Note on the stability of Gauss-Jordan elimination for diagonally dominant matrices

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Abstract

Peters and Wilkinson [4] state that "it is well known that Gauss-Jordan is stable" for a diagonally dominant matrix, but a proof does not seem to have been published [3]. The present note fills this gap. Gauss-Jordan elimination is backward stable for matrices diagonally dominant by rows and not backward stable for matrices diagonally dominant by columns. In either case it is forward stable.

Gauss-Jordan elimination (GJE) reduces a matrix $A \in R^{n\times n}$ to diagonal form in n successive steps. Starting from $A^{GJE(0)} = A$, the k-th elimination step transforms $A^{GJE(k-1)}$ into $A^{GJE(k)}$ such that the off-diagonal elements in the k-th column become zero. Let m_k be the column vector

$$m_k = \left[A_{1k}^{GJE(k-1)}, \ldots, A_{k-1,k}^{GJE(k-1)}, 0, A_{k+1,k}^{GJE(k-1)}, \ldots, A_{nk}^{GJE(k-1)}\right]^T / A_{kk}^{GJE(k-1)},$$

then GJE produces the following matrices:

$$A^{GJE(0)} = A, \qquad A^{GJE(k)} = (I - m_k e_k^T) A^{GJE(k-1)}, \quad k = 1:n, \qquad A^{GJE(n)} = D,$$

where $e_k = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ with 1 in the k-th entry. Solution of a linear system $Ax = b$ is given by

$$x = D^{-1}(I - m_n e_n^T) \ldots (I - m_1 e_1^T) b.$$

The analysis of GJE is simplified if one considers it as comprising two stages [1], [3], [4]. The first stage is the Gaussian elimination (GE) $U = L^{-1}A$, $y = L^{-1}b$. The second stage solves $Ux = y$ with the upper triangular matrix U by GJE. The latter stage computes a matrix V such that $VU = D = \mathrm{diag}(U)$. In this notation $x = D^{-1}VL^{-1}b$.

Values computed in a floating point arithmetic will be marked by the symbols fl or $\tilde{}$. To analyze rounding errors, we follow the standard model:

$$\mathrm{fl}(x \circ y) = (x \circ y)(1 + \delta), \quad |\delta| \le \epsilon = \epsilon_{\mathrm{machine}}, \quad \circ = +, -, \times, /,$$

and sometimes the model $\mathrm{fl}(x \circ y) = (x \circ y)/(1 + \delta)$. As in [3] we use the parameter $\gamma_n = n\epsilon_{\mathrm{machine}}/(1 - n\epsilon_{\mathrm{machine}})$. The matrix inequality $A \le B$ means that $A_{ij} \le B_{ij}$ for all i, j, and $|A| = \mathrm{abs}(A)$ denotes the matrix with elements $|A_{ij}|$.

1 General error analysis

It is well known that

$$\widetilde{L}\widetilde{U} = A + E_1, \qquad |E_1| \leq \gamma_{n-1}|\widetilde{L}||\widetilde{U}|. \tag{1}$$

Indeed, the equation $A_{ij} = \sum_{k=1}^{\min(i,j)} L_{ik}U_{kj}$ is solved by

$$\begin{aligned} \widetilde{L}_{ij} &= \mathrm{fl}([A_{ij} - \sum_{k=1}^{j-1} \widetilde{L}_{ik}\widetilde{U}_{kj}]/\widetilde{U}_{jj}), \quad i > j, \\ \widetilde{U}_{ij} &= \mathrm{fl}(A_{ij} - \sum_{k=1}^{i-1} \widetilde{L}_{ik}\widetilde{U}_{kj}), \quad i \leq j. \end{aligned}$$

The estimate of E_1 follows from

Lemma 1 [3] *If $y = (c - \sum_{i=1}^{k-1} a_i b_i)/b_k$ is evaluated in floating point arithmetic, then, no matter what the order of evaluation,*

$$\widetilde{y} b_k (1 + \theta_k^{(k)}) = c - \sum_{i=1}^{k-1} a_i b_i (1 + \theta_k^{(i)}),$$

where $|\theta_k^{(i)}| \leq \gamma_k$ for all i. If $b_k = 1$, so that there is no division, then $|\theta_k^{(i)}| \leq \gamma_{k-1}$ for all i.

