Stability radii of symplectic and Hamiltonian matrices

A.N. Malyshev
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Department of Informatics,
University of Bergen, 5020 Bergen, Norway

Abstract

We review the Krein-Gel'fand-Lidskii theory of strong stability of Hamiltonian and symplectic matrices in order to find a quantitative measure of the strong stability. As a result, formulas for the 2-norm distance from a strongly stable Hamiltonian or symplectic matrix to the set of unstable matrices are derived.

1 Definition and simple properties of symplectic and Hamiltonian matrices

Throughout the paper the superscript $*$ denotes the hermitian conjugation, and I is the identity matrix.

The Krein-Gel'fand-Lidskii theory ([4], Chapter 3), which is main tool in study of stability of symplectic and Hamiltonian matrices, is valid for general G-unitary and G-Hamiltonian matrices. Let $G \in \mathbf{C^{n\times n}}$ be a nonsingular hermitian matrix (i.e. $\det G \neq 0$, $G^* = G$) which is neither positive nor negative definite. In applications the matrix G often equals iJ, where i is the imaginary unit and $J = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$. Together with the standard scalar product (x, y) we define a new, indefinite scalar product $< x, y >= (Gx, y)$. Note that the number $< x, x >$ is real but unlike (x, x) it need not be positive, hence the term indefinite.

Given a matrix $A \in \mathbf{C^{n\times n}}$, the matrix A^G defined by $< Ax, y >=< x, A^G y >$ (x, y are arbitrary vectors in $\mathbf{C^{n\times n}}$) is said to be G-adjoint to A. The explicit formula for A^G is $A^G = G^{-1}A^*G$. A matrix $\mathcal{U} \in \mathbf{C^{n\times n}}$ is called G-unitary if $\mathcal{U}^G\mathcal{U} = I$. It is easy to show that $\mathcal{U}$ is G-unitary if and only if $\mathcal{U}^*G\mathcal{U} = G$. A matrix $\mathcal{H} \in \mathbf{C^{n\times n}}$ is called G-antihermitian (or G-Hamiltonian) if $\mathcal{H}^G = -\mathcal{H}$. Each G-Hamiltinian matrix $\mathcal{H}$ has the representation $\mathcal{H} = iG^{-1}H$ with some $H = H^*$.

Real iJ-unitary matrices are usually called symplectic matrices, and real iJ-Hamiltonian ones simply Hamiltonian.

Since $\mathcal{U} = G^{-1}(\mathcal{U}^{-1})^*G$, the spectrum of a G-unitary matrix $\mathcal{U}$ is symmetric with respect to the unit circle, i.e. together with any its eigenvalue ρ the matrix $\mathcal{U}$ has the eigenvalue $1/\bar{\rho}$. Similarly, from the identity $\mathcal{H} = -G^{-1}\mathcal{H}^*G$ it follows that the spectrum of a G-Hamiltonian matrix $\mathcal{H}$ is symmetric about the imaginary axis, i.e. if λ is an eigenvalue of $\mathcal{H}$, then $-\bar{\lambda}$ is an eigenvalue of $\mathcal{H}$ too.

One can show that $e^{\mathcal{H}}$ is G-unitary if $\mathcal{H}$ is G-Hamiltonian. The converse (the logarithm of a G-unitary matrix is G-Hamiltonian) is generally false. It is valid only if a special branch of logarithm is chosen.

Stability of a G-unitary matrix $\mathcal{U}$ means that $\|\mathcal{U}^n\| \leq C$ for all $n = 1, 2, \ldots$. From this definition it follows immediately that all eigenvalues of $\mathcal{U}$ lie on the unit circle ($|\rho| = 1$) and are not defective. Recall that a multiple eigenvalue λ of a matrix A is said to be defective if there is a Jordan chain of linear independent vectors v_1, v_2, $\ldots$, v_r, $r \geq 2$, such that $Av_1 = \lambda v_1$, $Av_i = \lambda v_i + v_{i-1}$, $i = 2, \ldots, r$.

Stability of a G-Hamiltonian matrix $\mathcal{H}$ means that $\|e^{t\mathcal{H}}\| \leq C$ for all $t \geq 0$. This definition implies that all eigenvalues of $\mathcal{H}$ lie on the imaginary axis and are not defective.

