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Classification of the weight hierarchies of binary linear codes of dimension 4*

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Abstract

The weight hierarchy of a binary linear $[n, k]$ code C is the sequence $(d_1, d_2, \ldots, d_k)$ where d_r is the smallest support of an r–dimensional subcode of C. The codes of dimension 4 are collected in classes and the possible weight hierarchies in each class is determined.

Keywords: Weight hierarchy, support weight, binary linear code, chain condition, difference sequence.

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I Introduction

The weight hierarchy of linear codes has been studied by a number of researchers. For a code of dimension k, it is a sequence of parameters $(d_1, d_2, \ldots, d_k)$. In particular, d_1 is the minimum distance of the code. The parameters were first introduced in [11]. In [17] it was shown that these parameters are important in the analysis of an application of linear codes to the wiretap channel of type II. Later, the weight hierarchy has been shown to be important in the analysis of the trellis complexity of linear codes, see e.g. [9], [13], [16]; and analysis of linear codes for error detection on the local binomial channel, see [15]. The possible weight hierarchies of binary linear codes of dimension up to 4 were determined in [14]. In [2]–[7] we studied the possible weight hierarchies of linear codes of dimension 4 or less over arbitrary finite fields. In [2] we studied the weight hierarchies of codes of dimension 4; we split them into two classes: the weight hierarchies of codes satisfying the so-called chain condition and other weight hierarchies. However, this is a quite crude classification of the weight hierarchies. In particular, it does not tell us what are the possible weight hierarchies of codes not satisfying the chain condition since it is well known that there exist pairs of codes C_1 and C_2 with the same weight hierarchy and such that C_1 satisfies the chain condition whereas C_2 does not. More information is obtained if the codes are classified according to how the subspaces of minimal support and different degrees are related. However, such an analysis is also more complicated.

In [6] we introduced a classification of the codes of dimension 4 into 9 classes. One of these classes is the class of codes satisfying the chain condition. Another is the class of extremal non-chain codes which we studied in [5] and [7]. In [6] we only gave upper bounds on the d_i in the weight hierarchies (d_1, d_2, d_3, d_4) of the codes in various classes (for q-ary linear codes). In this paper we will give a complete characterization of the possible weight hierarchies (d_1, d_2, d_3, d_4) for each class in the binary case.

II Notations and problem formulation

Throughout this paper, unless otherwise stated, C will be an $[n, 4]$ code, that is, a binary linear code of length n and dimension 4. For convenience we give all definitions below for 4-dimensional codes, rather than codes of general dimension, since we concentrate on 4-dimensional codes.

For any subcode D of C, the *support* of D is the set of positions where not all the codewords of D are zero, and it is denoted by $\chi(D)$. Further, the *support weight* of D is the size of $\chi(D)$, and it is denoted by $w_S(D)$.

For $1 \le r \le 4$, the *the r-th minimum support weight* (or Generalized Hamming weight) of C is defined by

$$d_r(C) = \min\{w_S(D) \mid D \text{ is an } [n, r] \text{ subcode of } C\}.$$

The sequence (d_1, d_2, d_3, d_4) is the *weight hierarchy* of C.

We note that if we add a zero-position to an $[n, 4]$ code C we get an $[n + 1, 4]$ code

$$C' = \{(\mathbf{c}|0) \mid \mathbf{c} \in C\}.$$

The codes C and C' have the same weight hierarchy. Therefore, without loss of generality, we can restrict ourselves to codes without zero-positions, that is, we will assume that $n = d_4$.

Our problems can be reformulated in terms of projective geometry and we do this next.

The *difference sequence* (DS) (i_0, i_1, i_2, i_3) of a $[d_4, 4]$ code is defined by

$$i_0 = d_4 - d_3, \quad i_1 = d_3 - d_2, \quad i_2 = d_2 - d_1, \quad i_3 = d_1.$$

The difference sequence can easily be computed from the weight hierarchy and vice versa. It was shown in [11] that

$$i_r \geq 1 \quad \text{for all } r. \tag{1}$$

Let G be a generator matrix for C. For any $\mathbf{x} \in GF(2)^4$, $m(\mathbf{x})$, the *value* of $\mathbf{x}$, will denote the number of occurrences of $\mathbf{x}$ as a column in G. In [12] it was shown that there is a one-one correspondence between the subspaces C of dimension r and the subspaces of $GF(2)^4$ of dimension $(4-r)$ such that if D corresponds to U, then

$$w_S(D) + \sum_{\mathbf{x} \in U} m(\mathbf{x}) = d_4.$$

We may view the vectors as points in the projective space $V_3 = PG(3,2)$. A *value assignment* is a function

$$m : V_3 \rightarrow \mathbb{N} = \{0, 1, 2, \ldots\}.$$

For $p \in V_3$ we call $m(p)$ the *value* of p. A value assignment defines a generator matrix and a code (up to equivalence). We define the value of a subset S of V_3 as follows:

$$m(S) = \sum_{p \in S} m(p).$$

Let C_1 be the set of lines and C_2 the set of planes in V_3. The existence of a code with weight hierarchy (d_1, d_2, d_3, d_4) is equivalent to the existence of a value assignment m such that

$$\begin{aligned}
\max\{m(p) \mid p \in V_3\} &= i_0, \\
\max\{m(l) \mid l \in C_1\} &= i_0 + i_1, \\
\max\{m(P) \mid P \in C_2\} &= i_0 + i_1 + i_2, \\
m(V_3) &= i_0 + i_1 + i_2 + i_3.
\end{aligned}$$

Let

$$\begin{aligned}
M_0 &= \{p \mid p \in V_3 \text{ and } m(p) = i_0\}, \\
M_1 &= \{l \mid l \in C_1 \text{ and } m(l) = i_0 + i_1\}, \\
M_2 &= \{P \mid P \in C_2 \text{ and } m(P) = i_0 + i_1 + i_2\}.
\end{aligned}$$

Following [6], we introduce some conditions which may or may not be true.

(Con1) There exist $p \in M_0$, $l \in M_1$, and $P \in M_2$ such that $p \in l \subset P$.

(Con2) There exist $p \in M_0$ and $l \in M_1$ such that $p \in l$.

(Con3) There exist $p \in M_0$ and $P \in M_2$ such that $p \in P$.

(Con4) There exist $l \in M_1$ and $P \in M_2$ such that $l \subset P$.

We split the codes (and value assignments) into classes. The classes are determined by if the conditions above are true (T) or false (F) according to the following table:

Class	**(Con1)**	**(Con2)**	**(Con3)**	**(Con4)**
A	T	T	T	T
B	F	T	T	T
C	F	T	T	F
D	F	T	F	T
E	F	T	F	F
F	F	F	T	T
G	F	F	T	F
H	F	F	F	T
I	F	F	F	F

The space $V_3 = PG(3,2)$

The space $PG(3,2)$ contains 15 points, 35 lines, and 15 planes. We assign the letters A–O to the points, l_1–l_{35} to the lines and P_1–P_{15} to the planes. The description of the lines and the planes are given in the following table. It is also illustrated in Fig. 1, where some of the lines are included. Some of the planes appears as triangles in the figure.

i	l_i	i	l_i	i	l_i
1	$\{A,D,G\}$	2	$\{D,E,H\}$	3	$\{B,D,F\}$
4	$\{C,D,J\}$	5	$\{D,I,K\}$	6	$\{D,L,N\}$
7	$\{D,M,O\}$	8	$\{A,B,E\}$	9	$\{A,C,I\}$
10	$\{A,F,H\}$	11	$\{A,J,K\}$	12	$\{A,L,M\}$
13	$\{A,N,O\}$	14	$\{B,C,L\}$	15	$\{B,G,H\}$
16	$\{B,I,M\}$	17	$\{B,J,N\}$	18	$\{B,K,O\}$
19	$\{C,E,M\}$	20	$\{C,F,N\}$	21	$\{C,G,K\}$
22	$\{C,H,O\}$	23	$\{E,F,G\}$	24	$\{E,I,L\}$
25	$\{E,J,O\}$	26	$\{E,K,N\}$	27	$\{F,I,O\}$
28	$\{F,J,L\}$	29	$\{F,K,M\}$	30	$\{G,I,J\}$
31	$\{G,L,O\}$	32	$\{G,M,N\}$	33	$\{H,I,N\}$
34	$\{H,J,M\}$	35	$\{H,K,L\}$		

i	P_i	i	P_i
1	$\{A,C,D,G,I,J,K\}$	2	$\{C,D,E,H,J,M,O\}$
3	$\{B,C,D,F,J,L,N\}$	4	$\{A,B,D,E,F,G,H\}$
5	$\{A,B,C,E,I,L,M\}$	6	$\{A,D,G,L,M,N,O\}$
7	$\{A,B,E,J,K,N,O\}$	8	$\{A,C,F,H,I,N,O\}$
9	$\{A,F,H,J,K,L,M\}$	10	$\{B,C,G,H,K,L,O\}$
11	$\{B,D,F,I,K,M,O\}$	12	$\{B,G,H,I,J,M,N\}$
13	$\{C,E,F,G,K,M,N\}$	14	$\{D,E,H,I,K,L,N\}$
15	$\{E,F,G,I,J,L,O\}$		

III Discussion of the various cases

Class A, which is the set of codes satisfying the chain condition, was studied in [2]. The possible weight hierarchies of codes in class I were determined in [7]. In this paper we study the remaining classes. We characterize the possible weight hierarchies of the codes in each class. For each case, we prove a set of necessary conditions for (i_0, i_1, i_2, i_3) to be a DS of a code in that class and then we give constructions which show that these conditions are also sufficient. For two of the constructions in class E (which is one of the most complicated), we give, as an illustration, the details of the verification that the given constructions have the stated properties. The remaining cases are similar and the details are omitted.

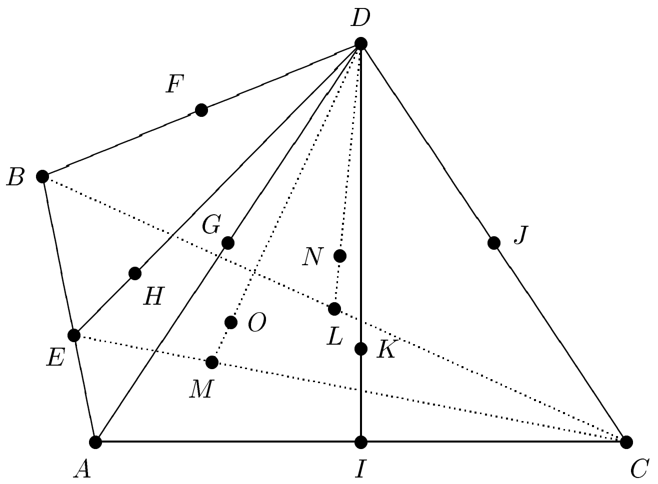

Figure 1: Representation of $PG(3,2)$

We start by giving some general results. Theorem 1 in [6], specialized to $q = 2$, gives the following lemma.

Lemma 1 *Let* (i_0, i_1, i_2, i_3) *be a DS and suppose* **(Con1)** *is false. Then*
i) $i_1 \leq 2i_0$,
ii) if **(Con2)** *is false or* **(Con4)** *is true, then* $i_1 \leq 2i_0 - 3$,
iii) if **(Con3)** *is true, then* $i_2 \leq 2i_1 - 3$,
iv) $i_3 \leq 2i_2$,
v) if **(Con2)** *is true or* **(Con4)** *is false, then* $i_3 \leq 2i_2 - 3$.

■

In the rest of the paper, the integers b, t, θ, and ϑ are defined as follows.

$$2i_0 - i_1 = 3b - t \tag{2}$$

where $t \in \{-1, 0, 1\}$,

$$\theta = \begin{cases} 1 & \text{if } t = 1, \\ 0 & \text{if } t \in \{-1, 0\}, \end{cases} \tag{3}$$

$$\vartheta = \begin{cases} 1 & \text{if } t = -1, \\ 0 & \text{if } t \in \{0, 1\}. \end{cases} \tag{4}$$

For the various classes or constructions, we will define further parameters c, u, etc. which will be defined differently from class to class and sometimes from construction to construction.

We first give the following additional upper bounds.

Lemma 2 *Let* (i_0, i_1, i_2, i_3) *be a DS and suppose* **(Con3)** *is false. Then*
i) if **(Con2)** *is true and* **(Con4)** *is false, then* $i_1 \leq 2i_0 - 1$,
ii) $i_2 \leq (4i_0 + 4i_1 - 4\theta)/3$,
iii) $i_2 \leq 6i_0 - i_1 - 7$,
iv) $i_2 \leq 6i_1 - i_0$,
v) if **(Con4)** *is false, then* $i_2 \leq (4i_0 + 4i_1 - 4\vartheta - 7)/3$,
vi) if **(Con2)** *is true and* **(Con4)** *is false, then* $i_2 \leq i_0 + 2i_1 - 2$,
vii) $i_3 \leq 6i_1 - i_2$,
viii) if **(Con2)** *is false, then* $i_3 \leq 6i_1 - i_2 - 7$,
ix) if **(Con2)** *is false and* **(Con4)** *is true, then* $i_3 \leq 2i_2 - 1$.