When computing $y = L^{-1}b$, the computed vector $\widetilde{y}$ satisfies

$$(\widetilde{L} + E_2)\widetilde{y} = b, \qquad |E_2| \leq \gamma_{n-1}|\widetilde{L}|. \tag{2}$$

Again Lemma 1 is applied to

$$\widetilde{y}_i = \mathrm{fl}(b_i - \sum_{k=1}^{i-1} \widetilde{L}_{ik}\widetilde{y}_k).$$

Since $\widetilde{v}_{ij} = -\mathrm{fl}([\sum_{k=i}^{j-1} \widetilde{v}_{ik}\widetilde{u}_{kj}]/\widetilde{u}_{jj})$ for $i < j$, the i-th row of the matrix $\widetilde{V}$, $e_i^T\widetilde{V}$, has the property that

$$e_i^T\widetilde{V}(\widetilde{U} + E_{3i}) = \widetilde{U}_{ii}e_i^T, \qquad |E_{3i}| \leq \gamma_{n-i}|\widetilde{U}|, \quad i = 1:n. \tag{3i}$$

From (3i) it follows that

$$\widetilde{V}\widetilde{U} = \widetilde{D} + E_3, \qquad \widetilde{D} = \mathrm{diag}(\widetilde{U}), \qquad |E_3| \leq \gamma_{n-1}|\widetilde{V}||\widetilde{U}|. \tag{3}$$

Similarly, when computing $x = D^{-1}Vy$, one obtain that

$$\widetilde{x}_i = \widetilde{U}_{ii}^{-1}(e_i^T\widetilde{V})(I + E_{4i})^{-1}\widetilde{y}, \qquad |E_{4i}| \leq \gamma_{n+2-i}I, \quad i = 1:n, \tag{4i}$$

and, consequently,

$$\widetilde{x} = \widetilde{D}^{-1}(\widetilde{V} + E_4)\widetilde{y}, \qquad |E_4| \leq \gamma_{n+1}|\widetilde{V}|. \tag{4}$$

All details concerning the derivation of these estimates are found in [3].

Theorem 1 *Let $\widetilde{x}$ be a computed by GJE solution to $Ax = b$. Then each component $\widetilde{x}_i$ of $\widetilde{x}$ satisfies*

$$\widetilde{x}_i = e_i^T (A + \Delta_i)^{-1} b, \qquad |\Delta_i| \le \gamma_{4n-2} |\widetilde{L}||\widetilde{U}|, \quad i = 1 : n.$$

Proof: Using estimates (1), (2), (3i), and (4i):

$$\widetilde{x}_i = \widetilde{U}_{ii}^{-1}(e_i^T \widetilde{V})(I + E_{4i})^{-1}(\widetilde{L} + E_2)^{-1} b = e_i^T (\widetilde{U} + E_{3i})^{-1}(I + E_{4i})^{-1}(\widetilde{L} + E_2)^{-1} b,$$

$$\Delta_i = (\widetilde{L} + E_2)(I + E_{4i})(\widetilde{U} + E_{3i}) - A,$$

$$|\Delta_i| \le |E_1| + [(1 + \gamma_{n-1})^2 (1 + \gamma_{n+1}) - 1]|\widetilde{L}||\widetilde{U}| \le \gamma_{4n-2}|\widetilde{L}||\widetilde{U}|. \quad \blacksquare$$

This result due to Peters and Wilkinson [4] shows that the forward error in $\widetilde{x}$ computed by GJE is of the same order of magnitude as that of GE.