2 Elements of the Krein-Gel'fand-Lidskii theory

We give here a self-contained, concise presentation of basic facts of the classical Krein-Gel'fand-Lidskii theory ([4], Chapter 3), which are necessary for characterization of strongly stable G-unitary and G-Hamiltonian matrices.

A G-unitary matrix $\mathcal{U}$ is said to be strongly stable if there exists $\varepsilon > 0$ such that any G-unitary matrix $\widetilde{\mathcal{U}}$ satisfying the inequality $\|\widetilde{\mathcal{U}} - \mathcal{U}\| < \varepsilon$ is stable. Similarly, a G-Hamiltonian matrix $\mathcal{H}$ is said to be strongly stable if there exists $\varepsilon > 0$ such that any G-Hamiltonian matrix $\widetilde{\mathcal{H}}$ satisfying the inequality $\|\widetilde{\mathcal{H}} - \mathcal{H}\| < \varepsilon$ is stable. Our goal is to derive convenient characterizations of strongly stable G-unitary and G-Hamiltonian matrices as well as suitable descriptions of the sets of all G-unitary or G-Hamiltonian matrices that are not strongly stable.

Definition 1 *Let S be the eigensubspace (or the root subspace) corresponding to an eigenvalue ρ' of a G-unitary matrix lying on the unit circle or to an eigenvalue λ' of a G-Hamiltonian matrix lying on the imaginary axis.*

a) If $< v, v > > 0$ for any $v \neq 0$ in S, then ρ' or λ' is called an eigenvalue of the first kind, and if $< v, v > < 0$ for any $v \neq 0$ in S, then ρ' or λ' is called an eigenvalue of the second kind. Eigenvalues of the first and second kind are said to be definite.

b) If $< v, v >$ changes sign on S, i.e. there exists $v \neq 0$ in S such that $< v, v >= 0$, then we shall say that the eigenvalue ρ' or λ' is indefinite or of mixed kind.

Let us consider the case of G-unitary matrices first.

Lemma 1 *Let ρ be an eigenvalue of a G-unitary matrix $\mathcal{U}$, not on the unit circle ($|\rho| \neq 1$). Then any corresponding eigenvector $v \neq 0$ satisfies the equality $< v, v >= 0$.*

Proof: By assumption $\mathcal{U}v = \rho v$, and so $< v, v >=< \mathcal{U}v, \mathcal{U}v >=< \rho v, \rho v >= |\rho|^2 < v, v >$, $(1 - |\rho|^2) < v, v >= 0$. Since $|\rho| \neq 1$, $< v, v >= 0$. □

Lemma 2 *Let ρ' be a defective eigenvalue of a G-unitary matrix $\mathcal{U}$ lying on the unit circle ($|\rho'| = 1$). Then there is an eigenvector $v \neq 0$ corresponding to ρ' such that $< v, v >= 0$.*

Proof: Choose a Jordan chain $v = v_1, v_2, \ldots, v_r$ of length at least 2 such that $\mathcal{U}v_1 = \rho' v_1$, $\mathcal{U}v_2 = \rho' v_2 + v_1, \ldots$. Since $< v_1, v_2 >=< \mathcal{U}v_1, \mathcal{U}v_2 >= |\rho'|^2 < v_1, v_2 > +\rho' < v_1, v_1 >$, we obtain $< v_1, v_1 >= (1 - |\rho'|^2)/\rho' < v_1, v_2 >= 0$. □

Note that by Lemma 2 the definite eigenvalues lying on the unit circle are not defective.

Theorem 1 *Let ρ' be a definite eigenvalue of a G-unitary matrix $\mathcal{U}$ lying on the unit circle ($|\rho'| = 1$). Then there exist numbers $\gamma > 0$, $\varepsilon > 0$ such that for any G-unitary matrix $\widetilde{\mathcal{U}}$ satisfying the inequality $\|\widetilde{\mathcal{U}} - \mathcal{U}\| < \varepsilon$ all eigenvalues in the neighbourhood $|\rho - \rho'| < \gamma$ lie on the unit circle and are not defective.*

