REPORTS
IN
INFORMATICS

ISSN 0333-3590

(Co)Institutions for Coalgebras

Uwe Wolter

REPORT NO 415 **October 2016**

Department of Informatics

UNIVERSITY OF BERGEN

Bergen, Norway

This report has URL
http://www.ii.uib.no/publikasjoner/texrap/pdf/2016-415.pdf

Reports in Informatics from Department of Informatics, University of Bergen, Norway, is available at http://www.ii.uib.no/publikasjoner/texrap/.

Abstract

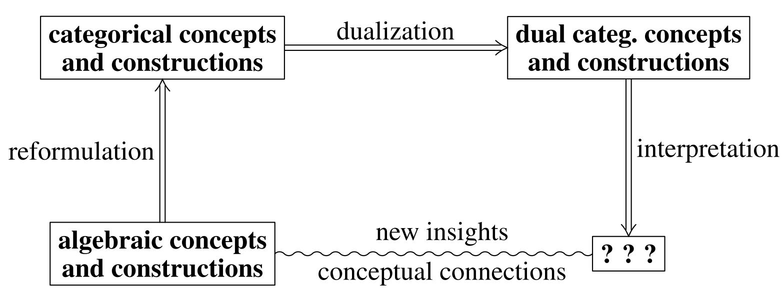
The paper presents a further step within a comprehensive program of "Dualizing Universal Algebra". The concept of equation has been dualized in [16] based on the concept of cofree coalgebras, the dual of term algebras. Here we show that this concept allows even for a dualization of the whole institution [6] of equational logic, i.e., allows to define a wide range of "coinstitutions" for coalgebras. Especially, we are able to define coinstitutions based on the model-theoretic extension of CSP developed in [17]. Thereby, coinstitutions are essentially the same as institutions only that the rôle of syntax and semantics is interchanged.

It has been shown that institutions are well-suited to present, to reformulate and to generalize the very many insights and results on structuring and modularization of algebraic specifications. In analogy, the results of the paper should provide a basis for developing general structuring formalisms for "system specifications" [13].

1 Introduction

The paper[1] presents a further step within a comprehensive program of "Dualizing Universal Algebra" [7, 10, 11, 13, 16, 17, 18]. Trusting the methodological power of Category Theory our part of the program is based on a clean three step strategy: In a first, most demanding, step we analyze the algebraic concepts, constructions, and results in question and reformulate them in a systematic categorical way. In a second step we dualize the categorical description in a quite formal way. And, in a third, creative, step, we try to interpret the abstract dual concepts, constructions, and results in terms of known, let us say set-theoretical, concepts.

It may happen that we end up with concepts already introduced and used in other fields of Computer Science. This will give new insights and conceptual connections that contribute to a "Unification of Theories" [9]. The interesting experience is that the new conceptual connections are often not apparent or even not expressible on the level of traditional concepts. The categorical level seems to be the natural level to organize and systematize the conglomeration of all our concepts and theories in Computer Science. Sometimes, we may end up with more general or even quite new concepts worth to investigate and to become used.



The experiences made in the area of Algebraic Specification have shown that the concept of institution [6] is well-suited to present, to reformulate and to generalize the very many insights and results on structuring and modularization of algebraic specifications. Especially, this concept reflects the insight that any reasonable structuring formalism has to be based on a certain compatibility between the structures on the level of specifications and structures on the level of models.

Based on these experiences in Algebraic Specification we should expect that structuring formalisms for "system specifications" [13] can be based on something similar as institutions. Dually to the concept of equations in Universal Algebra and in accordance with the

[1] The paper had been written in 2006 for publication in LNCS. Unfortunately, it was rejected and I decided to publish it now "for the records" as a Technical Report.

proposals in [7, 13] the concept of coequations introduced by the author in [16] is based on the existence of cofree coalgebras. In the present paper we show that this concept allows, indeed, to define quite naturally a range of coinstitutions for coalgebras. As an application of this general result we show, especially, that there are coinstitutions strongly related to CSP [8]. Thereby, the concept of coinstitution is essentially the same as the concept of institution only that the rôle of syntax and semantics is interchanged, as we should, of course, expect since coalgebras are the dual of algebras.

The paper gives a full exposition of the "unsorted" case, but outlines also, in some detail, a generalization to the "many-sorted" case.

A detailed presentation of the above mentioned three methodological steps seems to be not convenient for a conference paper. But we hope that, besides the cited literature, the included remarks will help the reader on his journey from Universal Algebra to Universal Coalgebra.