Proof: i), vi). Since **(Con2)** is true there exist $p^* \in M_0$ and $l^* \in M_1$ such that $p^* \in l^*$. Let $P^* \in M_2$. Since **(Con4)** is false, $l^* \not\subset P^*$. Let $l^* \cap P^* = \{p\}$. Since **(Con3)** is false we have $m(p) \leq i_0 - 1$. Let $l^* = \{p, p^*, p'\}$. Then

$$i_0 + i_1 = m(l^*) = m(p) + m(p^*) + m(p') \leq i_0 - 1 + i_0 + i_0,$$

and so $i_1 \leq 2i_0 - 1$ which proves i). Since **(Con3)** is false, $p^* \notin P^*$. Further,

$$i_0 + i_1 = m(l^*) \geq m(p) + m(p^*) = m(p) + i_0$$

and so $m(p) \leq i_1$. Hence

$$m(P^* \setminus \{p\}) = i_0 + i_1 + i_2 - m(p) \geq i_0 + i_2.$$

Therefore, there exists a line $l \subset (P^* \setminus \{p\})$ such that

$$m(l) \geq \frac{1}{4} m(P^* \setminus \{p\}) \geq \frac{1}{2}(i_0 + i_2).$$

Since **(Con4)** is false, $m(l) \leq i_0 + i_1 - 1$. Hence

$$\frac{1}{2}(i_0 + i_2) \leq i_0 + i_1 - 1$$

and vi) follows.
ii), iii), v). Let $P^* \in M_2$ and let p be a point in P^* of maximal value. Then

$$m(p) \geq \frac{1}{7} m(P^*) = \frac{1}{7}(i_0 + i_1 + i_2). \tag{5}$$

Since **(Con3)** is false, $m(p) \leq i_0 - 1$. Combining this with (5) we get iii). Next, since

$$i_0 + i_1 + i_2 = m(P^*) = m(p) + \sum_{p \in l \subset P} \big(m(l) - m(p)\big),$$

there exists a line l, such that $p \in l \subset P^*$, satisfying

$$m(l) - m(p) \geq \frac{1}{3}\big(i_0 + i_1 + i_2 - m(p)\big).$$

Hence

$$\frac{2m(p)}{3} \leq m(l) - \frac{i_0 + i_1 + i_2}{3} \leq i_0 + i_1 - \frac{i_0 + i_1 + i_2}{3} \tag{6}$$

and so

$$m(p) \leq i_0 + i_1 - \frac{i_2}{2}. \tag{7}$$

Combining (5) and (7) we get

$$\frac{1}{7}(i_0 + i_1 + i_2) \leq i_0 + i_1 - \frac{i_2}{2}$$

which proves

$$i_2 \leq (4i_0 + 4i_1)/3. \tag{8}$$

that is, ii) for $\theta = 0$. If $\theta = 1$, that is, $i_1 = 2i_0 - 3\alpha + 1$, then (8) implies

$$i_2 \leq \left\lfloor \frac{4i_0 + 4(2i_0 - 3\alpha + 1)}{3} \right\rfloor = 4i_0 - 4\alpha + 1.$$

Suppose $i_2 = 4i_0 - 4\alpha + 1$. Then (5) implies

$$m(p) \geq \frac{1}{7}(i_0 + i_1 + i_2) = i_0 - \alpha + \frac{2}{7} > i_0 - \alpha$$

and (7) implies

$$m(p) \leq i_0 + i_1 - \frac{i_2}{2} = i_0 - \alpha + \frac{1}{2} < i_0 - \alpha + 1,$$

but this is impossible since $m(p)$ is an integer. Hence

$$i_2 \leq 4i_0 - 4\alpha = \frac{4i_0 + 4i_1 - 4}{3}.$$

If, in addition, **(Con4)** is false, then $m(l) \leq i_0 + i_1 - 1$ in (6) which implies

$$m(p) \leq i_0 + i_1 - \frac{i_2 + 3}{2}. \tag{9}$$

Combined with (5) this implies

$$i_2 \leq (4i_0 + 4i_1 - 7)/3. \tag{10}$$

This proves v) for $\vartheta = 0$. If $\vartheta = 1$, then $i_1 = 2i_0 - 3\alpha - 1$ and (10) implies

$$i_2 \leq \left\lfloor \frac{4i_0 + 4(2i_0 - 3\alpha - 1) - 7}{3} \right\rfloor = 4i_0 - 4\alpha - 4.$$

Suppose $i_2 = 4i_0 - 4\alpha - 4$, then (5) and (9) implies

$$i_0 - \alpha + \frac{5}{7} \leq m(p) \leq i_0 - \alpha + \frac{1}{2},$$

which is impossible. Hence

$$i_2 \leq 4i_0 - 4\alpha - 5 = \frac{4i_0 + 4i_1 - 11}{3}.$$

iv) Let $p^* \in M_0$. Since **(Con3)** is false, $p^* \notin P^*$. In particular, $p^* \neq p$. Let l be the line determined by p and p^*. Then, by (5),

$$i_0 + i_1 \geq m(l) \geq i_0 + m(p) \geq i_0 + \frac{1}{7}(i_0 + i_1 + i_2)$$

and so $i_2 \leq 6i_1 - i_0$.
vii), viii). We have

$$i_0 + i_1 + i_2 + i_3 = m(V_3) = m(p^*) + \sum_{\substack{l \in C_1 \\ p^* \in l}} \big(m(l) - m(p^*)\big) \leq i_0 + 7i_1.$$

proving vii). Further, if **(Con2)** is false, then

$$m(l) - m(p^*) \leq i_1 - 1$$

for all l and so $i_3 \leq 6i_1 - i_2 - 7$ which proves viii).
ix). Since **(Con4)** is true, there exist $l^* \in M_1$ and $P^* \in M_2$ such that $l^* \subset P^*$. Let $p^* \in M_0$. Since **(Con2)** is false, $p^* \notin l^*$. Since **(Con3)** is false, the plane containing p^* and l^* has value at most $i_0 + i_1 + i_2 - 1$. Hence

$$i_0 + i_1 + i_2 + i_3 = m(V_3) = m(l^*) + \sum_{\substack{P \\ l^* \subset P}} m(P \setminus l^*) \leq (i_0 + i_1) + i_2 + i_2 + (i_2 - 1)$$

and so $i_3 \leq 2i_2 - 1$.

■

We now go on to give some lower bounds.

Lemma 3 *Let (i_0, i_1, i_2, i_3) be a DS and suppose* **(Con1)** *is false. Then*

i) if **(Con4)** *is true, then $i_1 \geq i_0/2$;*
ii) if **(Con4)** *is true and* **(Con2)** *is false, then $i_1 \geq (i_0 + 3)/2$;*
iii) if **(Con4)** *is true, then $i_2 \geq i_0$;*
iv) if **(Con4)** *is true and* **(Con3)** *is false, then $i_2 \geq i_0 + 1$;*
v) if **(Con3)** *is true and* **(Con2)** *and* **(Con4)** *are false, then $i_2 \geq i_0 + 1$;*
vi) if **(Con3)** *is false, then $i_3 \geq i_0$;*
vii) if **(Con3)** *is true and* **(Con2)** *and* **(Con4)** *are false, then $i_3 \geq i_0 + 1$;*
viii) if **(Con2)** *is true, then $i_3 \geq i_1$;*
ix) if **(Con2)** *is true and* **(Con3)** *is false, then $i_3 \geq i_1 + 1$;*
x) if **(Con4)** *is false and* **(Con2)** *or* **(Con3)** *is false, then $i_3 \geq i_1 + 1$;*
xi) if **(Con2)** *is true and* **(Con3)** *is false, then $i_3 \geq i_2/2$;*
xii) if **(Con2)** *is true and* **(Con3)** *and* **(Con4)** *are false, then $i_3 \geq (i_2 + 3)/2$.*

Proof: i)–iv) Since **(Con4)** is true, there exist $l^* \in M_1$ and $P^* \in M_2$ such that $l^* \subset P^*$. Let $p^* \in M_0$. Since **(Con1)** is false, $p^* \notin l^*$. Let $p \in l^*$ have maximal value. Then $m(p) \geq (i_0 + i_1)/3$. Let l be the line determined by p and p^*. Then

$$i_0 + i_1 \geq m(l) \geq m(p^*) + m(p) \geq i_0 + \frac{i_0 + i_1}{3},$$

and so $i_1 \geq i_0/2$ which proves i). If, in addition, **(Con2)** is false, then $m(l) \leq i_0 + i_1 - 1$ and ii) follows. Let P be the plane determined by p^* and l^*. Then

$$i_0 + (i_0 + i_1) = m(p^*) + m(l^*) \leq m(P) \leq i_0 + i_1 + i_2$$

and so $i_2 \geq i_0$ which proves iii). If, in addition, **(Con3)** is false, then $m(P) \leq i_0 + i_1 + i_2 - 1$ and so $i_2 \geq i_0 + 1$ which proves iv).
v) Since **(Con3)** is true there exist $p^* \in M_0$ and $P^* \in M_2$ such that $p^* \in P^*$. Let $l^* \in M_1$. Since **(Con2)** is false $p^* \notin l^*$. Let P be the plane determined by p^* and l^*. Since **(Con4)** is false, $m(P) \leq i_0 + i_1 + i_2 - 1$. Hence

$$i_0 + i_1 + i_2 - 1 \geq m(P) \geq m(p^*) + m(l^*) = i_0 + i_0 + i_1$$

and so $i_2 \geq i_0 + 1$.
vi) Let $p^* \in M_0$ and $P^* \in M_2$. Since **(Con3)** is false, $p^* \notin P^*$. Hence

$$i_0 + i_1 + i_2 + i_3 = m(V_3) \geq m(p^*) + m(P^*) = i_0 + (i_0 + i_1 + i_2)$$

and vi) follows.
vii) Since **(Con3)** is true there exist $p^* \in M_0$ and $P^* \in M_2$ such that $p^* \in P^*$. Let $l^* \in M_1$. Then $p^* \notin l^*$ and $l^* \not\subset P^*$. Let $p = l^* \cap P^*$ and let l be the line determined by p and p^*. Then $l \notin M_1$. Hence

$$i_0 + i_1 - 1 \geq m(l) \geq i_0 + m(p)$$

and so $m(p) \leq i_1 - 1$. This implies that

$$i_0 + i_1 + i_2 + i_3 = m(V_3) \geq m(P^*) + m\big(l^* \setminus \{p\}\big) \geq (i_0 + i_1 + i_2) + \big(i_0 + i_1 - (i_1 - 1)\big)$$

and so $i_3 \geq i_0 + 1$.
viii) Since **(Con2)** is true, there exist $p^* \in M_0$ and $l^* \in M_1$ such that $p^* \in l^*$. Let $P^* \in M_2$. Since **(Con1)** is false, $l^* \not\subset P^*$. Let $l^* \cap P^* = \{p\}$. Then

$$i_0 + i_1 + i_2 + i_3 = m(V_3) \geq m(l^* \cup P^*) = (i_0 + i_1 + i_2) + (i_0 + i_1) - m(p) \geq i_0 + 2i_1 + i_2$$

since $m(p) \leq i_0$. This proves viii). If, in addition, **(Con3)** is false, then $m(p) \leq i_0 - 1$ and ix) follows.

x) Let $l^* \in M_1$ and $P^* \in M_2$. Since **(Con4)** is false, $l^* \not\subset P^*$. Let $l^* \cap P^* = \{p\}$. Since **(Con2)** or **(Con3)** is false, $p \notin M_0$ and so $m(p) \leq i_0 - 1$. As in case ix) we get $i_3 \geq i_1 + 1$.
xi) Let p^*, p, l^*, and P^* be as in the proof of viii). Since

$$i_0 + i_1 + i_2 = m(P^*) = m(p) + \sum_{\substack{l \\ p \in l \subset P^*}} \big(m(l) - m(p)\big)$$

there exists a line l such that $p \in l \subset P^*$ and

$$m(l) - m(p) \geq \frac{1}{3}\big(i_0 + i_1 + i_2 - m(p)\big).$$

Since $m(l) \leq i_0 + i_1$, this implies

$$m(p) \leq i_0 + i_1 - \frac{i_2}{2}.$$

Since $\big(l^* \setminus \{p\} \subset V_3 \setminus P^*$, we get

$$i_3 = m(V_3 \setminus P^*) \geq m(l^* \setminus \{p\}) \geq \frac{i_2}{2}.$$

This proves xi). If, in addition, **(Con4)** is false, then $m(l) \leq i_0 + i_1 - 1$ and xii) follows.
■

As a final general lemma, we describe how a DS may be modified to give new DS. First be introduce some terminology. Consider a value assignment m in class B as an example. By definition, **(Con2)**, **(Con3)**, and **(Con4)** are true. Therefore, there exist points $p_2^*, p_3^* \in M_0$, lines $l_2^*, l_4^* \in M_1$, and planes $P_3^*, P_4^* \in M_2$, such that $p_2^* \in l_2^*$, $p_3^* \in P_3^*$, and $l_4^* \subset P_4^*$. We call the set

$$\Gamma = \{\{p_2^*\}, \{p_3^*\}, l_2^*, l_4^*, P_3^*, P_4^*\}$$

a *core* of m (it may not be unique). The points p_2^* and p_3^* may or may not be distinct, and similarly for the planes. However, note that we can not have $l_2^* = l_4^*$ since if $l_2^* = l_4^*$, then $p_2^* \in l_2^* \subset P_4^*$ and so **(Con1)** would be true which is not the case for class B. The value of the core given above is defined by

$$m(\Gamma) = m(\{p_2^*\} \cup \{p_3^*\} \cup l_2^* \cup l_4^* \cup P_3^* \cup P_4^*).$$

For the various constructions below, we will give one core explicitly (usually it is unique).