Proof: Suppose the contrary. Then there is a sequence of matrices $\mathcal{U}_1, \mathcal{U}_2, \ldots \to \mathcal{U}$ with respective eigenvalues $\rho_1, \rho_2, \ldots \to \rho'$ such that either $|\rho_m| \neq 1$, or $|\rho_m| = 1$ but $\mathcal{U}_m$ has a defective eigenvalue ρ_m. Let v_m be corresponding normalized eigenvectors, $\mathcal{U}_m v_m = \rho_m v_m$ ($\|v_m\| = 1$), so that by Lemmas 1 and 2 $< v_m, v_m >= 0$, $m = 1, 2, \ldots$. One can select a subsequence such that $v_{m_j} \to v$ as $m_j \to \infty$; by suitably changing the notation we may assume that $v_m \to v$ as $m \to \infty$. Letting $m \to \infty$ in the above equalities we obtain $\mathcal{U}v = \rho' v$, $\|v\| = 1$, $< v, v >= 0$. Hence ρ' is not definite, and this contradiction completes the proof. □

Theorem 2 *Let $\mathcal{U}$ be a G-unitary matrix with an eigenvalue ρ' of mixed kind on the unit circle. Then there is a G-unitary matrix $\widetilde{\mathcal{U}}$ arbitrarily near $\mathcal{U}$ which has eigenvalues outside the unit circle in the neighbourhood of ρ'.*

Proof: By assumption there is an eigenvector $v \neq 0$ such that $\mathcal{U}v = \rho' v$ and $< v, v >= 0$. We look for the matrix $\widetilde{\mathcal{U}}$ in the form $\widetilde{\mathcal{U}} = \mathcal{V}\mathcal{U}$, where $\mathcal{V}$ is a G-unitary matrix arbitrarily near I and v is an eigenvector of $\mathcal{V}$ corresponding to an eigenvalue outside the unit circle.

Note that G can be factorized as $G = T^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T$ with a nonsingular matrix T. The vector $Tv = \begin{bmatrix} v_+ \\ v_- \end{bmatrix}$ is partitioned into v_+ and v_- in conformity with the orders of I and $-I$ in the factorization of G. Then $< v, v >= 0$ implies that $\|v_+\|_2 = \|v_-\|_2$. The matrix

$$\mathcal{V}_\varepsilon = I + \frac{1}{\|v_+\|_2^2} T^{-1} \begin{bmatrix} [\cosh(\varepsilon) - 1]v_+ v_+^* & \sinh(\varepsilon) v_+ v_-^* \\ \sinh(\varepsilon) v_- v_+^* & [\cosh(\varepsilon) - 1] v_- v_-^* \end{bmatrix} T$$

is G-unitary for any real ε, $\mathcal{V}_\varepsilon v = e^\varepsilon v$ and $e^\varepsilon \neq 1$ when $\varepsilon \neq 0$. Since $\mathcal{V}_\varepsilon \to I$ as $\varepsilon \to 0$, the theorem is proven. □

Corollary 1 *A G-unitary matrix $\mathcal{U}$ is strongly stable if and only if all its eigenvalues lie on the unit circle and are definite.*

Recall that such eigenvalues are not defective.

Corollary 2 *A G-unitary matrix $\mathcal{U}$ is not strongly stable if and only if there exists an eigenvector $v \neq 0$ of $\mathcal{U}$ such that $< v, v >= 0$.*

Remark 1. If the matrix $\mathcal{U}$ in theorem 2 is symplectic, then theorem 2 is valid with a symplectic matrix $\widetilde{\mathcal{U}}$, i.e. real perturbations can be chosen in the case of real matrices. The proof is technically involved [4].

Let us consider now G-Hamiltonian matrices.

Lemma 3 *Let λ be an eigenvalue of a G-Hamiltonian matrix $\mathcal{H}$, not on the imaginary axis ($Re\lambda \neq 0$). Then any corresponding eigenvector $v \neq 0$ of $\mathcal{H}$ satisfies the equality $< v, v >= 0$.*

Proof: We take advantage of the identity $Re < \mathcal{H}v, v >= 0$ valid for any vector v. Since $\mathcal{H}v = \lambda v$, $< \mathcal{H}v, v >= \lambda < v, v >$. Therefore, $0 = Re < \mathcal{H}v, v >= Re\lambda < v, v >$ and $< v, v >= 0$. $\square$

Lemma 4 *Let λ' be a defective eigenvalue of a G-Hamiltonian matrix $\mathcal{H}$ lying on the imaginary axis ($Re\lambda' = 0$). Then there is an eigenvector $v \neq 0$ corresponding to λ' such that $< v, v >= 0$.*