Let us denote the matrices produced in GE as $A^{(1)} = A$, $A^{(2)}, \dots$, $A^{(n)} = U$ and define the growth factor of GE by

$$g = \max_{i,j,k} |A_{ij}^{(k)}| / \|A\|_\infty.$$

One can prove [3] that

$$\| |\widetilde{L}||\widetilde{U}| \|_\infty \le 2n(1 + \gamma_2)\widetilde{g}\|A\|_\infty. \tag{5}$$

Since $\widetilde{x}_i - x_i = -e_i^T (A + \Delta_i)^{-1} \Delta_i x$,

$$|\widetilde{x}_i - x_i| \le \|(A + \Delta_i)^{-1}\Delta_i x\|_\infty \le \frac{\|\Delta_i\|_\infty \|A^{-1}\|_\infty}{1 - \|\Delta_i\|_\infty \|A^{-1}\|_\infty} \|x\|_\infty.$$

As a result,

$$\frac{\|\widetilde{x} - x\|_\infty}{\|x\|_\infty} \le \frac{2n(1 + \gamma_2)\gamma_{4n-2}\widetilde{g}\kappa_\infty(A)}{1 - 2n(1 + \gamma_2)\gamma_{4n-2}\widetilde{g}\kappa_\infty(A)} = (8n^2 - 4n)\epsilon\widetilde{g}\kappa_\infty(A) + O(\epsilon^2), \tag{6}$$

where $\kappa_\infty(A) = \|A^{-1}\|_\infty \|A\|_\infty$. If the growth factor $\widetilde{g}$ is not large, then GJE is forward stable.

Theorem 2 *A solution $\widetilde{x}$ to $Ax = b$ computed by GJE satisfies*

$$(A + \Delta_{LU})(I + \Delta_V)^{-1}\widetilde{x} = b,$$

where

$$|\Delta_{LU}| \le \gamma_{2n-2}|\widetilde{L}||\widetilde{U}|, \qquad |\Delta_V| \le \gamma_{2n}|\widetilde{U}^{-1}||\widetilde{U}| \left(I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|\right)^{-1},$$

if $\gamma_{n-1}\| |\widetilde{U}^{-1}||\widetilde{U}| \|_\infty < 1$.

Proof: Derive from (3) that $(\widetilde{V} + E_4)\widetilde{U} = \widetilde{D} + E_3 + E_4\widetilde{U}$ and $\widetilde{D}^{-1}(\widetilde{V} + E_4) = (I + \widetilde{D}^{-1}E_3 + \widetilde{D}^{-1}E_4\widetilde{U})\widetilde{U}^{-1}$. Therefore, $\widetilde{x} = \widetilde{D}^{-1}(\widetilde{V} + E_4)(\widetilde{L} + E_2)^{-1}b = (I + \widetilde{D}^{-1}E_3 + \widetilde{D}^{-1}E_4\widetilde{U})\widetilde{U}^{-1}(\widetilde{L} + E_2)^{-1}b$, $(\widetilde{L} + E_2)\widetilde{U}(I + \widetilde{D}^{-1}E_3 + \widetilde{D}^{-1}E_4\widetilde{U})^{-1}\widetilde{x} = b$. Denote $\Delta_{LU} = E_1 + E_2\widetilde{U}$ and $\Delta_V = \widetilde{D}^{-1}E_3 + \widetilde{D}^{-1}E_4\widetilde{U}$. Then

$$(A + \Delta_{LU})(I + \Delta_V)^{-1}\widetilde{x} = b, \qquad |\Delta_{LU}| \le \gamma_{2n-2}|\widetilde{L}||\widetilde{U}|, \qquad |\Delta_V| \le \gamma_{2n}|\widetilde{D}^{-1}\widetilde{V}||\widetilde{U}|.$$

