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On the Growth Rate of Convolutional Codes

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Abstract

Let $\mathcal{C}$ be a rate $(n-r)/n$ convolutional code with canonical parity-check matrix $\mathbf{H}(D)$, and let w_0 denote the minimum average weight per edge over all nonzero cycles in the state diagram for $\mathcal{C}$. The rate of growth per edge of the minimum distance between two nonmerged codewords is given by the value of w_0. For any code $\mathcal{C}$, we show how to obtain an upper bound on w_0 from the columns of $\mathbf{H}(D)$.

Index Terms—convolutional codes, minimum average weight per edge.

I. Introduction

In this paper we investigate how fast the minimum weight of an unmerged codeword grows as a function of the codeword length. We are particularly interested in determining the growth rates for the codes constructed in [3].

We denote a convolutional code with symbols in F=GF(2) by $(n, n-r, m, d_{free})$ where n is the block length, $n-r$ is the dimension, m is the code degree (or constraint length), and d_{free} is the free distance [7, Sec. 3]. A code will be defined by a canonical parity-check matrix, $\mathbf{H}(D)$. The maximum degree of the polynomials in the ith row of $\mathbf{H}(D)$ is denoted by e_i, $1 \le i \le r$, and called the ith *row degree*. We assume that the rows of any matrix $\mathbf{H}(D)$ are ordered such that the row degrees satisfy $e_1 = e_2 = \cdots = e_\gamma = 0$ for some γ, $0 \le \gamma < r$, and $1 \le e_{\gamma+1} \le e_{\gamma+2} \le \cdots \le e_r$. All ordered canonical parity-check matrices for a

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given convolutional code have the same row degrees and the sum $e_1+e_2+\cdots+e_r$ of the row degrees is equal to the code degree m [7, Sec. 3].

If $\gamma > 0$, then the γ first rows of $\mathbf{H}(D)$ define a parity-check matrix for an $[n, n-\gamma]$ binary block code, $\mathcal{E}$, called the *embedded block code* [6]. For $\gamma = 0$, we define $\mathcal{E}$ to be the trivial $[n, n]$ block code consisting of all possible binary n-tuples. The minimum distance of $\mathcal{E}$ is denoted by $d_{\mathcal{E}}$. Note that if $\gamma = 0$, then $d_{\mathcal{E}} = 1$.

The labels on the edges in the state diagram obtained from a canonical parity-check matrix are the codewords in $\mathcal{E}$ [6]. Observe that no nonzero label can have weight less than $d_{\mathcal{E}}$. As an example, the 2×3 canonical parity-check matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1+D & D & 1 \end{bmatrix}$$

has row degrees $e_1 = 0$ and $e_2 = 1$, i.e. $\gamma = 1$. The first row of the matrix defines a $[3, 2]$ embedded block code $\mathcal{E}$ with $d_{\mathcal{E}} = 2$.

A path of length $\tau\,(\geq 1)$ in a state diagram consists of τ consecutive edges and the corresponding labels. A cycle of length τ is a path of length τ that starts and ends in the same state s and whose other states are different from each other as well as s. The average weight per edge of a cycle is equal to the total Hamming weight of all labels on the edges divided by τ. We denoted by w_0 the *minimum average weight per edge* over all cycles in a state diagram, excluding the cycle of length one and weight zero that starts and ends in the zero state.

The minimum weight nonzero codeword of length τ which is unmerged with the zero state is obtained, when τ is large, by first following the minimum weight connecting path from the zero state to the cycle with average weight w_0 and then repeating this cycle until τ edges have been traversed [8]. Hence, as pointed out in [2], the weight of the minimum weight unmerged codeword of length τ can be approximated by $w(\tau) \approx w_0\tau$ for large τ.

It was shown in [3] that any $(n, n-r, m)$ code has $w_0 \geq d_{\mathcal{E}}/e_{max}$ where $e_{max} = \max\{e_i\}$. To upper bound w_0, we have found it very convenient to use a novel vector notation for $(n, n-r, m)$ codes first introduced by Ytrehus [9]. A somewhat simplified version of this notation is presented in Section II.

