \Phi$$

8 Related work.

The first, to our knowledge, reasoning system for multialgebras was introduced by Hußmann in [Hus90]. It dealt only with simple (atomic) inclusions and, more significantly, was not complete. A complete extension of this calculus was given in [Wal93]. The present system NEQ_5 is a significant improvement of the original calculi from [Wal93] and [WM95b], particularly, due to cut-freeness.

Another cut-free, Gentzen-style calculus was introduced in [BK95]. It addresses proofs of tautologies (no specific axioms) and so it is a counterpart of our NEQ_6.

- The main difference is that [BK95] address partial multialgebras, i.e. ones where some terms may denote empty set. Our calculi work only for total multialgebras.
- There are two differences in the language. The first one concerns the special "let" term-construction: '$let\ x := t\ in\ f(x,x)$' is a possible term. Although absent from our language, it is expressible, except for the cases when a term with *let* occurs in the RHS of inclusion.
- On the other hand, our system deals not only with inclusion, but also with the determinisitc equality which is not a part of the language in [BK95]. Our results show that, restricting the language to inclusions, the only rules needed are the first axiom and the inclusion rules from figure 7.
- The proofs in the calculus from [BK95] proceed essentially in a bottom-up fashion by 1) replacing (sub)terms of concern by variables and introducing additional assumptions about the relations between them, and then 2) performing the deductions using rules handling variables. In general, this leads to complex proof-*trees* – in NEQ_6 all the proofs are linear.
- The restrictions on the inclusion and elimination rules are essentially the same in both systems. However, we have made it explicit that the elimination rules (which are present in [BK95], even though no specific axioms are admitted) are needed only in the presence of specific axioms – when no such axioms are involved they can be eliminated and NEQ_6 is a complete system for deriving all multialgebraic tautologies.

The sufficiency of NEQ_6 is a significant strengthening of these earlier results. In addition, the cut-freeness of the system from [BK95] corresponds to the situations when no specific axioms are present (i.e., cut-freeness of NEQ_6). We have shown how such axioms can be treated explicitly by the more specific cut rules instead of general cut. Finally, other rules of the calculus have been given very specific restrictions. All these results enhance significantly proof search.

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