Let us estimate Δ_V in terms of $\widetilde{U}$. Since $\widetilde{V} = (\widetilde{D} + E_3)\widetilde{U}^{-1}$, one can derive that $|\widetilde{D}^{-1}E_3| \le \gamma_{n-1}|I + \widetilde{D}^{-1}E_3||\widetilde{U}^{-1}||\widetilde{U}| \le \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}| + \gamma_{n-1}|\widetilde{D}^{-1}E_3||\widetilde{U}^{-1}||\widetilde{U}|$, $|\widetilde{D}^{-1}E_3|(I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|) \le \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|$. The condition $\gamma_{n-1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty < 1$ implies that $I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|$ is nonsingular and $(I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|)^{-1} = \sum_{k=0}^{\infty}\left(\gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|\right)^k$. Hence all entries of $(I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|)^{-1}$ are nonnegative. At last, $|\widetilde{D}^{-1}E_3| \le \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|(I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|)^{-1}$, $|\Delta_V| \le \gamma_{2n}|I + \widetilde{D}^{-1}E_3||\widetilde{U}^{-1}||\widetilde{U}| \le (I - \gamma_{n-1}|\widetilde{U}^{-1}||\widetilde{U}|)^{-1}\gamma_{2n}|\widetilde{U}^{-1}||\widetilde{U}|$. ∎

The following asymptotic estimate holds:

$$|b - A\widetilde{x}| \le \left[(2n-2)\epsilon|\widetilde{L}||\widetilde{U}| + 2n\epsilon|A||\widetilde{U}^{-1}||\widetilde{U}|\right]|\widetilde{x}| + O(\epsilon^2). \tag{7}$$

The sequence of formulas $\|\Delta_V\|_\infty \le \gamma_{2n}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty(1 - \gamma_{n-1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty)^{-1}$,

$$\|(I + \Delta_V)^{-1}\|_\infty \le (1 - \|\Delta_V\|_\infty)^{-1} \le (1 - \gamma_{n-1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty)(1 - \gamma_{3n-1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty)^{-1},$$

$$(A + \Delta_{LU})(I + \Delta_V)^{-1} - A = (\Delta_{LU} - A\Delta_V)(I + \Delta_V)^{-1},$$

$$\|(A + \Delta_{LU})(I + \Delta_V)^{-1} - A\|_\infty \le \frac{\gamma_{2n-2}\||\widetilde{L}||\widetilde{U}|\|_\infty + \gamma_{2n}\|A\|_\infty\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty}{1 - \gamma_{3n-1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty}$$

leads to an estimate of the backward error:

$$\|(A + \Delta_{LU})(I + \Delta_V)^{-1} - A\|_\infty \le \frac{2n\gamma_{2n-2}\widetilde{g} + \gamma_{2n}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty}{1 - \gamma_{3n+1}\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty}\|A\|_\infty. \tag{8}$$

The residual $b - A\widetilde{x}$ satisfies the asymptotic estimate

$$\|b - A\widetilde{x}\|_\infty \le \left[(4n^2 - 4n)\epsilon\widetilde{g} + 2n\epsilon\||\widetilde{U}^{-1}||\widetilde{U}|\|_\infty\right]\|A\|_\infty\|\widetilde{x}\|_\infty + O(\epsilon^2), \tag{9}$$

which "means that the Gauss-Jordan solution gives a residual which is commonly of the order of magnitude one naturally associates with its accuracy" [4].

2 GJE for matrices diagonally dominant by rows

We say that $A \in R^{n\times n}$ is diagonally dominant by rows if

$$|A_{ii}| \ge \sum_{\substack{j=1\\ j\ne i}}^{n} |A_{ij}|, \qquad i = 1:n.$$

It is well known [2] that the diagonal dominance is preserved under Gaussian elimination.

Theorem 3 *Assume that $A \in R^{n\times n}$ satisfies the condition*

$$|A_{ii}| \ge \alpha_i \sum_{\substack{j=1\\ j\ne i}}^{n} |A_{ij}| \quad \text{with } \alpha_i \ge (1+\gamma_2)^{n-1}. \tag{10}$$

Then the upper triangular matrix $\widetilde{U}$ computed by GE in a floating point arithmetic is diagonally dominant by rows. Moreover, the growth factor $\widetilde{g}$ is bounded from above by $1 + \gamma_{n-1}$.