Lemma 4 *Let (i_0, i_1, i_2, i_3') be the DS corresponding to the value assignment m and let Γ be a core of m. Then (i_0, i_1, i_2, i_3) is a DS in the same class if $i_3 \geq 1$ and*

$$m(\Gamma) - (i_0 + i_1 + i_2) \leq i_3 \leq i_3'.$$

Proof: Suppose $i_3' > 1$

$$m(\Gamma) < m(V_3) = i_0 + i_1 + i_2 + i_3'.$$

Then there exists a $p' \in V_3 \setminus \Gamma$ such that $m(p') > 0$. Define m' by

$$m'(p) = \begin{cases} m(p) - 1 & \text{if } p = p', \\ m(p) & \text{otherwise.} \end{cases}$$

The corresponding DS is $(i_0, i_1, i_2, i_3' - 1)$ and it is clearly in the same class as the original DS and has the same core. We can repeat this modification as long as $i_3 > 1$ and $m(\Gamma) < m(V_3)$. ■

Class A: (Con1), (Con2), (Con3), (Con4) true.

This is the case of codes satisfying the chain condition, and it was treated in [14] where the following result was shown (in a slightly different notation).

Theorem 1 *(i_0, i_1, i_2, i_3) is a class A DS if and only if*
i) $1 \leq i_1 \leq 2i_0$,
ii) $1 \leq i_2 \leq 2i_1$,
iii) $1 \leq i_3 \leq 2i_2$.

Class B: (Con1) false; (Con2), (Con3), (Con4) true.

Theorem 2 *(i_0, i_1, i_2, i_3) is a class B DS if and only if*
i) $i_1 \leq 2i_0 - 3$,
ii) $i_0 \leq i_2 \leq 2i_1 - 3$,
iii) $i_1 \leq i_3 \leq 2i_2 - 3$.

Proof: We first show the "only if" part, that is, that the bounds must be satisfied. The upper bounds follow from Lemma 1 ii), iii) and v) and the lower bounds from Lemma 3 iii) and viii).

Construction for class B.

Suppose i) and ii) are satisfied and $i_3 = 2i_2 - 3$. We show, by an explicit construction, that $(i_0, i_1, i_2, 2i_2 - 3)$ is a class B DS. For the verification of the construction it is useful to observe that $i_0 \leq i_2 \leq 2i_1 - 3$ implies that $i_1 \geq \frac{1}{2}(i_0 + 3)$ and $\frac{1}{2}(i_0 + 3) \leq i_1 \leq 2i_0 - 3$ implies $i_0 \geq 3$.

We will use the following notation.

$$2i_1 - i_2 - 3 = 3c + u \text{ where } u \in \{0, 1, 2\},$$

$$\alpha = \begin{cases} 1 & \text{if } i_2 = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Further, $\delta(i_1, p)$, $\varepsilon(u, p)$, and $m(p)$ are defined by the following tables.

$\delta(p) = \delta(i_1, p)$				
i_1	D	F, G, O	H, K, N	J
even	0	-4	-2	0
odd	-1	-3	-3	1

$\varepsilon(p) = \varepsilon(u, p)$				
u	F, G, M, O	H	I, N	K, L
0	0	0	0	0
1	0	1	$1 - \alpha$	α
2	1	0	1	0

$m(p) =$	for
i_0	$p = C$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
$\frac{1}{2}\big(i_1 + \delta(p)\big)$	$p \in \{D, J\}$
$\frac{1}{2}\big(i_1 + \delta(p)\big) - c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$
$i_1 - i_0 + b - 1 - c - \varepsilon(p) - t$	$p = L$
$i_1 - i_0 + b - 1 - c - \varepsilon(p)$	$p \in \{I, M\}$

We skip the details of the proof, but we mention a few main points of the construction. We have

$$M_0 = \{C\}, \quad M_1 = \{l_4, l_8\}, \quad P_5 \in M_2, \quad P_1, P_2, P_3 \notin M_2,$$

$$m(V_3) = i_0 + i_1 + i_2 + (2i_2 - 3).$$

These facts imply that m is in class B with DS $(i_0, i_1, i_2, i_3 = 2i_2 - 3)$.

A core of m is $\Gamma = \{\{C\}, l_4, l_8, P_5\}$ and $m(\Gamma) = i_0 + 2i_1 + i_2$. Using Lemma 4 we can conclude that (i_0, i_1, i_2, i_3) is a class B DS for $i_1 \le i_3 \le 2i_2 - 3$. ■

Class C: (Con1), (Con4) false; (Con2), (Con3) true.

Theorem 3 *(i_0, i_1, i_2, i_3) is a class C DS if and only if*

i) $i_1 \le 2i_0$,

ii) $i_2 \le 2i_1 - 3$,

iii) $i_1 \le i_3 \le 2i_2 - 3$.

Proof: The upper bounds follow from Lemma 1 i), iii) and v), and the lower bound from Lemma 3 viii).

Construction for class C.

Assume that i)–iii) are satisfied. We note that this implies that

$$i_0 \ge 2, \quad 3 \le i_1 \le 2i_0, \quad \text{and} \quad \frac{1}{2}(i_1 + 3) \le i_2 \le 2i_1 - 3.$$

Let

$$2i_1 - i_2 - 3 = 6c + u \text{ where } u \in \{0, 1, 2, 3, 4, 5\},$$

and

$$\alpha = \begin{cases} 2 & \text{if } i_1 = 2i_2 - 4, \\ 0 & \text{otherwise.} \end{cases}$$

Let δ, ε and m be defined by the following tables.

$\delta(p) = \delta(i_1, p)$					
i_1	A, B, E	D	F, G, H	I, J, L, M	K, N, O
even	-2	0	$-\alpha - 2$	0	$\alpha - 4$
odd	-3	-1	-3	1	-3

$\varepsilon(p) = \varepsilon(u, p)$									
u	A	B, G, H	E	F	I	K	L	M	N, O
0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	1
2	1	0	1	1	0	1	0	0	1
3	1	1	1	1	0	1	0	0	1
4	1	1	1	1	1	1	0	0	2
5	1	1	1	2	1	2	0	1	2

$m(p) =$	for
i_0	$p = C$
$\frac{1}{2}(i_1 + \delta(p))$	$p \in \{D, J\}$
$\frac{1}{2}(i_1 + \delta(p)) - c - \varepsilon(p)$	$p \in \{A, B, E, I, L, M\}$
$\frac{1}{2}(i_1 + \delta(p)) - 2c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$

This m corresponds to the class C DS $(i_0, i_1, i_2, 2i_2 - 3)$. For this construction we have

$$M_0 = \{C\}, \quad M_1 = \{l_4\}, \quad M_2 = \{P_5\}.$$

A core of m is $\Gamma = \{\{C\}, l_4, P_5\}$ and $m(\Gamma) = i_0 + 2i_1 + i_2$. Using Lemma 4 we can conclude that (i_0, i_1, i_2, i_3) is a class C DS for $i_1 \le i_3 \le 2i_2 - 3$. ■

Class D: (Con1), (Con3) false; (Con2), (Con4) true.

Theorem 4 *(i_0, i_1, i_2, i_3) is a class D DS if and only if*

i) $i_0/2 \le i_1 \le 2i_0 - 3$,

ii) $i_0 + 1 \le i_2 \le (4i_0 + 4i_1 - 4\theta)/3$,

iii) $\max(i_0, i_1 + 1, i_2/2) \le i_3 \le \min(2i_2 - 3, 6i_1 - i_2)$.

Proof: The upper bounds follow from Lemma 1 ii) and v) and Lemma 2 ii) and vii). The lower bounds follow from Lemma 3 i) iv), vi), ix), xi).

Construction 1 for class D.

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \le 2i_1$. In particular, this implies that

$$\max(i_0, i_1 + 1, i_2/2) = \max(i_0, i_1 + 1),$$

$$\min(2i_2 - 3, 6i_1 - i_2) = 2i_2 - 3,$$

and so

$$i_0 \ge 3, \quad (i_0 + 1)/2 \le i_1 \le 2i_0 - 3, \quad \text{and} \quad i_0 + 1 \le i_2 \le 2i_1.$$

Let

$$2i_1 - i_2 = 3c + u \quad \text{where} \quad u \in \{0, 1, 2\},$$

and

$$\alpha = \begin{cases} 1 & \text{if } t = 1 \text{ and } i_2 \in \{i_0 + 1, i_0 + 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\delta(p)$, $\varepsilon(p)$ and $m_1(p)$ as follows.

$\delta(p) = \delta(i_1, p)$			
i_1	C, I	D, G, L, M	F, H
even	0	0	-2
odd	1	-1	-1

$\varepsilon(p) = \varepsilon(u, p)$				
u	F, G, M, O	H	I, N	L
0	0	0	0	0
1	0	1	1	0
2	1	0	1	0

$m_1(p) =$	for
i_0	$p = J$
$i_0 - b$	$p \in \{B, E\}$
$i_0 - b + t$	$p = A$
$\frac{1}{2}(i_1 + \delta(p))$	$p \in \{C, D\}$
$\frac{1}{2}(i_1 + \delta(p)) - \alpha - c - \varepsilon(p)$	$p = G$
$\frac{1}{2}(i_1 + \delta(p)) + \alpha - c - \varepsilon(p)$	$p = H$
$\frac{1}{2}(i_1 + \delta(p)) - c - \varepsilon(p)$	$p \in \{F, I, L, M\}$
$i_1 - i_0 + b - 1 + \alpha - t - c$	$p = K$
$i_1 - i_0 + b - c - \varepsilon(p)$	$p = N$
$i_1 - i_0 + b - \alpha - c - \varepsilon(p)$	$p = O$

For this construction (and the other constructions for class D) we have

$$M_0 = \{J\}, \quad l_4, l_8 \in M_1, \qquad M_2 = \{P_5\}.$$

A core of m is $\Gamma = \{\{J\}, l_4, P_5\}$.

The corresponding DS, therefore, is in class D and it is $(i_0, i_1, i_2, 2i_2 - 3)$. We have

$$m(\Gamma) = 2i_0 + i_1 + i_2 + m(D) = (i_0 + i_1 + i_2) + i_0 + \lfloor i_1/2 \rfloor.$$

From Lemma 4 we get a class D DS (i_0, i_1, i_2, i_3) for $i_0 + \lfloor i_1/2 \rfloor \le i_3 \le 2i_2 - 3$.

Construction 2 for class D.

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \ge 2i_1 + 1$. In particular, this implies that

$$\max\bigl(i_0, i_1 + 1, i_2/2\bigr) = \max\bigl(i_0, i_2/2\bigr),$$

$$\min\bigl(2i_2 - 3, 6i_1 - i_2\bigr) = 6i_1 - i_2.$$

and so

$$i_0 \ge 2 + \theta, \quad i_0/2 \le i_1 \le 2i_0 - 3 - \theta,$$

and

$$2i_1 + 1 \le i_2 \le (4i_0 + 4i_1 - 4\theta)/3.$$

Let

$$i_2 - 2i_1 - 1 = 4c + u \text{ where } u \in \{0, 1, 2, 3\}.$$

Further, $\delta(i_1, p)$, $\varepsilon(u, p)$ and $m_2(p)$ are defined by the following tables.