2 Coalgebras

It is an old observation that unsorted signatures Σ used in Universal Algebra and Algebraic Specifications can be coded by functors $\mathcal{F}_\Sigma : \mathbf{Set} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of sets and total maps. The crucial idea is to collect all the operations of a corresponding Σ-algebra with carrier A into a single map $\alpha : \mathcal{F}_\Sigma(A) \to A$, where $\mathcal{F}_\Sigma(A)$ is just the disjoint union of the domains of all the operations of the corresponding Σ-algebra, i.e., a finite coproduct of finite products. Generalizing and dualizing this observation we obtain

Definition 1 (Coalgebras)**.** *Given a functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ an* $\mathcal{F}$-coalgebra (A, α) *consists of a set A, called the* carrier*, and a map $\alpha : A \to \mathcal{F}(A)$, called the* (coalgebraic) structure map[2]*. An* $\mathcal{F}^c$-homomorphism $f : (A, \alpha) \to (B, \beta)$ *between $\mathcal{F}$-coalgebras is a map $f : A \to B$ such that $\beta \circ f = \mathcal{F}(f) \circ \alpha$.*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathcal{F}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ B & \xrightarrow{\beta} & \mathcal{F}(B) \end{array}$$

By $\mathbf{Alg}^c(\mathcal{F})$ we denote the category of all $\mathcal{F}$-coalgebras and all $\mathcal{F}^c$-homomorphisms between them.

Immediately by definition we have

Corollary 1 (Coforgetful functor)**.** *The assignments $(A, \alpha) \mapsto A$ and $(f : (A, \alpha) \to (B, \beta)) \mapsto (f : A \to B)$ extend to a functor $U^c_\mathcal{F} : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Set}$.*

Note, that dually to algebras $U^c_\mathcal{F}$ creates colimits [11, 13, 16].

Remark 1 (Many-sorted Coalgebras)**.** *We can define, of course, $\mathcal{F}$-coalgebras for any category $\mathbf{C}$ and any functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}$. Many-sorted coalgebras would be defined, in such a way, as $\mathcal{F}$-coalgebras for functors $\mathcal{F} : \mathbf{Set}^n \to \mathbf{Set}^n$, where n is the corresponding number of sorts. We are not aware of any explicit investigations or applications of many-sorted coalgebras. But it is quite natural to interpret many-sorted $\mathcal{F}$-coalgebras as systems with n components and a certain kind of interaction between these components.*

Definition 2 (Subcoalgebra)**.** *An $\mathcal{F}$-coalgebra (A, α) is an* $\mathcal{F}$-subcoalgebra *of an $\mathcal{F}$-coalgebra (B, β) iff $A \subseteq B$ and the inclusion map $\subseteq : A \to B$ defines an $\mathcal{F}^c$-homomorphism $\subseteq : (A, \alpha) \to (B, \beta)$.*

[2] Note, that the position of A in (A, α) indicates that A is the domain of α.

Many applications of coalgebras in system theory are based on *extended polynomial functors*, i.e., functors that can be build from constant functors $M : \mathbf{Set} \to \mathbf{Set}$ (where M is any set), the identical functor $ID : \mathbf{Set} \to \mathbf{Set}$, the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$, the product functor $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, the coproduct functor $+ : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, and the function space functor $[M \Rightarrow \_] : \mathbf{Set} \to \mathbf{Set}$, where M is again an arbitrary set [10, 13].

More recently we have seen that coalgebras allow to define a loose semantics for CSP:

Example 1. *Given a set I of "input symbols" we can define a functor $\mathcal{D}_I : \mathbf{Set} \to \mathbf{Set}$ with $\mathcal{D}_I(X) \overset{def}{=} [I \Rightarrow \mathbf{1} + X]$ for any set X, where $\mathbf{1} = \{*\}$ is a fixed singleton set, i.e., the empty product and thus the terminal object in* **Set**. *$\mathcal{D}_I$-coalgebras represent* (partial) deterministic automata without output *in a curried fashion. The uncurried representation of a $\mathcal{D}_I$-coalgebra $(A, \alpha : A \to [I \Rightarrow \mathbf{1}+A])$ would be given by the partial state transition $\delta_\alpha : I \times A \to \mathbf{1}+A$ with $\delta_\alpha(i, a) = \alpha(a)(i)$ for all $i \in I, a \in A$. These "curried automata" are exactly the systems CSP has dealt with [17]. For any $\mathcal{D}_I^c$-homomorphism $f : (A, \alpha) \to (B, \beta)$ and any map $g \in \mathcal{D}_I(A) = [I \Rightarrow \mathbf{1} + A]$ we have $\mathcal{D}_I(f)(g) = (id_{\mathbf{1}} + f) \circ g$*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & [I \Rightarrow \mathbf{1} + A] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (id_{\mathbf{1}}+f)\circ\_} \\ B & \xrightarrow{\beta} & [I \Rightarrow \mathbf{1} + B] \end{array}$$