Proof: Consider the first elimination step $A^{(1)} = A \to A^{(2)}$. The matrix $\widetilde{A}^{(2)}$ has the entries

$$\widetilde{A}^{(2)}_{ij} = \left[A_{ij} - \frac{A_{i1}A_{1j}}{A_{11}}(1+\delta')(1+\delta'')\right](1+\delta'''), \qquad i = 2:n,\ j = 2:n,$$

with $|\delta'|$, $|\delta''|$, $|\delta'''| \le \epsilon$. From this representation we derive that

$$|\widetilde{A}^{(2)}_{ii}| \ge |A_{ii}|(1-\epsilon) - \frac{|A_{i1}||A_{1i}|}{|A_{11}|}(1+\epsilon)^3$$

and

$$\sum_{\substack{j=2\\ j\ne i}}^{n} |\widetilde{A}^{(2)}_{ij}| \le (1+\epsilon)\sum_{\substack{j=2\\ j\ne i}}^{n} |A_{ij}| + (1+\epsilon)^3 \sum_{\substack{j=2\\ j\ne i}}^{n} \frac{|A_{i1}||A_{1j}|}{|A_{11}|} \le$$

$$(1+\epsilon)\left(|A_{ii}|/\alpha_i - |A_{i1}|\right) + (1+\epsilon)^3 \frac{|A_{i1}|}{|A_{11}|}\left(|A_{11}|/\alpha_1 - |A_{1i}|\right) =$$

$$\frac{1+\epsilon}{\alpha_i}|A_{ii}| - (1+\epsilon)^3\frac{|A_{i1}||A_{1i}|}{|A_{11}|} + (1+\epsilon)\left[\frac{(1+\epsilon)^2}{\alpha_1} - 1\right]|A_{i1}| \le$$

$$\frac{1+\epsilon}{\alpha_i(1-\epsilon)}\left[|\widetilde{A}^{(2)}_{ii}| + (1+\epsilon)^3\frac{|A_{i1}||A_{1i}|}{|A_{11}|}\right] - (1+\epsilon)^3\frac{|A_{i1}||A_{1i}|}{|A_{11}|} + (1+\epsilon)\left[\frac{(1+\epsilon)^2}{\alpha_1} - 1\right]|A_{i1}| \le$$

$$\frac{1+\gamma_2}{\alpha_i}|\widetilde{A}^{(2)}_{ii}| + (1+\epsilon)^3\left(\frac{1+\gamma_2}{\alpha_i} - 1\right)\frac{|A_{i1}||A_{1i}|}{|A_{11}|} + (1+\epsilon)\left(\frac{1+\gamma_2}{\alpha_1} - 1\right)|A_{i1}|.$$

Therefore,

$$\sum_{\substack{j=2\\ j\ne i}}^{n} |\widetilde{A}^{(2)}_{ij}| \le \frac{1+\gamma_2}{\alpha_i}|\widetilde{A}^{(2)}_{ii}|, \quad \text{when } \alpha_i \ge 1+\gamma_2.$$

Denote $\alpha_i^{(2)} = \alpha_i/(1+\gamma_2)$, then $\alpha_i^{(2)} \sum_{\substack{j=2\\ j\ne i}}^{n} |\widetilde{A}^{(2)}_{ij}| \le |\widetilde{A}^{(2)}_{ii}|$. After $n-1$ elimination steps $\alpha_i^{(n)} = \alpha_i/(1+\gamma_2)^{n-1} \ge 1$.

In order to estimate $\widetilde{g}$, we proceed as follows:

$$\sum_{j=2}^{n} |\widetilde{A}^{(2)}_{ij}| \le (1+\epsilon)\sum_{j=2}^{n} |A_{ij}| + (1+\epsilon)^3 \sum_{j=2}^{n} \frac{|A_{i1}||A_{1j}|}{|A_{11}|} \le$$

$$(1+\epsilon)\sum_{j=2}^{n}|A_{ij}| + (1+\epsilon)^3|A_{i1}|/\alpha_1 \le (1+\epsilon)\sum_{j=1}^{n}|A_{ij}|, \qquad \text{if } \alpha_1 \ge (1+\epsilon)^2.$$