In Section III, the vector notation is used to show how w_0 for an arbitrary $(n, n-r, m)$ code, $\mathcal{C}$, can be upper bounded by studying the columns of a canonical parity-check matrix for $\mathcal{C}$. The bounding technique is a generalization of a technique for $(n, n-1, m)$ codes given in [4]. To demonstrate the usefulness of the technique, we prove that, for each value of e_{max}, there exist codes with least possible $w_0 = 1/e_{max}$ for all values of the code parameters $(n, n-r, m)$. It is furthermore shown that all the codes with free distances 3, 4 and 5 constructed in [3] have $w_0 = 1$ for all m.

II. A general vector notation

Let an $(n, n-r, m)$ convolutional code $\mathcal{C}$ be defined by a $r \times n$ canonical parity-check matrix $\mathbf{H}(D)$, and let $h_j^{(i)}(D) = h_{j0}^{(i)} + h_{j1}^{(i)} D + h_{j2}^{(i)} D^2 + \cdots + h_{je_i}^{(i)} D^{e_i}$, $h_{js}^{(i)} \in \mathrm{F}$, denote the jth polynomial in the ith row of $\mathbf{H}(D)$. The coefficients of the polynomial define a *column* vector $\mathbf{h}_j^{(i)} = (h_{j0}^{(i)}\, h_{j1}^{(i)}\, h_{j2}^{(i)} \cdots h_{je_i}^{(i)})^T$. Together, the n polynomials in the ith row of the parity-check matrix define a $(e_i+1) \times n$ binary matrix $\mathbf{H}^{(i)} = [\mathbf{h}_1^{(i)} \cdots \mathbf{h}_n^{(i)}]$.

Let $\mathbf{x} = (x_a x_{a+1} \cdots x_b)^T$ be a $(b-a+1)$-dimensional binary column vector. The yth shift of $\mathbf{x}$, denoted by $\mathbf{x}^{\overset{y}{\leftarrow}}$, is defined by $\mathbf{x}^{\overset{y}{\leftarrow}} = (x_{a+y} \cdots x_b 0 \cdots 0)^T$ where $y > 0$. Note that y zeros are added to the bottom of the column vector as its bits are shifted up. As an example, let $\mathbf{x} = (10010111)^T$, then $\mathbf{x}^{\overset{3}{\leftarrow}} = (10111000)^T$.

Let $\mathbf{u}_t = (u_t^1 \cdots u_t^n)^T$ denote a binary *column* vector. A codeword that "starts" at time $t = 0$ can be represented by a semi-infinite sequence $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1, \cdots)$. The ith *syndrome vector* $\mathbf{S}_t^{(i)} = (s_{t,0}^{(i)} \cdots s_{t,e_i}^{(i)})^T$ at time t can now be defined as

$$\mathbf{S}_t^{(i)} = (\mathbf{S}_{t-1}^{(i)})^{\overset{1}{\leftarrow}} + \mathbf{H}^{(i)} \mathbf{u}_t, \ t \geq 0, \tag{1}$$

with $\mathbf{S}_{-1}^{(i)}$ equal to some fixed vector, usually the all-zero vector, $1 \leq i \leq r$. The ith *syndrome sequence* is equal to $(s_{0,0}^{(i)}\, s_{1,0}^{(i)} \cdots)$ where $s_{t,0}^{(i)}$, $t \geq 0$, is the zeroth element in the (e_i+1)-dimensional syndrome vector $\mathbf{S}_t^{(i)}$.

The rows of the matrix $\mathbf{H}(D)$ define r binary $(e_i+1) \times n$ matrices $\mathbf{H}^{(i)}$. Since a binary sequence $\mathbf{u}$ is a codeword if and only if all the r syndrome sequences obtained from (1) are equal to the all-zero sequence, the code is given by $\mathcal{C} = \{\mathbf{u} \mid s_{t,0}^{(i)} = 0, t \geq 0, 1 \leq i \leq r\}$.