$\delta(p) = \delta(i_1, p)$				
i_1	C, G, L	D, F, I	H	M
even	0	0	-2	2
odd	1	-1	-1	1

$\varepsilon(p) = \varepsilon(u, p)$				
u	C, D	F, L	G, I	H, M
0	0	0	0	0
1	0	0	1	0
2	0	1	1	0
3	1	1	1	0

$m_2(p) =$	for
i_0	$p = J$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
$\frac{1}{2}\bigl(i_1 + \delta(p)\bigr) + c + \varepsilon(p)$	$p \in \{C, I, L, M\}$
$\frac{1}{2}\bigl(i_1 + \delta(p)\bigr) - c - \varepsilon(p)$	$p \in \{D, F, G, H\}$
$i_1 - i_0 + b$	$p \in \{K, O\}$
$i_1 - i_0 + b - t$	$p = N$

The corresponding DS is in class D and it is $(i_0, i_1, i_2, 6i_1 - i_2)$. From Lemma 4 we get a class D DS (i_0, i_1, i_2, i_3) for $i_0 + i_1 - m_2(C) \le i_3 \le 6i_1 - i_2$.

Construction 3 for class D.

For this construction we assume that i)–iii) are satisfied and, in addition,

$$\lambda \geq \max(0, i_2 - i_0 - i_1 + \theta, i_1 - i_0 + 1)$$

where $\lambda = i_0 + i_1 - i_3$. Define $\zeta(p)$ and $m_3(p)$ by the following tables.

$\zeta(p) = \zeta(\theta, i_2 - \lambda, p)$				
$(i_2 - \lambda) \pmod 3$	θ	I	L	M
0		0	0	0
1	0	1	-2	1
1	1	-2	1	1
2		-1	2	-1

$m_3(p) =$	for
i_0	$p = J$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
λ	$p = C$
$i_1 - \lambda$	$p = D$
0	$p \in \{F, G, H, K, N, O\}$
$\frac{1}{3}\big(i_2 - \lambda - \zeta(p)\big)$	$p \in \{I, L, M\}$

Then the DS corresponding to m_3 is in class D and it is (i_0, i_1, i_2, i_3). If $i_2 \leq 2i_1$, then

$$\max(0, i_2 - i_0 - i_1 + \theta, i_1 - i_0 + 1) \leq \lceil i_1/2 \rceil$$

and if $i_2 \geq 2i_1 + 1$, then

$$\max(0, i_2 - i_0 - i_1 + \theta, i_1 - i_0 + 1) \leq m_2(C).$$

Hence, Construction 3 covers the cases not covered by Constructions 1 and 2.

Class E: (Con1), (Con3), (Con4) false; (Con2) true.

Theorem 5 *(i_0, i_1, i_2, i_3) is a class E DS if and only if*

i) $i_1 \leq 2i_0 - 1$,

ii) $i_2 \leq \min(i_0 + 2i_1 - 2, 6i_0 - i_1 - 7, 6i_1 - i_0, (4i_0 + 4i_1 - 4\vartheta - 7)/3)$,

iii) $\max(i_0, i_1 + 1, (i_2 + 3)/2) \leq i_3 \leq \min(2i_2 - 3, 6i_1 - i_2)$.

Proof: The upper bounds follow from Lemma 1 v) and Lemma 2 iii), iv), v), vi), vii). The lower bounds follow from Lemma 3 vi), ix), xii).

To show that i)–iii) are also sufficient, we give 4 constructions. Detailed proofs that the constructions have the stated properties are given in an appendix.

Construction 1 for class E.

Let

$$\alpha = \begin{cases} 0 & \text{if } i_1 \leq 2i_0 - 5 \text{ and } i_2 = 2i_1, \\ -2 & \text{otherwise,} \end{cases}$$

$$\beta = \begin{cases} -2 & \text{if } i_1 = 2i_0 - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\pi = \begin{cases} 1 & \text{if } i_1 \leq 2i_0 - 5 \text{ and } i_2 = 2i_1, \\ 0 & \text{otherwise.} \end{cases}$$

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \leq 2i_1$ and $i_1 \geq i_0$. Under these conditions,

$$\max(i_0, i_1+1, (i_2+3)/2) = i_1 + 1 + \pi,$$

and

$$\min(2i_2-3, 6i_1-i_2) = 2i_2-3.$$

It is not to hard to check that this implies that

$$i_0 \geq 2, \quad i_0 \leq i_1 \leq 2i_0-1,$$

and

$$(i_1+4+\pi)/2 \leq i_2 \leq 2i_1 + (\alpha+3\beta)/2.$$

Let

$$2i_1 + (\alpha+3\beta)/2 - i_2 = 6c+u \quad \text{where } u \in \{0,1,2,3,4,5\}.$$

Define $\delta(p)$, $\varepsilon(p)$, and $m_1(p)$ by the following tables.

$\delta(p) = \delta(i_1, p)$										
i_1	A	B	E	F	G,H	I	K	L	M	N,O
even	0	0	2	-2	-2	0	α	α+2	0	-2
odd	α+1	-1	1	α-1	-3	β+1	-1	β+1	1	-1

$\varepsilon(p) = \varepsilon(u, p)$								
u	A	B,G,H	E	F,K	I	L	M	N,O
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	1
2	1	0	1	1	0	0	0	1
3	1	1	1	1	0	0	0	1
4	1	1	1	1	1	0	0	2
5	1	1	1	2	1	0	1	2

$m_1(p) =$	for
i_0	$p = J$
$\frac{1}{2}\big(i_1+\delta(p)\big) - c - \varepsilon(p)$	$p \in \{A, B, I, L\}$
$i_0 - 2 - \frac{1}{2}\alpha$	$p = C$
$i_1 - i_0 + 2 + \frac{1}{2}\alpha$	$p = D$
$\frac{1}{2}\big(i_1+\delta(p)+\alpha\big) - c - \varepsilon(p)$	$p = E$
$\frac{1}{2}\big(i_1+\delta(p)+\beta\big) - 2c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$
$\frac{1}{2}\big(i_1+\delta(p)+\beta\big) - c - \varepsilon(p)$	$p = M$

The corresponding DS is in class E and it is $(i_0, i_1, i_2, 2i_2-3)$. A detailed proof of this fact (and similarly for Construction 3 in class E) is given in an appendix. The set $\Gamma = \{\{J\}, l_4, P_5\}$ is a core for the constructions in class E. From Lemma 4 we get class E DS (i_0, i_1, i_2, i_3) for $i_1 + 1 + \pi \leq i_3 \leq 2i_2 - 3$.

Construction 2 for class E.

Let

$$\begin{aligned}
\alpha &= \begin{cases} 0 & \text{if } i_1 \le 2i_0 - 5, \\ -1 & \text{otherwise,} \end{cases} \\
\rho &\equiv i_0 + 2i_1 \pmod 6, \quad 0 \le \rho \le 5, \\
\sigma &= \begin{cases} 1 & \text{if } i_1 = i_0 - 1 \text{ and } i_2 = 2i_1, \\ 0 & \text{otherwise,} \end{cases} \\
\mu &= \begin{cases} 1 & \text{if } \rho = 1, \\ 0 & \text{otherwise,} \end{cases} \\
\beta &= \begin{cases} 1 & \text{if } \rho \in \{0,4,5\}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \le 2i_1$ and $i_1 \le i_0 - 1$. Under these conditions,

$$\max(i_0, i_1 + 1, (i_2 + 3)/2) = i_0 + \sigma,$$

and

$$\min(2i_2 - 3, 6i_1 - i_2) = 2i_2 - 3.$$

This implies that

$$i_0 \ge 3, \quad (i_0 + 3 + \sigma - 2\alpha)/4 \le i_1 \le i_0 - 1,$$

and

$$(i_0 + 3 + \sigma)/2 \le i_2 \le 2i_1 + \alpha.$$

Let

$$2i_1 + \alpha - i_2 = 6c + u \quad \text{where } u \in \{0,1,2,3,4,5\}.$$

Let

$$\begin{aligned}
\gamma &= \begin{cases} 1 & \text{if } \rho = 5 \text{ and } u = 2, \\ 0 & \text{otherwise,} \end{cases} \\
\omega &= \frac{1}{6}(i_0 + 2i_1 - \rho).
\end{aligned}$$

Define $\varepsilon(p)$, $\zeta(p)$, and $m_2(p)$ by the following tables.

$\varepsilon(p) = \varepsilon(u,p)$										
u	A	B,E,G	F	H	I	K	L	M	N	O
0	0	0	0	0	0	0	0	0	0	0
1	0	0	1-μ	μ	0	1	μ	1-μ	0	0
2	0	1	1	0	0	1	0	0	0	1
3	1	1	1	1	0	1	0	0	1	1
4	1	1	1	2-β	0	2	1	0	1	1+β
5	1	1	2	1	1	2	0	1	2	2

$\zeta(p) = \zeta(\rho,p)$												
ρ	A	B	E	F	G	H	I	K	L	M	N	O
0	0	0	0	0	-1	-1	0	0	0	0	-1	0
1	0	1	0	-1	-1	α-σ	0	α	α	σ	-1	-1
2	1	0	1	0	-1	-1	0	-1	0	0	-1	-1
3	0	1	1	-1	-1	-1	1	-1	0	0	-1	-1
4	0	1	1	-1-σ	-1	-2	1	-1	σ	1	-1	-1
5	γ	1	1	-1	-2+γ	-2	2-2γ	-1	γ	1	-1-γ	-1

$m_2(p) =$	for
i_0	$p = J$
$i_1 - \sigma$	$p = C$
σ	$p = D$
$\omega + \zeta(p) - c - \varepsilon(p)$	$p \in \{A, B, E, I, L, M\}$
$i_1 - \omega + \zeta(p) - 2c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$

The corresponding DS is in class E and it is $(i_0, i_1, i_2, 2i_2 - 3)$. From Lemma 4 we get class E DS (i_0, i_1, i_2, i_3) for $i_0 + \sigma \le i_3 \le 2i_2 - 3$.

Construction 3 for class E.

Let

$$i_2 - 2i_1 - 1 = 4c + u \quad \text{where } u \in \{0, 1, 2, 3\},$$

$$\alpha = \begin{cases} 1 & \text{if } i_1 \text{ is even,} \\ 0 & \text{if } i_1 \text{ is odd.} \end{cases}$$

$\delta(p) = \delta(i_1, p)$		
i_1	A, E, F, K, L, O	B, G, H, I, M, N
even	2	0
odd	1	1

$\varepsilon(p) = \varepsilon(u, p)$					
u	A, E, K, O	B, G, I, N	C, D	F, L	H, M
0	0	0	0	0	0
1	0	1	1	0	0
2	1-α	α	1	1-α	α
3	1	1	2	0	1

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \ge 2i_1 + 1$ and $i_2 \ge 2i_0 - 3$. Under these conditions,

$$\max(i_0, i_1 + 1, (i_2 + 3)/2) = (i_2 + 3)/2,$$

and

$$\min(2i_2 - 3, 6i_1 - i_2) = 6i_1 - i_2.$$

This implies that

$$i_0 \ge 3 + 2\vartheta, \; (i_0 - 1)/2 < i_1 \le 2i_0 - 5 - 2\vartheta,$$

and

$$\max(2i_0 - 3, 2i_1 + 1) \le i_2 \le (4i_0 + 4i_1 - 4\vartheta - 7)/3.$$

Let

$m_3(p) =$	for
i_0	$p = J$
$i_0 - 2c - \varepsilon(C) - 2$	$p = C$
$i_1 - i_0 + 2c + \varepsilon(D) + 2$	$p = D$
$\frac{1}{2}\big(i_1 + \delta(p)\big) + c + \varepsilon(p)$	$p \in \{A, B, E, I, L, M\}$
$\frac{1}{2}\big(i_1 - \delta(p)\big) - c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$

The corresponding DS is in class E and it is $(i_0, i_1, i_2, 6i_1 - i_2)$. A detailed proof of this fact is given in the appendix. From Lemma 4 we get class E DS (i_0, i_1, i_2, i_3) for $\lceil (i_2 + 3)/2 \rceil \le i_3 \le 6i_1 - i_2$. (Note that $\lceil (i_2 + 3)/2 \rceil = i_1 + 2c - \varepsilon(D) + 2$).

Construction 4 for class E.