Proof: Choose a Jordan chain $v = v_1, v_2, \ldots, v_r$, $r \geq 2$, such that $\mathcal{H}v_1 = \lambda' v_1$, $\mathcal{H}v_2 = \lambda' v_2 + v_1, \ldots$. From $< \mathcal{H}v_2, v_1 >= \lambda' < v_2, v_1 > + < v_1, v_1 >$ it follows that $< v_1, v_1 >=< \mathcal{H}v_2, v_1 > -\lambda' < v_2, v_1 >=< v_2, \mathcal{H}^G v_1 > -\lambda' < v_2, v_1 >= - < v_2, \mathcal{H}v_1 > -\lambda' < v_2, v_1 >= -2Re\lambda' < v_2, v_1 >= 0$. $\square$

Theorem 3 *Let λ' be a definite eigenvalue of a G-Hamiltonian matrix $\mathcal{H}$ lying on the imaginary axis. Then there exist numbers $\gamma > 0$, $\varepsilon > 0$ such that for any G-Hamiltonian matrix $\widetilde{\mathcal{H}}$ satisfying the inequality $\|\widetilde{\mathcal{H}} - \mathcal{H}\| < \varepsilon$ all eigenvalues in the neighbourhood $|\lambda - \lambda'| < \gamma$ lie on the imaginary axis and are not defective.*

Proof of theorem 3 is analogous to that of theorem 1.

Theorem 4 *Let $\mathcal{H}$ be a G-Hamiltonian matrix with an eigenvalue λ' of mixed type on the imaginary axis. Then there is a G-Hamiltonian matrix $\widetilde{\mathcal{H}}$ arbitrarily near $\mathcal{H}$ which has eigenvalues outside the imaginary axis in the neighbourhood of λ'.*

Proof: There is an eigenvector $v \neq 0$ such that $\mathcal{H}v = \lambda' v$ and $< v, v >= 0$. The vector $Tv = \begin{bmatrix} v_+ \\ v_- \end{bmatrix}$ is partitioned according to the orders of I and $-I$ in the factorization $G = T^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T$, $\det T \neq 0$. Then $\|v_+\|_2 = \|v_-\|_2$ and the matrix

$$\mathcal{F}_\varepsilon = \frac{1}{\|v_+\|_2^2} T^{-1} \begin{bmatrix} 0 & \varepsilon v_+ v_-^* \\ \varepsilon v_- v_+^* & 0 \end{bmatrix} T$$

is G-Hamiltonian if $\varepsilon > 0$. Since $\mathcal{F}_\varepsilon v = \varepsilon v$, the matrix $\widetilde{\mathcal{H}} = \mathcal{H} + \mathcal{F}_\varepsilon$ satisfies the theorem. $\square$

Corollary 3 *A G-Hamiltonian matrix $\mathcal{H}$ is strongly stable if and only if all its eigenvalues lie on the imaginary axis and are definite.*

Recall that such eigenvalues are not defective.

Corollary 4 *A G-Hamiltonian matrix $\mathcal{H}$ is not strongly stable if and only if there exists an eigenvector $v \neq 0$ of $\mathcal{H}$ such that $< v, v >= 0$.*

Remark 2. If the matrix $\mathcal{H}$ in theorem 4 is Hamiltonian, then theorem 4 is valid with a Hamiltonian matrix $\widetilde{\mathcal{H}}$, i.e. real perturbations can be chosen in the case of real matrices. The proof is found in [4].

3 Stability radius of Hamiltonian matrices

Given a strongly stable G-Hamiltonian matrix $\mathcal{H}$, let $\widetilde{\mathcal{H}}$ be a nearest to $\mathcal{H}$ G-Hamiltonian matrix, which is not strongly stable. Owing to the continuity of eigenvalues, $\widetilde{\mathcal{H}}$ has all its eigenvalues on the imaginary axis. In the previous section we proved that there is a vector $v \neq 0$ such that $\widetilde{\mathcal{H}}v = i\xi v$ with real ξ and $(Gv, v) = 0$.