An example illustrates how the expression in (1) can be used to determine subsequences of codewords. Consider the $(4, 2, 4)$ convolutional code defined by the canonical parity-check matrix

$$\mathbf{H}(D) = \begin{bmatrix} 1+D+D^2 & 1+D^2 & 1+D & D+D^2 \\ 0 & D & 1+D+D^2 & 1+D+D^2 \end{bmatrix}.$$

The coefficients of the polynomials in the two rows define two 3×4 binary matrices

$$\mathbf{H}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \ \mathbf{H}^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let the syndrome vectors at time $t-1$ be $\mathbf{S}_{t-1}^{(1)} = (010)^T$ and $\mathbf{S}_{t-1}^{(2)} = (001)^T$. Furthermore, let $\mathbf{u}_t = (0100)^T$ and $\mathbf{u}_{t+1} = (0000)^T$. Then, from (1), the syndrome vectors at time t are

$$\mathbf{S}_t^{(1)} = (100)^T + (101)^T = (001)^T, \ \mathbf{S}_t^{(2)} = (010)^T + (010)^T = (000)^T,$$

and the syndrome vectors at time $t+1$ are

$$\mathbf{S}_{t+1}^{(1)} = (010)^T + (000)^T = (010)^T, \ \mathbf{S}_{t+1}^{(2)} = (000)^T + (000)^T = (000)^T.$$

The binary vectors, or labels, $\mathbf{u}_t$ and $\mathbf{u}_{t+1}$ are contained in a codeword since the first symbol in each syndrome vector is equal to zero, i.e., $s_{t,0}^{(1)} = s_{t,0}^{(2)} = s_{t+1,0}^{(1)} = s_{t+1,0}^{(2)} = 0$.

Notice that when a label $\mathbf{u}_{t+1}$ is the all-zero label, then the new syndrome vectors $\mathbf{S}_{t+1}^{(i)}$ are obtained by simply taking a shift of the old syndrome vectors $\mathbf{S}_t^{(i)}$. We will use this observation repeatedly in the next section.

A more compact form of the notation is also available. Let $\mathbf{H}$ be the combined binary matrix whose e_1+1 first rows are the rows of $\mathbf{H}^{(1)}$, the next e_2+1 rows are the rows of $\mathbf{H}^{(2)}$, and so on for all $\mathbf{H}^{(i)}$ matrices, $1 \le i \le r$. Note that the combined matrix has $\sum_{i=1}^{r}(e_i+1) = m+r$ rows and n columns. Furthermore, let $\mathbf{S}_t = (\mathbf{S}_t^{(1)}, \cdots, \mathbf{S}_t^{(r)})^T$ be the combined syndrome vector at time t, and let $\mathbf{S}_t^{\overset{1}{\leftarrow}} = ((\mathbf{S}_t^{(1)})^{\overset{1}{\leftarrow}}, \cdots, (\mathbf{S}_t^{(r)})^{\overset{1}{\leftarrow}})$. The combined syndrome vector at time t can now be written as

$$\mathbf{S}_t = (\mathbf{S}_{t-1})^{\overset{1}{\leftarrow}} + \mathbf{H}\mathbf{u}_t, \ t \ge 0. \tag{2}$$

The combined syndrome vector in (2) contains $\sum_{i=1}^{r}(e_i+1) = m+r$ bits. Since $\mathbf{S}_t$ is a syndrome vector for a codeword if and only if the r subvectors $\mathbf{S}_t^{(i)}$ have $s_{t,0}^{(i)} = 0$, $\forall i, t$, there are 2^m possible syndrome vectors. It should be clear that we can obtain a state diagram representation for any convolutional code from (2) where each state in the diagram represents a combined syndrome vector.

III. Upper bounds on w_0

In the following we show how w_0 for an arbitrary $(n, n-r, m)$ code $\mathcal{C}$ can be upper bounded studying the columns of the combined matrix $\mathbf{H}$. Several of the results stated below are generalizations of results for $(n, n-1, m)$ codes presented in [4], [5].

Consider the state diagram for $\mathcal{C}$ obtained from (2). For any $\tau \in \{0, 1, \cdots e_{max} - 1\}$, let

$$\mathbf{B}_\tau = (s_0^{(1)} \cdots s_{e_1}^{(1)},\ s_0^{(2)} \cdots s_{e_2}^{(2)},\ \cdots,\ s_0^{(r)} \cdots s_{e_r}^{(r)})^T$$

denote a nonzero state in the state diagram such that $s_j^{(i)} = 0$ for all $0 \leq j \leq \min\{\tau, e_i\}$ and all $1 \leq i \leq r$, and such that there exists at least one value of i for which $s_{\tau+1}^{(i)} = 1$ (we have removed the subscript t from the elements in $\mathbf{B}_\tau$ to simplify the notation). In other words, the ith subvector of $\mathbf{B}_\tau$ has $\min\{\tau, e_i\} + 1$ leading zeros. As an example, for $r = 2$, $e_1 = e_2 = 3$, and $\tau = 1$, two possible states are $\mathbf{B}_1 = (0010,\ 0001)^T$ and $\widetilde{\mathbf{B}}_1 = (0000,\ 0011)^T$.