For this construction we assume that i)–iii) are satisfied and, in addition, $i_2 \geq 2i_1+1$ and $i_2 \leq 2i_0 - 4$. We first show that under these conditions we have $(4i_0 + 4i_1 - 4\vartheta - 7)/3 > \min(2i_0 - 4, 6i_1 - i_0)$. First, if $i_0 = 2i_1 + 1$, then $2i_0 - i_1 = 3(i_1+1) - 1$, and so $t = 1$ and $\vartheta = 0$. Hence

$$\frac{4i_0 + 4i_1 - 4\vartheta - 7}{3} = 2i_0 - 3 > 2i_0 - 4.$$

If $i_0 \leq 2i_1$, then

$$\frac{4i_0 + 4i_1 - 11}{3} - (2i_0 - 4) = \frac{4i_1 + 1 - 2i_0}{3} > 0,$$

and if $i_0 \geq 2i_1 + 2$, then

$$\frac{4i_0 + 4i_1 - 11}{3} - (6i_1 - i_0) = \frac{7i_0 - 14i_1 - 11}{3} > 0.$$

Similarly,

$$\min(2i_0 - 4, 6i_1 - i_0) < \min(i_0 + 2i_1 - 2, 6i_0 - i_1 - 7, (4i_0 + 4i_1 - 4\vartheta - 7)/3),$$

$$\max(i_0, i_1 + 1, (i_2 + 3)/2) = i_0,$$

and

$$\min(2i_2 - 3, 6i_1 - i_2) = 6i_1 - i_2.$$

This implies that

$$i_0 \geq 5,\ \ (i_0 + 1)/4 \leq i_1 \leq i_0 - 3,$$

and

$$2i_1 + 1 \leq i_2 \leq \min(2i_0 - 4, 6i_1 - i_0).$$

Let

$$\alpha \equiv i_0 + i_2 \pmod 6, \quad 0 \leq \alpha \leq 5.$$

Define $\zeta(p)$ and $m_4(p)$ by the following tables.

$\zeta(p) = \zeta(\alpha, p)$					
α	A, K	B, G, I, N	E, O	F, L	H, M
0	0	0	0	0	0
1	1	0	0	0	0
2	1	0	1	0	0
3	1	0	1	1	0
4	1	1	0	1	0
5	1	1	1	0	1

$m_4(p) =$	for
$\frac{1}{6}(i_0 + i_2 - \alpha) + \zeta(p)$	$p \in \{A, B, E, I, L, M\}$
i_1	$p = C$
0	$p = D$
$i_1 - \frac{1}{6}(i_0 + i_2 - \alpha) - \zeta(p)$	$p \in \{F, G, H, K, N, O\}$
i_0	$p = J$

The corresponding DS is in class E and it is $(i_0, i_1, i_2, 6i_1 - i_2)$. From Lemma 4 we get class E DS (i_0, i_1, i_2, i_3) for $i_0 \leq i_3 \leq 6i_1 - i_2$.

Class F: (Con1), (Con2) false; (Con3), (Con4) true.

Theorem 6 *(i_0, i_1, i_2, i_3) is a class F DS of if and only if*

i) $i_1 \leq 2i_0 - 3$,

ii) $i_0 \leq i_2 \leq 2i_1 - 3$,

iii) $1 \leq i_3 \leq 2i_2$.

Proof: The upper bounds follow from Lemma 1 ii), iii) and iv) and the lower bounds from (1) and Lemma 3 iii).

Construction for class F.

For this construction we assume that i)–iii) are satisfied. Then

$$i_0 \geq 3, \quad (i_0+3)/2 \leq i_1 \leq 2i_0 - 3,$$

and

$$i_0 \leq i_2 \leq 2i_1 - 3.$$

Let

$$2i_1 - i_2 - 3 = 3c + u \text{ where } u \in \{0, 1, 2\},$$

$$\alpha = \begin{cases} 1 & \text{if } i_2 = i_0, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\delta(i_1, p)$, $\varepsilon(u, p)$, and $m(p)$ be defined by the following tables.

$\delta(p) = \delta(i_1, p)$			
i_1	D, K	F, G, J, O	H, N
even	0	-2	-2
odd	-1	-1	-3

$\varepsilon(p) = \varepsilon(u, p)$				
u	F, G, M, O	H	I, N	K, L
0	0	0	0	0
1	0	1	$1-\alpha$	α
2	1	0	0	1

$m(p) =$	for
i_0	$p = C$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
$\frac{1}{2}(i_1 + \delta(p))$	$p \in \{D, J\}$
$\frac{1}{2}(i_1 + \delta(p)) - c - \varepsilon(p)$	$p \in \{F, G, H, K, N, O\}$
$i_1 - i_0 + b - 1 - c - \varepsilon(p) - t$	$p = L$
$i_1 - i_0 + b - 1 - c - \varepsilon(p)$	$p \in \{I, M\}$

For this construction

$$M_0 = \{C\}, \quad l_8 \in M_1, \quad P_5 \in M_2,$$

$$\text{if } C \in l \in C_1, \text{ then } l \notin M_1.$$

Hence the corresponding DS is in class F and it is $(i_0, i_1, i_2, 2i_2)$. A core is $\Gamma = \{\{C\}, l_8, P_5\}$, and $m(\Gamma) = i_0 + i_1 + i_2$. By Lemma 4 we get DS (i_0, i_1, i_2, i_3) in class F for $1 \leq i_3 \leq 2i_2$.

Class G: (Con1), (Con2), (Con4) false; (Con3) true.

Theorem 7 *(i_0, i_1, i_2, i_3) is a class G DS of if and only if*

i) $i_1 \le 2i_0 - 3$,

ii) $i_0 + 1 \le i_2 \le 2i_1 - 3$,

iii) $\max(i_0 + 1, i_1 + 1) \le i_3 \le 2i_2 - 3$.

Proof: The upper bounds follow from Lemma 1 ii), iii) and v). The lower bounds follow from Lemma 3 v), vii), x).

Construction 1 for case G.

Assume that i)–iii) are satisfied. Then

$$i_0 \ge 4, \quad (i_0 + 4)/2 \le i_1 \le 2i_0 - 3,$$

and

$$i_0 + 1 \le i_2 \le 2i_1 - 3.$$

Let

$$2i_1 - i_2 - 3 = 3c + u \text{ where } u \in \{0, 1, 2\},$$

and let $\delta(i_1, p)$, $\varepsilon(u, p)$, $\zeta(p)$, and $m_1(p)$ be defined by the following tables.

$\delta(p) = \delta(i_1, p)$				
i_1	B, H, N	E, F, O	L	M
even	0	-2	0	2
odd	-1	-1	1	1

$\varepsilon(p) = \varepsilon(u, p)$				
u	F, G, M, O	H	I, N	K, L
0	0	0	0	0
1	0	1	1	0
2	1	0	1	0

$\zeta(p) = \zeta(t, p)$			
t	G	I	K
-1	0	0	0
0	0	0	-1
1	-1	0	-1

$m_1(p) =$	for
i_0	$p = A$
$i_0 - b$	$p \in \{C, J\}$
$i_0 - b + t$	$p = D$
$\frac{1}{2}(i_1 + \delta(p))$	$p \in \{B, E\}$
$\frac{1}{2}(i_1 + \delta(p)) - 1 - c - \varepsilon(p)$	$p \in \{F, H, L, M, N, O\}$
$i_1 - i_0 + b - 1 - c - \varepsilon(p) + \zeta(p)$	$p \in \{G, I, K\}$

For this construction, and the next, we have

$$M_0 = \{A\}, \quad M_1 = \{l_4\}, \quad P_5 \in M_2,$$

$$\text{if } A \in l \in C_1, \text{ then } l \notin M_1,$$

$$\text{if } A \in P \in C_2, \text{ then } P \notin M_2.$$

Hence the corresponding DS is in class G and it is $(i_0, i_1, i_2, 2i_2 - 3)$. A core is $\Gamma = \{\{A\}, l_4, P_5\}$, and $m(\Gamma) = (i_0 + i_1 + i_2) + i_0 + i_1 - m_1(C)$.

By Lemma 4 we get DS (i_0, i_1, i_2, i_3) in class G for $i_0 + i_1 - m_1(C) = i_1 + b \le i_3 \le 2i_2 - 3$.

Construction 2 for case G.

With the same notations as in Construction 1 above, let

$$1 \le \eta \le b - 1 + \min(0, i_1 - i_0),$$

and

$\lambda(p) = \lambda(c,\eta,p)$					
η	D	J	$\min(c,\eta)$	B	E
even	0	0	even	0	0
odd	1	-1	odd	-1	1

$m_2(p) =$	for
0	$p \in \{F,G,H,K,N,O\}$
$m_1(p)$	$p \in \{A,L,M\}$
$m_1(p) + \eta$	$p = C$
$m_1(p) - \max(0, \eta - c)$	$p = I$
$m_1(p) - \frac{1}{2}\big(\eta - \lambda(p)\big)$	$p \in \{D,J\}$
$m_1(p) - \frac{1}{2}\big(\min(c,\eta) - \lambda(p)\big)$	$p \in \{B,E\}$

Together with the Lemma 4 modification of Construction 1 above, this gives a class G (i_0, i_1, i_2, i_3) for $\max(i_0+1, i_1+1) \le i_3 \le 2i_2 - 3$.

Class H: (Con1), (Con2), (Con3) false; (Con4) true.

Theorem 8 *(i_0, i_1, i_2, i_3) is a class H DS if and only if*

i) $\frac{1}{2}(i_0+3) \le i_1 \le 2i_0 - 3$,

ii) $i_0 + 1 \le i_2 \le 4(i_0 + i_1 - \theta)/3$,

iii) $i_0 \le i_3 \le \min(2i_2 - 1, 6i_1 - i_2 - 7)$.

Proof: The upper bounds follow from Lemma 1 ii) and Lemma 2 ii), viii), ix). The lower bounds follow from Lemma 3 ii), iv), vi).

Construction 1 for class H.

Let

$$\pi = \begin{cases} 1 & \text{if } i_2 = 2i_1, \\ 0 & \text{otherwise.} \end{cases}$$
$$\alpha = \begin{cases} 1 & \text{if } i_2 \in \{2i_1 - 1, 2i_1\}, \\ 0 & \text{otherwise.} \end{cases}$$
$$\rho = \begin{cases} 1 & \text{if } i_0 = 3 \text{ and } i_2 = 2i_1, \\ 0 & \text{otherwise.} \end{cases}$$
$$\sigma = \begin{cases} 1 & \text{if } i_0 \in \{3,4\} \text{ and } i_2 = 2i_1, \\ 0 & \text{otherwise.} \end{cases}$$
$$\beta = \begin{cases} 1 & \text{if } t = 1 \text{ and } i_2 = i_0 + 2, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that i)–iii) are satisfied and in addition $i_2 \le 2i_1$. Then

$$i_0 \ge 3, \quad (i_0+3)/2 + \rho \le i_1 \le 2i_0 - 3,$$

and

$$i_0 + 1 + \sigma \le i_2 \le 2i_1.$$

In this case

$$\min(2i_2 - 1, 6i_1 - i_2 - 7) = 2i_2 - 1 - 3\pi - 3\alpha.$$

Let

$$2i_1 - i_2 - 3 = 3c + u \text{ where } u \in \{0, 1, 2\}.$$

Further, $\delta(i_1, p)$, $\varepsilon(u, p)$, and $m_1(p)$ are defined by the following tables.

$\delta(p) = \delta(i_1, p)$						
i_1	C, G	D, I	F	H	L	M
even	0	-2	-2α	-2	$2\alpha - 2$	0
odd	-1	-1	-1	$-1 - 2\alpha$	-1	$2\alpha - 1$

$\varepsilon(p) = \varepsilon(u, p)$						
u	F, K	G	H	L, O	M	N
0	0	0	0	0	0	0
1	$1 - \beta$	β	0	0	1	β
2	1	1	0	1	1	0

$m_1(p) =$	for
i_0	$p = J$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
$\frac{1}{2}\big(i_1 + \delta(p)\big)$	$p \in \{C, D\}$
$\frac{1}{2}\big(i_1 + \delta(p)\big) - c - \varepsilon(p)$	$p \in \{F, H, L, M\}$
$\frac{1}{2}\big(i_1 + \delta(p)\big) - c - \varepsilon(p) - \pi$	$p = G$
$\frac{1}{2}\big(i_1 + \delta(p)\big) - c + \pi$	$p = I$
$i_1 - i_0 + b - 1 - c - \varepsilon(p)$	$p \in \{K, O\}$
$i_1 - i_0 + b - 1 - c - \varepsilon(p) - t$	$p = N$

This construction as well as the following construction has the following properties.

$$M_0 = \{J\}, \quad l_8 \in M_1, \quad P_5 \in M_2,$$

$$\text{if } J \in l \in C_1, \text{ then } l \notin M_1,$$

$$\text{if } J \in P \in C_2, \text{ then } P \notin M_2.$$

Hence it is in class H. The corresponding DS is $(i_0, i_1, i_2, 2i_2 - 1 - 3\pi - 3\alpha)$. A core is $\Gamma = \{\{J\}, l_8, P_5\}$, and $m(\Gamma) = (i_0 + i_1 + i_2) + i_0$.