Let us introduce a stability radius for a strongly stable G-Hamiltonian matrix $\mathcal{H}$ as follows:

$$rstab(\mathcal{H}) = \min\{\|\Delta\|_2 \mid \mathcal{H} + \Delta \text{ is } G\text{-Hamiltonian and not strongly stable}\}.$$

We can not derive formulas of $rstab$ for a general matrix G, so we assume now that $G = iJ$. This means that $\mathcal{H}$ is Hamiltonian and, therefore, $\mathcal{H} = J^{-1}H$ and $\Delta = J^{-1}F$, where $H = H^*$ and $F = F^*$. Since $(\mathcal{H} + \Delta)v = i\xi v$, it follows that $\Delta v = i\xi v - \mathcal{H}v$, $Fv = (i\xi J - H)v$. When v and ξ are fixed, the minimum of $\|\Delta\|_2 = \|F\|_2$ attains $\|(i\xi J - H)v\|_2/\|v\|_2$. The latter is a simple case of the dilation theorem [1], which, however, can be proved directly. As a result,

$$rstab(\mathcal{H}) = \min_{\xi \in R} \min_{v \neq 0,\, (Jv,v)=0} \frac{\|(i\xi J - H)v\|_2}{\|v\|_2} = \min_{\xi \in R} \min_{v \neq 0, (Jv,v)=0} \frac{\|(i\xi I - \mathcal{H})v\|_2}{\|v\|_2}.$$

4 Stability radius of symplectic matrices

Consider a strongly stable G-unitary matrix $\mathcal{U}$. Denote by $\widetilde{\mathcal{U}}$ a G-unitary matrix that is nearest to $\mathcal{U}$ in a special sense, to be defined below, and not strongly stable. Owing to the continuity of eigenvalues, $\widetilde{\mathcal{U}}$ has all its eigenvalues on the unit circle. Therefore, there exists a vector $v \neq 0$ such that $\widetilde{\mathcal{U}}v = e^{i\varphi}v$ with real φ and $(Gv, v) = 0$.

Let us represent $\widetilde{\mathcal{U}}$ in the form

$$\widetilde{\mathcal{U}} = (I - \Delta/2)^{-1}(I + \Delta/2)\mathcal{U},$$

where Δ is a G-Hamiltonian matrix. Accordingly, a stability radius for a strongly stable G-unitary matrix $\mathcal{U}$ is defined as follows:

$$rstab(\mathcal{U}) = \min\{\|\Delta\|_2 \mid (I - \Delta/2)^{-1}(I + \Delta/2)\mathcal{U} \text{ is } G\text{-unitary and not strongly stable}\}.$$

Again we assume that $G = iJ$, that is $\mathcal{U}$ is symplectic. Therefore, $\Delta = J^{-1}F$, where $F = F^*$. From $(I + \Delta/2)e^{-i\varphi}\mathcal{U}v = (I - \Delta/2)v$ it follows that $F(I + e^{-i\varphi}\mathcal{U})v = 2J(I -$

$e^{-i\varphi}\mathcal{U})v$. When v and φ are fixed, the minimum of $\|\Delta\|_2 = \|F\|_2$ attains the value $2\|J(I - e^{-i\varphi}\mathcal{U})v\|_2/\|(I + e^{-i\varphi}\mathcal{U})v\|_2$. Hence

$$rstab(\mathcal{U}) = \min_{\varphi\in R} \min_{v\neq 0,\,(Jv,v)=0} 2\frac{\|(I - e^{-i\varphi}\mathcal{U})v\|_2}{\|(I + e^{-i\varphi}\mathcal{U})v\|_2}.$$

5 Stability of stable eigendecompositions of Hamiltonian and symplectic matrices

In [3] the author proposed an eigendecomposition of a Hamiltonian matrix $\mathcal{H}$, which is the partition of its spectrum into the following three groups: all eigenvalues on the imaginary axis, all eigenvalues in the open left half-plane, and all eigenvalues in the open right half-plane. When continuously changing the entries of $\mathcal{H}$, the eigendecomposition introduced above may change in the sense that some eigenvalues in the open left half-plane or in the open right half-plane may come onto the imaginary axis, or some eigenvalues on the imaginary axis may jump off it. More precisely, by symmetry of the spectrum about the imaginary axis the change of the structure may happen only if a pair of symmetric eigenvalues outside the imaginary axis transforms into a multiple eigenvalue on the imaginary axis or, conversely, a multiple eigenvalue on the imaginary axis replicates into a pair of symmetric eigenvalues outside the imaginary axis. The Krein-Gel'fand-Lidskii theory says that the multiple eigenvalue on the imaginary axis, which appears at the moment of change of the structure, must be indefinite.