A. Analysis of codes with $e_{max} > 1$

Lemma 1: Let $d(\tau) = \sum_{i:e_i \geq \tau}(e_i - \tau)$ for $\tau \in \{1, 2, \cdots, e_{max} - 1\}$ and $e_{max} > 1$. The state diagram for $\mathcal{C}$ contains $2^{d(\tau)} - 1$ distinct all-zero paths of length τ whose states are all nonzero. Each path starts in some state $\mathbf{B}_\tau$ and ends in state $(\mathbf{B}_\tau)^{\overset{\tau}{\leftarrow}}$.

Remark: Forney [1, Theorem 2] was the first to prove a nearly identical result and Lemma 1 should therefore be attributed to him. The difference is that we only consider all-zero paths that do not contain the zero state. Our Lemma makes explicit which states are the starting states of the all-zero paths. This information is needed to upper bound w_0.

Proof: We first note that all shifts $(\mathbf{B}_\tau)^{\overset{i}{\leftarrow}}$ for $i = 1, 2, \cdots, \tau$ result in valid state vectors since, for each value of i, all r subvectors of $(\mathbf{B}_\tau)^{\overset{i}{\leftarrow}}$ have zeroth elements equal to zero. Setting $\mathbf{S}_0 = \mathbf{B}_\tau$ and $\mathbf{u}_t = (0 \cdots 0)^T$ in (2), we find that the new state for each time instant $t = 1, 2, \cdots \tau$ is given by $\mathbf{S}_t = (\mathbf{B}_\tau)^{\overset{t}{\leftarrow}}$. Consequently, there is an all-zero path from $\mathbf{B}_\tau$ to $(\mathbf{B}_\tau)^{\overset{\tau}{\leftarrow}}$.

The number of starting states $\mathbf{S}_0$ containing $\tau + 1$ leading zeros in each of the r subvectors is $2^{d(\tau)}$ since the $e_i - \tau$ last elements in each subvector can be chosen freely. This number must be reduced by one since we require that $\mathbf{S}_0$ is different from the zero state. Q.E.D.

Let the jth (combined) column of a combined matrix $\mathbf{H}$ be denoted by

$$\mathbf{h}_j = (h_{j0}^{(1)} h_{j1}^{(1)} \cdots h_{je_1}^{(1)},\ h_{j0}^{(2)} h_{j1}^{(2)} \cdots h_{je_2}^{(2)},\ \cdots,\ h_{j0}^{(r)} h_{j1}^{(r)} \cdots h_{je_r}^{(r)})^T.$$

We are now ready to prove the following upper bound on w_0.

Theorem 1: Let $\mathbf{H}$ be a combined matrix with maximum row degree $e_{max} > 1$. If there exists a sum of $l \geq d_{\mathcal{E}}$ columns in $\mathbf{H} = [\mathbf{h}_j]$ such that

$$\sum_{k=1}^{l} \mathbf{h}_{j_k} = \mathbf{B}_\tau + (\mathbf{B}_\tau)^{\overset{\tau+1}{\longleftarrow}} \tag{3}$$

for some state $\mathbf{B}_\tau$, then $w_0 \leq l/(\tau+1)$.

Proof: From Lemma 1, if $e_{max} > 1$ then the state diagram obtained from (2) contains an all-zero path of length $\tau \geq 1$ from a nonzero state $\mathbf{B}_\tau$ to the state $(\mathbf{B}_\tau)^{\overset{\tau}{\leftarrow}}$. The all-zero path is on a cycle of length $\tau+1$ if there is a branch from $(\mathbf{B}_\tau)^{\overset{\tau}{\leftarrow}}$ back to $\mathbf{B}_\tau$. It follows from (2) that this is the case if for some l there is a sum of columns $\sum_{k=1}^{l} \mathbf{h}_{j_k}$ such that

$$\mathbf{B}_\tau = ((\mathbf{B}_\tau)^{\overset{\tau}{\leftarrow}})^{\overset{1}{\leftarrow}} + \sum_{k=1}^{l} \mathbf{h}_{j_k} = (\mathbf{B}_\tau)^{\overset{\tau+1}{\longleftarrow}} + \sum_{k=1}^{l} \mathbf{h}_{j_k},$$

or equivalently,

$$\sum_{k=1}^{l} \mathbf{h}_{j_k} = \mathbf{B}_\tau + (\mathbf{B}_\tau)^{\overset{\tau+1}{\longleftarrow}}.$$