After step k we have $\sum_{j=k+1}^{n} |A^{(k+1)}_{ij}| \le (1+\epsilon)^k \sum_{j=1}^{n} |A_{ij}|$ and finally $\max_{i,j,k} |A^{(k)}_{ij}| \le (1+\gamma_{n-1})\|A\|_\infty$. ■

Since the matrix $\widetilde{U}$ is diagonally dominant by rows, one can derive nice estimates of $|\widetilde{U}^{-1}||\widetilde{U}|$ that allow one to show the backward stability of GJE for matrices, which satisfy (10). Let us denote $W = \widetilde{D}^{-1}\widetilde{U}$, then $|\widetilde{U}^{-1}||\widetilde{U}| = |W^{-1}||W|$. The matrix W is unit upper triangular and $\sum_{j=i+1}^{n} |W_{ij}| \le 1$ due to the diagonal dominance by rows of $\widetilde{U}$.

Lemma 2

$$|W^{-1}| \le T = \begin{bmatrix} 1 & 1 & \dots & 1 \\ & 1 & \dots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

Proof: Denote the j-th column of the matrix W^{-1} by z. We have $z_j = 1$, $z_i = 0$ for $i > j$, and $\sum_{k=i}^{j} W_{ik} z_k = 0$ for $i < j$, so $z_i = -\sum_{k=i+1}^{j} W_{ik} z_k$. Assume, by induction, that $|z_k| \le 1$ for $k = i+1 : j$, then $|z_i| \le \sum_{k=i+1}^{j} |W_{ik}| \le 1$. ■

By the lemma, $|\tilde{U}^{-1}||\tilde{U}| \le T|W|$ and

$$|\Delta_V| \le \gamma_{2n} T|W|(I - \gamma_{n-1} T|W|)^{-1}. \tag{11}$$

Since $\|T|W|\|_\infty \le 2n - 1$,

$$\|(A + \Delta_{LU})(I + \Delta_V)^{-1} - A\|_\infty \le (8n^2 - 6n)\epsilon\|A\|_\infty + O(\epsilon^2). \tag{12}$$

3 GJE for matrices diagonally dominant by columns

In general GJE for such matrices is not backward stable. Consider, for instance, the following example:

$$A = \begin{bmatrix} 1+\delta & 1 & 0 \\ 1 & 1+\delta & 1 \\ 0 & 0 & 1+\delta \end{bmatrix}, \qquad b = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+\delta \\ 3+\delta \\ 1+\delta \end{bmatrix}.$$

Choose $\delta = 10^{-10}$ and compute all with 17 decimal digits. Then

$$\tilde{x} = [1 - \delta, 1, 1]^T, \qquad r = b - A\tilde{x} = [\delta(1+\delta), \delta, 0]^T,$$

$$\tilde{U} = \begin{bmatrix} 1+\delta & 1 & 0 \\ 0 & 2\delta & 1 \\ 0 & 0 & 1+\delta \end{bmatrix}, \qquad |A||\tilde{U}^{-1}||\tilde{U}| = \begin{bmatrix} 1+\delta & 3 & 2/\delta \\ 1 & 3 - \delta\frac{1-\delta}{1+\delta} & \frac{2}{\delta} + \frac{1+2\delta}{1+\delta} \\ 0 & 0 & 1+\delta \end{bmatrix}.$$

Since $\epsilon_{\text{machine}} = 10^{-16}$ and $\kappa_\infty(A) = \|A^{-1}\|_\infty \|A\|_\infty \approx 9/(2\delta)$, we have

$$\frac{\|\tilde{x} - x\|_\infty}{\|x\|_\infty} = \delta \approx \frac{2}{9} \cdot 10^{-4} \epsilon_{\text{machine}} \kappa_\infty(A),$$

$$\frac{\|r\|_\infty}{\|A\|_\infty \|\tilde{x}\|_\infty} \approx \frac{1}{3} \cdot 10^{-10} = \frac{1}{3} \cdot 10^{6} \epsilon_{\text{machine}}.$$

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