As a result, the above defined eigenstructure of $\mathcal{H}$, called a spectral trichotomy, is stable if and only if all eigenvalues of $\mathcal{H}$ lying on the imaginary axis are definite. When continuously changing $\mathcal{H}$, the change of the structure may happen if and only if an indefinite eigenvalue appears on the imaginary axis. Thus, the set of Hamiltonian matrices with unstable spectral trichotomy consists of the matrices with at least one indefinite eigenvalue on the imaginary axis.

If $\mathcal{H}$ has a stable spectral trichotomy, then the 2-norm distance from $\mathcal{H}$ to the set of Hamiltonian matrices with unstable spectral trichotomy equals

$$\min_{\xi\in R} \min_{v\neq 0,(Jv,v)=0} \frac{\|(i\xi I - \mathcal{H})\|_2}{\|v\|_2}.$$

The proof of this formula is analogous to that of $rstab$.

The spectral trichotomy is introduced in the same manner for symplectic matrices, where the imaginary axis is replaced by the unit circle, the left open half-plane by the open interior of the unit circle, and the open right half-plane by the open exterior of the unit circle. Similarly, the spectral trichotomy of $\mathcal{U}$ is stable if and only if all eigenvalues of $\mathcal{U}$ on the unit circle are definite. The 2-norm distance from $\mathcal{U}$ to the set of symplectic matrices with unstable spectral trichotomy equals

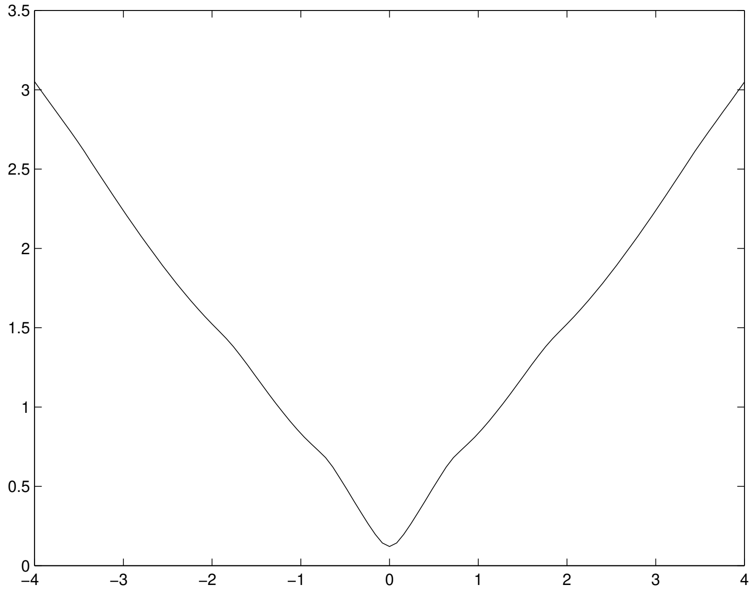
$$\min_{\phi\in R} \min_{v\neq 0,(Jv,v)=0} 2\frac{\|(I - e^{-i\phi}\mathcal{U})v\|_2}{\|(I + e^{-i\phi}\mathcal{U})v\|_2}.$$

6 An example

For demonstration purposes we have computed the function

$$r(\xi) = \min_{v \neq 0, (Jv,v)=0} \frac{\|(i\xi I - \mathcal{H})\|_2}{\|v\|_2}$$

using the MATLAB Optimization Toolbox. The Hamiltonian matrix $\mathcal{H}$ has been chosen in the form $\mathcal{H} = J^{-1}H$, where H is symmetric positive definite. It is easy to show that such Hamiltonian matrices are always strongly stable. Below is a plot of $r(\xi)$ for $\mathcal{H} = J^{-1}H$, where H is the tridiagonal matrix of order 8 with 2 on the main diagonal and -1 on the subdiagonal and superdiagonal. Its eigenvalues computed by MATLAB are $\pm 3.6601i$, $\pm 2.6307i$, $\pm 1.2740i$, and $\pm 0.2446i$.



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