The average weight per branch of this cycle is $l/(\tau+1)$ since the cycle has τ all-zero labels and one label of weight l. Consequently, $w_0 \leq l/(\tau+1)$. Since the labels in the state diagram are the codewords in the embedded block code $\mathcal{E}$ [6], any nonzero label has weight at least $d_{\mathcal{E}}$ and $l \geq d_{\mathcal{E}}$. Q.E.D.

Theorem 1 defines a simple technique for determining upper bounds on w_0 by combining columns in canonical parity-check matrices. To illustrate the usefulness of this technique, we prove two corollaries.

Corollary 1: If the combined matrix $\mathbf{H}$ has row degrees $1 \leq e_1 \leq \cdots \leq e_r = e_{max}$ with $e_{max} > 1$, and one combined column $\mathbf{h}_j$ is equal to $(0\cdots 0, \cdots, 0\cdots 0, 10\cdots 01)^T$, then $w_0 = 1/e_{max}$, and no other code with the same maximum row degree e_{max} can have smaller w_0.

Proof: For $\tau = e_{max} - 1 = e_r - 1$, there exists a nonzero state

$$\mathbf{B}_{e_r - 1} = (0\cdots 0, \cdots, 0\cdots 0, 0\cdots 01)^T$$

in the state diagram. For this particular state, the right-hand side of (3) is given by $\mathbf{B}_{e_r-1} + (\mathbf{B}_{e_r-1})^{\overset{e_r}{\leftarrow}} = (0\cdots0,\cdots,0\cdots0,10\cdots01)^T$. Since $l \geq d_{\mathcal{E}} = 1$ and there exists a column in $\mathbf{H}$ such that the left-hand side of (3) is also equal to $(0\cdots0,\cdots,0\cdots0,10\cdots01)^T$, we have $w_0 \leq 1/e_{max}$. On the other hand, it follows from Lemma 1 that the longest all-zero path nowhere merged with the zero state has length $e_{max}-1$. Hence, a path of length e_{max} has weight at least one and $w_0 \geq 1/e_{max}$. Q.E.D.

In [3] we constructed a general parity-check matrix $\mathbf{H}_m^*(D)$ (valid for all m) with row degrees $e_1 = 0$, $e_2 = \cdots e_{m-1} = 1$ and $e_m = 2$. The matrix defines a $(2^{2m-1}, 2^{2m-1}-m, m, 4)$ convolutional code with $w_0 \geq 1$ for any $m \geq 2$. The embedded block code has minimum distance $d_{\mathcal{E}} = 2$. We use the fact that $\mathbf{H}_m^*(D)$ contains the columns $E_{1+2^{2m-3}}^T = (1,00,\cdots,00,100)^T$ and $E_{1+2^{2m-1}}^T = (1,00,\cdots,00,001)^T$ to prove that $w_0 = 1$.

Corollary 2: The $(2^{2m-1}, 2^{2m-1}-m, m, 4)$ convolutional codes constructed in [3] have $w_0 = 1$ for all $m \geq 2$.

Proof: We first note that the canonical parity-check matrix $\mathbf{H}_m^*(D)$ has maximum row degree $e_{max} = 2$. For $\tau = e_{max} - 1 = 1$, there exists a nonzero state $\mathbf{B}_1 = (0,00,\cdots,00,001)^T$. The right-hand side of (3) for this state is equal to $\mathbf{B}_1 + (\mathbf{B}_1)^{\overset{2}{\leftarrow}} = (0,00,\cdots,00,101)^T$. On the other hand, the sum of the $l = 2$ columns $E_{1+2^{2m-3}}^T$ and $E_{1+2^{2m-1}}^T$ is also equal to the column vector $(0,00,\cdots,00,101)^T$, and we have from Theorem 1 that $w_0 \leq l/(\tau+1) = 2/2 = 1$. Since it was shown in [3] that $w_0 \geq 1$, the result follows. Q.E.D.