From Lemma 4 we get DS is (i_0, i_1, i_2, i_3) for $i_0 \leq i_3 \leq 2i_2 - 1 - 3\pi - 3\alpha$ in this case.

Construction 2 for class H.

Assume that i)–iii) are satisfied and in addition $i_2 \geq 2i_1 + 1$. Then

$$i_0 \geq 3, \quad (i_0 + 3)/2 \leq i_1 \leq 2i_0 - 3,$$

and

$$2i_1 + 1 \leq i_2 \leq (4i_0 + 4i_1 - 4\theta)/3.$$

In this case

$$\min(2i_2 - 1, 6i_1 - i_2 - 7) = 6i_1 - i_2 - 7.$$

We use the notations

$$i_2 - 2i_1 - 1 = 4c + u \text{ where } u \in \{0, 1, 2, 3\}.$$

Further, $\delta(i_1, p)$, $\varepsilon(u, p)$, and $m_2(p)$ are defined by the following tables.

$\delta(p) = \delta(i_1, p)$						
i_1	C, L	D, F	G	H	I	M
even	0	-2	-2	-4	0	2
odd	1	-3	-1	-3	-1	1

$\varepsilon(p) = \varepsilon(u, p)$				
u	C, D	F, L	G, I	H, M
0	0	0	0	0
1	0	0	1	0
2	0	1	1	0
3	1	1	1	0

$m_2(p) =$	for
i_0	$p = J$
$i_0 - b$	$p \in \{A, E\}$
$i_0 - b + t$	$p = B$
$\frac{1}{2}\big(i_1 + \delta(p)\big) + c + \varepsilon(p)$	$p \in \{C, I, L, M\}$
$\frac{1}{2}\big(i_1 + \delta(p)\big) - c - \varepsilon(p)$	$p \in \{D, F, G, H\}$
$i_1 - i_0 + b - 1$	$p \in \{K, O\}$
$i_1 - i_0 + b - 1 - t$	$p = N$

The corresponding DS is in class H and it is $(i_0, i_1, i_2, 6i_1 - i_2 - 7)$. By Lemma 4 we get DS (i_0, i_1, i_2, i_3) for $i_0 \le i_3 \le 6i_1 - i_2 - 7$.

Class I: (Con1), (Con2), (Con3), (Con4) false.

From [7] we have the following result.

Theorem 9 *(i_0, i_1, i_2, i_3) is a class I DS if and only if*
i) $i_1 \le 2i_0 - 3$,
ii) $i_2 \le 2i_1 - 4$,
iii) $\max\big(i_0 + i_1 + 1, 2i_0 + 3\big) \le i_3 \le 2i_2 - 3$.

The proof that these bounds are necessary goes along the same lines as the proofs of Lemmas 2 and 3 and we omit them here. For completeness we include constructions which shows that the bounds are sufficient. The first construction is taken from [7]. The second construction is simpler than the corresponding constructions in [7] (where five constructions were given to cover the same case).

Construction 1 for class I.

Assume i)–iii) are satisfied and in addition $i_1 \le i_0 + 2$. Then

$$i_0 \ge 5, \quad (i_0 + 7)/2 \le i_1 \le i_0 + 2,$$

and

$$i_0 + 3 \le i_2 \le 2i_1 - 4.$$

Further,

$$\max\big(i_0 + i_1 + 1, 2i_0 + 3\big) = 2i_0 + 3.$$

Let $\delta(p)$ and $m_1(p)$ be defined by the following tables.

$\delta(p) = \delta(i_0, i_2, p)$							
i_0	B, E	C	F, H	J	$i_2 - i_0$	A, N, O	G, L, M
odd	1	0	-1	0	odd	0	0
even	0	-1	0	1	even	1	-1

$m_1(p) =$	for
i_0	$p = I$
$(i_2 - i_0 - 1 + \delta(p))/2$	$p \in \{A, G\}$
$(i_0 + 2 + \delta(p))/2$	$p \in \{B, E, F, H\}$
$(i_0 + 3 + \delta(p))/2$	$p \in \{C, J\}$
$i_1 - 3$	$p = D$
0	$p = K$
$(i_2 - i_0 - 3 + \delta(p))/2$	$p \in \{L, M, N, O\}$

For this construction, and the construction below,

$$M_0 = \{I\}, \quad M_1 = \{l_4\}, \quad M_2 = \{P_5\}.$$

Hence it is in class I. The corresponding DS is $(i_0, i_1, i_2, 2i_2 - 3)$. A core is $\Gamma = \{\{I\}, l_4, P_4\}$, and $m_1(\Gamma) = (i_0 + i_1 + i_2) + 2i_0 + 3$. From Lemma 4 we get class I DS (i_0, i_1, i_2, i_3) for $2i_0 + 3 \le i_3 \le 2i_2 - 3$.

Construction 2 for class I.

Assume i)–iii) are satisfied and in addition $i_1 \ge i_0 + 3$. Then

$$i_0 \ge 6, \quad i_0 + 3 \le i_1 \le 2i_0 - 3,$$

and

$$(i_0 + i_1 + 4)/2 \le i_2 \le 2i_1 - 4.$$

Further,

$$\max\bigl(i_0 + i_1 + 1, 2i_0 + 3\bigr) = i_0 + i_1 + 1.$$

Let

$\varepsilon(p) = \varepsilon(u, p)$										
u	A	B	E	F, H	G	K	L	M	N	O
0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	1	1
2	1	0	0	0	1	0	1	1	1	1
3	1	0	0	1	0	2	1	1	1	1
4	1	0	0	1	1	2	1	2	2	1
5	1	0	1	1	1	3	2	2	2	1

and let κ be minimal such that

$$\kappa \ge i_2$$

and

$$i_1 - i_0 - 2 - 4\left\lfloor \frac{2i_1 - \kappa - 4}{6} \right\rfloor - \varepsilon\bigl((2i_1 - \kappa - 4)(\mathrm{mod}\ 6), K\bigr) \ge 0.$$

Let

$$2i_1 - \kappa - 4 = 6c + u \quad \text{where} \quad u \in \{0,1,2,3,4,5\}.$$
$$\kappa - i_2 = 2c_5 + u_2 \quad \text{where} \quad u_2 \in \{0,1\}.$$
$$\alpha = \begin{cases} 2 & \text{if } u = 4, \\ 0 & \text{otherwise,} \end{cases}$$
$$\beta = \begin{cases} 2 & \text{if } u = 5, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\delta(p)$, $\zeta(p)$, and $m_2(p)$ by the following tables.

$\delta(p) = \delta(i_1, p)$					
i_1	A	B, H	C	E, F, J	G
even	-4	0	2	0	-2
odd	-3	-1	1	1	-3
i_1	L	M	N	O	
even	$-6+\beta$	$-6+\alpha$	$-4-\beta$	$-4-\alpha$	
odd	-5	-5	-5	-5	

$\zeta(p) = \zeta(u, u_2, p)$							
u_2	u	A	G	L	M	N	O
0		0	0	0	0	0	0
1	0,2	1	0	0	0	1	1
1	1	1	0	1	1	0	0
1	3,4,5	0	1	1	0	0	1

$m_2(p) =$	for
i_0	$p = I$
$i_0 - 1$	$p = D$
$i_1 - i_0 - 2 - 4c - \varepsilon(K)$	$p = K$
$\frac{1}{2}(i_1 + \delta(p))$	$p \in \{C, J\}$
$\frac{1}{2}(i_1 + \delta(p)) - c - \varepsilon(p)$	$p \in \{B, E, F, H\}$
$\frac{1}{2}(i_1 + \delta(p)) - c - \varepsilon(p) - c_5 - \zeta(p)$	$p \in \{A, G\}$
$\frac{1}{2}(i_1 + \delta(p)) - 2c - \varepsilon(p) - c_5 - \zeta(p)$	$p \in \{L, M, N, O\}$

The corresponding DS is $(i_0, i_1, i_2, 2i_2 - 3)$. Further, $m_2(\Gamma) = (i_0 + i_1 + i_2) + (i_0 + i_1 + 1)$. From Lemma 4 we get class I DS (i_0, i_1, i_2, i_3) for $i_0 + i_1 + 1 \leq i_3 \leq 2i_2 - 3$.

IV Summary

We have given a classification in nine classes of the binary linear codes of dimension 4 based on how the subcodes with minimum support are related. For each class we have given an explicit characterization the possible weight hierarchies for the codes in each class.

References

[1] A. I. Barbero and J. G. Tena, "Weight hierarchy of a product code", *IEEE Trans. Inform. Theory*, vol. 41, pp. 1475-1479, 1995.

[2] W. Chen and T. Kløve, "The weight hierarchies of q-ary codes of dimension 4", *IEEE Trans. Inform. Theory*, vol. 42, pp. 2265-2272, 1996.

[3] W. Chen and T. Kløve, "The weight hierarchies of q-ary codes of dimension 2 and 3", *The Second Shanghai Conference on Designs, Codes and Finite Geometries*, May 15–19, 1996, pp. 15-16.

[4] W. Chen and T. Kløve, "Weight hierarchies of linear codes of dimension 3", to appear in *Journal of Statistical Planning and Inference.*

[5] W. Chen and T. Kløve, "Bounds on the weight hierarchies of extremal non-chain codes of dimension 4", *Applicable Algebra in Eng., Commun. and Computing*, vol. 8, pp. 379-386, 1997.

[6] W. Chen and T. Kløve, "Bounds on the weight hierarchies of linear codes of dimension 4", *IEEE Trans. Inf. Theory*, vol. 43, Nov. 1997.

[7] W. Chen and T. Kløve, "Weight hierarchies of extremal non-chain binary codes of dimension 4", submitted for publication 1997.

[8] S. Encheva and T. Kløve, "Codes satisfying the chain condition", *IEEE Trans. Inform. Theory*, vol. 40, pp. 175-180, 1994.

[9] G. D. Forney, "Dimension/length profiles and trellis complexity of linear block codes", *IEEE Trans. Inform. Theory*, vol. 40, pp. 1741-1752, 1994.

[10] T. Helleseth and T. Kløve, "The weight hierarchies of some product codes", *IEEE Trans. Inform. Theory*, vol. 42, pp. 1029-1034, 1996.

[11] T. Helleseth, T. Kløve, J. Mykkeltveit, "The weight distribution of irreducible cyclic codes with block lengths $n_1((q^l-1)/N)$", *Discrete Math.*, vol. 18, pp. 179-211, 1977.

[12] T. Helleseth, T. Kløve, Ø. Ytrehus, "Generalized Hamming weights of linear codes", *IEEE Trans. Inform. Theory*, vol. 38, pp. 1133-1140, 1992.

[13] T. Kasami, T. Takata, T. Fujiwara, S. Lin, "On the optimum bit orders with respect to the state complexity of trellis diagrams for binary linear codes", *IEEE Trans. Inform. Theory*, vol. 39, pp. 242-243, 1993.

[14] T. Kløve, "Minimum support weights of binary codes", *IEEE Trans. Inform. Theory*, vol. 39, pp. 648-654, 1993.

[15] T. Kløve, "The worst-case probability of undetected error for linear codes on the local binomial channel", *IEEE Trans. Inf. Theory*, vol. 42, pp. 172-179, 1996.

[16] A. Vardy and Y. Be'ery, "Maximum-likelihood soft decision decoding of BCH codes", *IEEE Trans. Inform. Theory*, vol. 40, pp. 546-554, 1994.

[17] V. K. Wei, "Generalized Hamming Weights for Linear Codes", *IEEE Trans. Inform. Theory*, vol. 37, pp. 1412-1418, 1991.

[18] V. K. Wei and K. Yang, "On the generalized Hamming weights of product codes", *IEEE Trans. Inform. Theory*, vol. 39, pp. 1709-1713, 1993.

Appendix. Proof for three of the constructions

In this appendix, as an illustration of the proofs that constructions have the stated properties, we give detailed proofs in three cases, namely, for Construction 3 in class D and Constructions 1 and 3 in Class E.

Class D, Construction 3.

This construction has a simple structure which simplifies the proof. A direct computation shows that $m_3(P_5) = i_0 + i_1 + i_2$. There are only two points outside P_5 which have positive values, namely J and (sometimes) D.

Let
$$\mu \equiv i_2 - \lambda \pmod 3, \quad \mu \in \{0, 1, 2\}.$$
Note that $\max\{\zeta(I), \zeta(L), \zeta(M)\} = \mu$. Further, note that $\zeta(I) + \zeta(L) + \zeta(M) = 0$ in all cases. In some cases below we have used this relation to simplify expressions.