B. Analysis of codes with $e_{max} = 1$

When $e_{max} = 1$, the r subvectors of a state $\mathbf{S}$ have maximum length 2. Since the zeroth element of each subvector must be equal to zero for $\mathbf{S}$ to be a valid state, a shift $\mathbf{S}^{\overset{1}{\leftarrow}}$ only result in a new valid state if $\mathbf{S}$ is the all-zero state. In other words, a code with $e_{max} = 1$ has a state diagram with only one edge of weight zero, namely the edge constituting the zero state cycle. Hence, we can no longer consider cycles containing all-zero paths to upper bound w_0.

Lemma 2: Let $\mathbf{H}$ be a combined matrix with maximum row degree $e_{max} = 1$. If there exists a sum of $l \geq d_{\mathcal{E}}$ columns in $\mathbf{H} = [\mathbf{h}_j]$ such that

$$\sum_{k=1}^{l} \mathbf{h}_{j_k} = \mathbf{S} + \mathbf{S}^{\overset{1}{\leftarrow}}$$

for some nonzero state $\mathbf{S}$, then $w_0 \leq l$.

Proof: It follows from (2) that there is a cycle of length one starting and ending in $\mathbf{S}$ if for some l there is a sum of columns $\sum_{k=1}^{l}\mathbf{h}_{j_k}$ such that

$$\mathbf{S} = \mathbf{S}^{\overset{1}{\leftarrow}} + \sum_{k=1}^{l}\mathbf{h}_{j_k},$$

or equivalently,

$$\sum_{k=1}^{l}\mathbf{h}_{j_k} = \mathbf{S} + \mathbf{S}^{\overset{1}{\leftarrow}}.$$

The label on the cycle has weight l and hence $w_0 \leq l$. Q.E.D.

We illustrate how to use Lemma 2 by proving two corollaries. In [3] we constructed a class of $(2^{2m}-2^m, 2^{2m}-2^m-m, m, 3)$ convolutional codes with $w_0 \geq 1$ for all $m \geq 1$. The general canonical parity-check matrix has row degrees $e_1 = e_2 = \cdots = e_m = 1$, and hence the embedded block code has minimum distance $d_{\mathcal{E}} = 1$. The last combined column of the matrix is equal to $B^T_{2^{2m}-1} = (11, 11, \cdots, 11)^T$.

Corollary 3: The $(2^{2m}-2^m, 2^{2m}-2^m-m, m, 3)$ convolutional codes constructed in [3] have $w_0 = 1$ for all $m \geq 1$.

Proof: We only need to show that $w_0 \leq 1$. The state diagram contains the nonzero state $\mathbf{S} = (01, 01, \cdots, 01)^T$. Since $\mathbf{S} + \mathbf{S}^{\overset{1}{\leftarrow}} = B^T_{2^{2m}-1}$, we have from Lemma 2 that $w_0 \leq l = 1$. Q.E.D.

In [3] we also gave a general construction for a code class with $d_{free} \geq 2t+1$ for many $t \geq 2$. For $t = 2$, the class contains $(2^m-1, 2^m-1-m, m, 5)$ convolutional codes with $w_0 \geq 1$. The row degrees are $e_1 = e_2 = \cdots = e_m = 1$, and the first combined column is $\mathbf{h}_1 = (11, 00, \cdots, 00)^T$. The embedded block code has $d_{\mathcal{E}} = 1$ since all row degrees are nonzero.

Corollary 4: The $(2^m-1, 2^m-1-m, m, 5)$ convolutional codes constructed in[3] have $w_0 = 1$ for all $m \geq 3$.

Proof: We must show that $w_0 \leq 1$. The state diagram contains the nonzero state $\mathbf{S} = (01, 00, \cdots, 00)^T$. Since $\mathbf{S} + \mathbf{S}^{\overset{1}{\leftarrow}} = \mathbf{h}_1$, we have from Lemma 2 that $w_0 \leq l = 1$. Q.E.D.

C. Discussion

The bounding technique used to prove the upper bounds on w_0 can of course be used to prove many other upper bounds. For some codes it may be necessary to consider cycles containing more than one all-zero subpath to obtain tight bounds.

Examples of such bounds for $(n, n-1, m)$ codes are given in [4]. It is not difficult to generalize these bounds to $(n, n-r, m)$ codes.

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