First we show that
$$0 \le m_3(p) < i_0 \tag{11}$$
for all $p \in V_3$, that is, that m_3 is well defined and $M_0 = \{J\}$. From the lower bound on i_3 we get
$$\lambda \le \min(i_1, i_0 - 1, i_0 + i_1 - i_2/2).$$
In particular, this shows that (11) is satisfied for $p = C$ and $p = D$. For $p \in \{A, E\}$ we get
$$m_3(p) = (3i_0 - 2i_0 + i_1 - t)/3 = (i_0 + i_1 - t)/3.$$
In particular, we immediately get $m_3(p) > 0$. Also we get
$$m_3(p) \le (i_0 + (2i_0 - 3) - t)/3 < i_0.$$
Similarly, we show that (11) is satisfied for $p = B$. Finally, let $p \in \{I, L, M\}$. We have
$$(i_2 - \lambda - \zeta(p))/3 \ge ((i_0 + 1) - (i_0 - 1) - \zeta(p))/3 = (2 - \zeta(p))/3 \ge 0,$$
and
$$\begin{aligned}
(i_2 - \lambda - \zeta(p))/3 &\le (i_2 - (i_2 - i_0 - i_1 + \theta) - \zeta(p))/3 \\
&= (i_0 + i_1 - \theta - \zeta(p))/3 \\
&\le (i_0 + (2i_0 - 3) - \theta - \zeta(p))/3 \\
&= i_0 - (3 + \theta + \zeta(p))/3 < i_0.
\end{aligned}$$

Now, consider the lines. If p is a point outside P_5 and l is a line containing p, then l meets P_5 in some point p'. If l' is the line determined by J and p', then $m_3(l') \le m_3(l)$. Hence, it remains to check that the lines containing J and the lines contained in P_5 have values at most $i_0 + i_1$. First, consider the lines containing J. The values of these are given in the following table.

i	$m_3(l_i) =$
4	$i_0 + i_1$
11,25	$2i_0 - b$
17	$2i_0 - b + t$
28	$i_0 + (i_2 - \lambda - \zeta(L))/3$
30	$i_0 + (i_2 - \lambda - \zeta(I))/3$
34	$i_0 + (i_2 - \lambda - \zeta(M))/3$

For $l \in \{l_{11}, l_{17}, l_{25}\}$ we have

$$\begin{aligned} m_3(l) &\leq 2i_0 - b + \theta \\ &= 2i_0 - (2i_0 - i_1 + t)/3 + \theta \\ &= i_0 + i_1 - (2i_1 - i_0 + t - 3\theta)/3. \end{aligned}$$

By assumption, $i_0 \leq 2i_1$. The possible combinations of t and θ are given by the following table.

t	θ	$t - 3\theta$
1	1	-2
0	0	0
-1	0	-1

If $2i_1 - i_0 \geq 2$, then clearly $2i_1 - i_0 + t - 3\theta \geq 0$. If $2i_1 - i_0 = 1$, then $t = -1$ and so $2i_1 - i_0 + t - 3\theta = 0$. If $2i_1 - i_0 = 0$, then $t = 0$ and so again $2i_1 - i_0 + t - 3\theta = 0$. Therefore, $m_2(l) \leq i_0 + i_1$ for $l \in \{l_{11}, l_{17}, l_{25}\}$.

For $l \in \{l_{28}, l_{30}, l_{34}\}$, we have

$$\begin{aligned} m_3(l) &\leq i_0 + (i_2 - \lambda - \mu)/3 \\ &\leq i_0 + (i_2 - (i_2 - i_0 - i_1 + \theta) - \mu)/3 \\ &= i_0 + i_1 - (2i_1 - i_0 + \theta + \mu)/3 \leq i_0 + i_1. \end{aligned}$$

Next, consider the lines contained in P_5. The values of these are given by the following table.

i	$m_3(l_i) =$
8	$i_0 + i_1$
9	$i_0 - b + (i_2 - \lambda - \zeta(I))/3 + \lambda$
14	$i_0 - b + (i_2 - \lambda - \zeta(L))/3 + t + \lambda$
19	$i_0 - b + (i_2 - \lambda - \zeta(M))/3 + \lambda$
12	$i_0 - b + (2i_2 - 2\lambda + \zeta(I))/3$
16	$i_0 - b + (2i_2 - 2\lambda + \zeta(L))/3 + t$
24	$i_0 - b + (2i_2 - 2\lambda + \zeta(M))/3$

We have

$$\begin{aligned} (i_0 + i_1) - m_3(l_9) &= i_1 + b - (i_2 + 2\lambda - \zeta(I))/3 \\ &= (2i_0 + 2i_1 - i_2 - 2\lambda + t + \zeta(I))/3. \end{aligned}$$

Similarly

$$(i_0 + i_1) - m_3(l_{14}) = (2i_0 + 2i_1 - i_2 - 2\lambda - 2t + \zeta(L))/3,$$

$$(i_0 + i_1) - m_3(l_{19}) = (2i_0 + 2i_1 - i_2 - 2\lambda + t + \zeta(M))/3.$$

By assumption, $\lambda = i_0 + i_1 - i_3 \leq i_0 + i_1 - i_2/2$ and so $2i_0 + 2i_1 - i_2 - 2\lambda \geq 0$. In the following table we give the possible values of $t + \zeta(I)$ etc.

$(i_2 - \lambda)$ (mod 3)	t	$t + \zeta(I)$	$-2t + \zeta(L)$	$t + \zeta(M)$
0		$-1, 0, 1$	$-2, 0, 2$	$-1, 0, 1$
1	-1	0	0	0
1	0	1	-2	1
1	1	-1	-1	2
2		$-2, -1, 0$	$0, 2, 4$	$-2, -1, 0$

From this table we see that $(t+\zeta(I))/3 > -1$. Hence $(i_0+i_1) - m_3(l_9) \geq 0$. Similarly, $m_3(l_{14}) \leq i_0 + i_1$ and $m_3(l_{19}) \leq i_0 + i_1$.

Finally, we have

$$\begin{aligned}(i_0+i_1) - m_3(l_{12}) &= i_1 + b - (2i_2 - 2\lambda + \zeta(I))/3 \\ &= (2i_0 + 2i_1 - 2i_2 + 2\lambda + t - \zeta(I))/3 \\ &\geq (2\theta + t - \zeta(I))/3.\end{aligned}$$

Similarly

$$(i_0+i_1) - m_3(l_{16}) \geq (2\theta - 2t - \zeta(L))/3,$$

$$(i_0+i_1) - m_3(l_{24}) \geq (2\theta + t - \zeta(M))/3.$$

In the following table we give the possible values of $2\theta + t - \zeta(I)$ etc.

$(i_2-\lambda)$ (mod 3)	t	θ	$2\theta+t$ $-\zeta(I)$	$2\theta-2t$ $-\zeta(L)$	$2\theta+t$ $-\zeta(M)$
0	-1	0	-1	2	-1
0	0	0	0	0	0
0	1	1	3	0	3
1	-1	0	-2	0	-2
1	0	0	-1	2	-1
1	1	1	5	-1	2
2	-1	0	-2	0	-2
2	0	0	1	-2	1
2	1	1	4	-2	4

As above, this proves that $m_3(l) \leq i_0 + i_1$ for $l \in \{l_{12}, l_{16}, l_{24}\}$.

We now consider the planes. As noted above, $m_3(P_5) = i_0 + i_1 + i_2$. Any plane $P \neq P_5$ is uniquely determined by a point $p \notin P_5$ and a line $l \subset P_5$. If P' is the plane determined by J and l, then $m_3(P) \leq m_3(P')$.

It remains to consider the planes containing J. For these planes we will show that $m_3(P) < i_0 + i_1 + i_2$. For the planes P determined by J and $l \subset P_5$ such that $C \notin l$, we have

$$m_3(P) = m_3(l) + m_3(J) \leq (i_0 + i_1) + i_0 \leq (i_0 + i_1) + (i_2 - 1).$$

Finally, the values of the three planes containing C and J are given in the following table.

i	$m_3(P_i) =$
1	$2i_0 - b + i_1 + (i_2 - \lambda - \zeta(I))/3$
2	$2i_0 - b + i_1 + (i_2 - \lambda - \zeta(M))/3$
3	$2i_0 - b + i_1 + (i_2 - \lambda - \zeta(L))/3 + t$

We have

$$\begin{aligned}&(i_0+i_1+i_2) - m_3(P_1) \\ &= -i_0 + b + i_2 - (i_2 - \lambda - \zeta(I))/3 \\ &= -i_0 + (2i_0 - i_1 + t)/3 + i_2 - (i_2 - \lambda - \zeta(I))/3 \\ &= (2i_2 + \lambda - i_0 - i_1 + t + \zeta(I))/3 \\ &\geq (2(i_0+1) + (i_1 - i_0 + 1) - i_0 - i_1 + t + \zeta(I))/3 \\ &= (3 + t + \zeta(I))/3 \geq (3-2)/3 > 0\end{aligned}$$

and so $m_3(P_1) \leq i_0 + i_1 + i_2 - 1$. Similarly,

$$m_3(P) \leq i_0 + i_1 + i_2 - 1 \text{ for } P \in \{P_2, P_3\}.$$

We can now summarize the main results shown above:

$$M_0 = \{J\}, \quad l_4, l_8 \in M_1, \quad M_2 = \{P_5\}.$$

Since $J \in l_4$, **(Con2)** is true. Since $l_8 \subset P_5$, **(Con4)** is true. Since $P \notin M_2$ for all planes P containing J, **(Con3)** false. Hence, the construction is in Class D.

Class E, Construction 1.

First we show that $m_1(p) \geq 0$ for all $p \in V_3$. For C and J this is obvious. Since $i_1 \geq i_0$,

$$m_1(D) \geq 2 + \alpha/2 \geq 1.$$

Consider $p \in \{F, G, H, K, N, O\}$. By the definition of $m_1(p)$ and the fact that $2i_2 \geq i_1 + 4 + \pi$ for this construction we get

$$\begin{aligned} m_1(p) &= \frac{i_1 + \delta(p) + \beta}{2} - 2c - \varepsilon(p) \\ &= \frac{i_1 + \delta(p) + \beta}{2} - \frac{4i_1 + \alpha + 3\beta - 2i_2 - 2u}{6} - \varepsilon(p) \\ &\geq \frac{i_1}{2} + \frac{\delta(p)}{2} - \frac{2i_1}{3} - \frac{\alpha}{6} + \frac{i_1}{6} + \frac{4}{6} + \frac{\pi}{6} + \frac{u}{3} - \varepsilon(p) \\ &= \frac{4 - \alpha + \pi}{6} + \frac{\delta(p)}{2} + \frac{u - 3\varepsilon(p)}{3}. \end{aligned}$$

From the definition of $\varepsilon(p)$ we get the following lower bounds on $u - 3\varepsilon(p)$:

p	G, H	F, K	N, O
$u - 3\varepsilon(p) \geq$	0	-1	-2

Let

$$s = 2i_0 - 1 - i_1.$$

The possible combinations of s, α, β, and π are given by the following table:

s	α	β	π	$(4 - \alpha + \pi)/6$
0	-2	-2	0	1
2	-2	0	0	1
≥ 4	-2	0	0	1
≥ 4	0	0	1	5/6

If $\alpha = -2$, then $\delta(p) \geq -3$ for $p \in \{F, G, H\}$ and $\delta(p) \geq -2$ for $p \in \{K, N, O\}$. Hence, we get the following lower bounds on $m_1(p)$ in these cases:

p	F	G, H	K	N, O
$m_1(p) \geq$	$-5/6$	$-1/2$	$-1/3$	$-2/3$

In all these cases $m_1(p) > -1$ and so $m_1(p) \geq 0$. A similar proof shows that $m_1(p) \geq 0$ also when $\alpha = 0$.

Next, consider $p \in \{A, B, I, L\}$. From the definition we see that $\varepsilon(p) \leq \varepsilon(N)$ and $\delta(p) \geq \delta(N)$. Hence $m_1(p) \geq m_1(N) \geq 0$. Similarly $m_1(E) \geq m_1(N)$ and $m_1(M) \geq m_1(N)$. Hence, $m_1(p) \geq 0$ for all $p \in V_3$.

Now, consider upper bounds on $m_1(p)$. We have

$$\begin{aligned} 2m_1(E) &= i_1 + \delta(E) + \alpha - 2c - \varepsilon(E) \\ &\leq i_1 + \delta(E) + \alpha \\ &\leq \begin{cases} 2i_0 - 1 & \text{if } 2i_0 - 4 \leq i_1 \leq 2i_0 - 1, \\ 2i_0 - 5 + 2 & \text{if } i_1 \leq 2i_0 - 5. \end{cases} \end{aligned}$$

Hence $m_1(E) < i_0$. Similarly $m_1(p) < i_0$ for $p \in \{A, B, F, G, H, I, L, M, N, O\}$. Clearly $m_1(C) \leq i_0 - 1$. Finally,

$$m_1(D) \leq \begin{cases} 2i_0 - 1 - i_0 + 2 - 1 & \text{if } 2i_0 - 4 \leq i_1 \leq 2i_0 - 1, \\ 2i_0 - 5 - i_0 + 2 & \text{if } i_1 \leq 2i_0 - 5. \end{cases}$$

Hence $m_1(D) \leq i_0$ (and $m_1(D) = i_0$ only if $i_1 = 2i_0 - 1$). We conclude that $m_1(p) \leq i_0$ for all p and $m_1(p) < i_0$ except for $p = J$ and sometimes for $p = D$.

Next, we consider the lines and planes. By assumption, $s \geq 0$. We consider $i_0 + i_1 - m_1(l)$ for all lines l and $i_0 + i_1 + i_2 - m_1(P)$ for all planes P. We give a detailed discussion of the case i_1 odd, i_1 even is similar and is omitted. From the expression for m_1, we see that $i_0 + i_1 - m_1(l)$ is a sum of one part which depends on u (and can be written as some linear combination of some $\varepsilon(p)$) and one part which does not depend on u. For example

$$i_0 + i_1 - m_1(l_6) = (s - 1 - \alpha/2 - \beta + 3c) + (\varepsilon(L) + \varepsilon(N)).$$

We write $i_0 + i_1 - m_1(l) = \mu(l) + \nu(l)$, where ν is the part dependent on u. Using the fact that $m(l_4) = i_0 + i_1$, we get

$$\nu(l) = \sum_{p \in l \setminus \{C,D,J\}} \varepsilon(p) \geq 0.$$

The values of $\mu(l)$ are given in the following table.

i	$\mu(l_i)$
4	0
11,25	$-\alpha/2 - \beta/2 + 3c$
28	$-\alpha/2 - \beta + 3c$
9,19	$1 - \beta/2 + 2c$
17	$1 - \beta/2 + 3c$
30,34	$1 - \beta + 3c$
14	$2 + \alpha/2 - \beta/2 + 2c$
20	$3 - \beta + 4c$
21,22	$4 + \alpha/2 - \beta + 4c$
12,24	$s/2 - 1 - \alpha/2 - \beta + 3c$
8	$s/2 - \alpha + 3c$
16	$s/2 - \beta + 3c$
13,26	$s/2 + 1 - \alpha/2 - \beta + 5c$
27,29	$s/2 + 1 - \alpha/2 - 3\beta/2 + 5c$
31,32,33,35	$s/2 + 2 - 3\beta/2 + 5c$
18	$s/2 + 2 - \beta + 5c$
10,23	$s/2 + 2 - \alpha - \beta + 5c$
15	$s/2 + 4 - \beta + 5c$
5,6,7	$s - 1 - \alpha/2 - \beta + 3c$
1,2,3	$s - \alpha - \beta/2 + 3c$

Similarly, we write $i_0 + i_1 + i_2 - m_1(P) = \mu(P) + \nu(P)$ for the planes. Using the fact that $m(P_5) = i_0 + i_1 + i_2$, we get

$$\nu(P) = \sum_{p \in P \setminus \{C,D,J\}} \varepsilon(p) - \sum_{p \in P_5 \setminus \{C,D,J\}} \varepsilon(p).$$

Note that $\nu(P)$ may be negative. The values of $\mu(P)$ and $\nu(P)$ are given in the following tables.

i	$\mu(P_i)$
5	0
9,15	$1-\alpha/2-\beta+3c$
7	$1-\alpha/2+3c$
1,2,3	1
12	$3+\alpha/2-\beta+3c$
8,13	$4-\beta+4c$
10	$6+\alpha-\beta+4c$
6,11,14	$s-\alpha/2-\beta+3c$
4	$s+2-3\alpha/2+3c$

$\nu(P_i)$						
	u					
i	0	1	2	3	4	5
1,2,3,5	0	0	0	0	0	0
4	0	0	1	3	2	2
6	0	2	1	1	2	2
7	0	2	3	3	4	4
8	0	2	2	2	4	4
9	0	0	1	1	0	2
10	0	0	0	2	2	2
11	0	0	1	1	2	4
12	0	0	-1	1	2	2
13	0	0	2	2	2	4
14,15	0	0	1	1	2	2

First, consider the planes. From the tables above we see that $m_1(P) < i_0+i_1+i_2$ for all $P \neq P_5$. For example, consider P_6. If $s=0$, then $s-\alpha/2-\beta=3$, if $s=2$ then $s-\alpha/2-\beta=3$, and if $s \geq 4$, then $s-\alpha/2-\beta \geq 4$. Hence $\mu(P_6) > 0$. Also $\nu(P_6) \geq 0$ for all values of u. Since $m_1(P_5) = i_0+i_1+i_2$ we have $M_2 = \{P_5\}$. Since $M_1 \subset \{D, J\}$, $D \notin P_5$ and $J \notin P_5$, we see that **(Con3)** is false.

Next, consider the lines. From the tables above we see that $m_1(l) \leq i_0+i_1$ for all l and $m_1(l) < i_0+i_1$ except for $l = l_4$ and possibly for $l \in \{l_{11}, l_{25}, l_{28}\}$. None of these four lines are contained in P_5 and so **(Con4)** is false. Finally **(Con2)** is true since $J \in l_4 \in M_1$. Hence, the construction is in class E.

Class E, Construction 3.

For this construction, $i_1 \leq 2i_0 - 5$. Let

$$s = 2i_0 - 5 - i_1,$$

where $s \geq 0$. Further,

$$i_2 = 2i_1 + 1 + 4c + u = 4i_0 - 10 - 2s + 1 + 4c + u,$$

and

$$\begin{aligned} i_2 &\leq (4i_0 + 4i_1 - 4\vartheta - 7)/3 \\ &= (4i_0 + 8i_0 - 20 - 4s - 4\vartheta - 7)/3 \\ &= 4i_0 - 9 - 4s/3 - 4\vartheta/3. \end{aligned}$$

Combining these two relations, we get

$$-2s + 4c + u \leq -4s/3 - 4\vartheta/3$$

and so

$$s \geq 6c + 3u/2 + 2\vartheta. \tag{12}$$

For the further analysis, we consider i_1 odd in detail (i_1 even is similar and is omitted). First, we show that $m_3(p) \geq 0$ for all p. For $p \in \{A, B, E, I, J, L, M\}$ this is obvious. Consider $p \in \{F, G, H, K, N, O\}$. In these cases $\delta(p) = 1$. Hence

$$\begin{aligned}
12m_3(p) &= 6(i_1 - 1) - 3(i_2 - 2i_1 - 1 - u) - 12\varepsilon(p) \\
&= 12i_1 - 3i_2 - 3 + 3u - 12\varepsilon(p) \\
&\geq 12i_1 - (4i_0 + 4i_1 - 4\vartheta - 7) - 3 + 3u - 12\varepsilon(p) \\
&= 8i_1 - 4i_0 + 4\vartheta + 4 + 3u - 12\varepsilon(p) \\
&\geq 8(i_0 - 1)/2 - 4i_0 + 4\vartheta + 4 + 3u - 12\varepsilon(p) \\
&= 4\vartheta + 3u - 12\varepsilon(p) \geq -9.
\end{aligned}$$

Hence $m_3(p) \geq 0$. Finally,

$$\begin{aligned}
6m_3(C) &= 6i_0 - 3(i_2 - 2i_1 - 1 - u) - 12 - 6\varepsilon(C) \\
&= 3(i_0 + 2i_1 - 2 - i_2) + 3i_0 + 3u - 3 - 6\varepsilon(C) \\
&\geq 3(i_0 - 1 + u - 2\varepsilon(C)) \\
&\geq 3(3 - 1 - 1) > 0, \\
6m_3(D) &= 6i_1 - 6i_0 + 3(i_2 - 2i_1 - 1 - u) + 12 + 6\varepsilon(D) \\
&= 3(i_2 - 2i_0 + 3) + 6\varepsilon(D) \\
&\geq 6\varepsilon(D) \geq 0.
\end{aligned}$$

In the following table we give the values of $i_0 - m_3(p)$. Combining this with (12) and the definition of $\varepsilon(p)$, we see that $m_3(p) < i_0$ for all $p \neq J$.

p	$i_0 - m_3(p) =$
J	0
A,E	$2 + s/2 - c - \varepsilon(A)$
B,I	$2 + s/2 - c - \varepsilon(B)$
C	$2 + 2c + \varepsilon(A) + \varepsilon(B)$
D	$3 + s - 2c - \varepsilon(A) - \varepsilon(B)$
F	$3 + s/2 + c + \varepsilon(F)$
G,N	$3 + s/2 + c + \varepsilon(B)$
H	$3 + s/2 + c + \varepsilon(H)$
K,O	$3 + s/2 + c + \varepsilon(A)$
L	$2 + s/2 - c - \varepsilon(F)$
M	$2 + s/2 - c - \varepsilon(H)$

Next, consider the planes. From the following table of $i_0 + i_1 + i_2 - m_3(P_i)$, we see that if $m_3(P) < i_0 + i_1 + i_2$ for $P \neq P_5$. Hence $M_2 = \{P_5\}$.

i	$i_0 + i_1 + i_2 - m_3(P_i) =$
5	0
1,2,3,7,9,12,15	$1 + 4c + \varepsilon(A) + \varepsilon(B) + \varepsilon(F) + \varepsilon(H)$
4	$4 + 2c + s - 2\varepsilon(A) + 2\varepsilon(F) + 2\varepsilon(H)$
6	$4 + 2c + s + 2\varepsilon(B)$
8	$4 + 8c + 2\varepsilon(A) + 2\varepsilon(B) + 2\varepsilon(F) + 2\varepsilon(H)$
10	$4 + 8c + 4\varepsilon(A) + 2\varepsilon(B) + 2\varepsilon(H)$
11	$4 + 2c + s + 2\varepsilon(A) - 2\varepsilon(B) + 2\varepsilon(F)$
13	$4 + 8c + 2\varepsilon(A) + 4\varepsilon(B) + 2\varepsilon(F)$
14	$4 + 2c + s + 2\varepsilon(H)$

Finally, consider the lines. From the following table of $i_0 + i_1 - m_3(l_i)$, we see that if $i \notin \{4, 11, 17, 25, 28, 30, 34\}$, then $m_3(l_i) < i_0 + i_1$. None of the lines l for which $m_3(l) = i_0 + i_1$ are contained in P_5. As for construction 1, it follows that m_3 is in class E.

i	$i_0 + i_1 - m_3(l_i) =$
4,11,17,25,28,30,34	0
9	1
14	$1 + \varepsilon(A) - \varepsilon(F)$
19	$1 + \varepsilon(B) - \varepsilon(H)$
1	$3 - 2c + s - 2\varepsilon(A)$
2	$3 - 2c + s - 2\varepsilon(A) - \varepsilon(B) + \varepsilon(H)$
3	$3 - 2c + s - \varepsilon(A) - 2\varepsilon(B) + \varepsilon(F)$
5	$3 - 2c + s - 2\varepsilon(B)$
6	$3 - 2c + s - \varepsilon(A) - \varepsilon(F)$
7	$3 - 2c + s - \varepsilon(B) - \varepsilon(H)$
10	$3 + c + s/2 - \varepsilon(A) + \varepsilon(F) + \varepsilon(H)$
13,26	$3 + c + s/2 + \varepsilon(B)$
15,33	$3 + c + s/2 + \varepsilon(H)$
18	$3 + c + s/2 + 2\varepsilon(A) - \varepsilon(B)$
23	$3 + c + s/2 - \varepsilon(A) + \varepsilon(B) + \varepsilon(F)$
27	$3 + c + s/2 + \varepsilon(A) - \varepsilon(B) + \varepsilon(F)$
29	$3 + c + s/2 + \varepsilon(A) + \varepsilon(F) - \varepsilon(H)$
31	$3 + c + s/2 + \varepsilon(A) + \varepsilon(B) - \varepsilon(F)$
32	$3 + c + s/2 + 2\varepsilon(B) - \varepsilon(H)$
35	$3 + c + s/2 + \varepsilon(A) - \varepsilon(F) + \varepsilon(H)$
20	$3 + 4c + \varepsilon(A) + 2\varepsilon(B) + \varepsilon(F)$
21	$3 + 4c + 2\varepsilon(A) + 2\varepsilon(B)$
22	$3 + 4c + 2\varepsilon(A) + \varepsilon(B) + \varepsilon(H)$
8	$1 - 3c + s/2 - 2\varepsilon(A) - \varepsilon(B)$
12	$1 - 3c + s/2 - \varepsilon(A) - \varepsilon(F) - \varepsilon(H)$
16	$1 - 3c + s/2 - 2\varepsilon(B) - \varepsilon(H)$
24	$1 - 3c + s/2 - \varepsilon(A) - \varepsilon(B) - \varepsilon(F)$