This means, especially, that for any $i \in I, a \in A$ the result $\beta(f(a))(i)$ is defined ($\neq *$) iff $\alpha(a)(i)$ is defined. That is, in terms of partial algebras, the $\mathcal{D}_I^c$-homomorphisms are exactly the closed homomorphisms *[12, 15].*

3 Process Coalgebras and Coequations

In Universal Algebra and Algebraic Specifications (conditional) equations are used to axiomatize classes of algebras. (Conditional) equations are build up by terms, and terms are constructed in a canonical way for any given signature Σ and any set X of variables. A crucial insight is that the set $T_\Sigma(X)$ of all Σ-terms over X is the carrier of a Σ-algebra and that this Σ-algebra can be characterized, in categorical terms, as the Σ-algebra *freely generated by X*. Generalizing and dualizing this insight we obtain

Definition 3 (Cofree Functor)**.** *Let C be a set (of* 'colors'*). An $\mathcal{F}$-coalgebra $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ together with a* 'coloring' *$\varepsilon_{\mathcal{F},C} : P_{\mathcal{F}}(C) \to C$ is* cofree over C w.r.t. $U_{\mathcal{F}}^c : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Set}$ *if for every $\mathcal{F}$-coalgebra (A, α) and for every coloring $c : A \to C$ there exists a unique in $\mathcal{F}^c$-homomorphism $c^{\mathcal{F}} : (A, \alpha) \to (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ such that $\varepsilon_{\mathcal{F},C} \circ U_{\mathcal{F}}^c(c^{\mathcal{F}}) = \varepsilon_{\mathcal{F},C} \circ c^{\mathcal{F}} = c$.*

$$\begin{array}{c} (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C}) \\ \Big\uparrow {\scriptstyle c^{\mathcal{F}}} \\ (A, \alpha) \end{array} \qquad\qquad \begin{array}{ccc} P_{\mathcal{F}}(C) & \xrightarrow{\varepsilon_{\mathcal{F},C}} & C \\ {\scriptstyle c^{\mathcal{F}}}\Big\uparrow & \nearrow{\scriptstyle c} & \\ A & & \end{array}$$

In general, the 'elements' of $P_{\mathcal{F}}(C)$ can be interpreted as the 'observable behaviours' of systems of typ $\mathcal{F}$ [7, 10, 11, 13] or as '$\mathcal{F}$-processes' [17]. Therefore, we will call $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ the *$\mathcal{F}$-process coalgebra over C*. Moreover, $\varepsilon_{\mathcal{F},C} : P_{\mathcal{F}}(C) \to C$ will be called the *$\mathcal{F}^c$-counit for C* and $c^{\mathcal{F}} : (A, \alpha) \to (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ the *(unique) $\mathcal{F}^c$-extension of c*. The image $c^{\mathcal{F}}(A) \subseteq P_{\mathcal{F}}(C)$ can be interpreted as a behaviour "observable in A" or "realized by A".

Standard categorical arguments show that cofree coalgebras define a functor

Corollary 2 (Cofree functor)**.** *If for all sets C there exists an $\mathcal{F}$-coalgebra cofree over C w.r.t. $U^{c}_{\mathcal{F}}$, then the assignments $C \mapsto (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ and $(s : Y \to C) \mapsto (s \circ \varepsilon_{\mathcal{F},Y})^{\mathcal{F}}$: $(P_{\mathcal{F}}(Y), \phi_{\mathcal{F},Y}) \longrightarrow (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ define a functor $\mathcal{P}_{\mathcal{F}} : \mathbf{Set} \to \mathbf{Alg}^{c}(\mathcal{F})$ right-adjoint to $U^{c}_{\mathcal{F}} : \mathbf{Alg}^{c}(\mathcal{F}) \to \mathbf{Set}$ and called the* cofree functor for $\mathcal{F}$*. Moreover, $\mathcal{F}$ will be called* co-syntactical *in this case.*

The possibility to construct Σ-terms is reflected, categorically, by the fact that the corresponding functor $\mathcal{F}_{\Sigma} : \mathbf{Set} \to \mathbf{Set}$ is ω-continuous [14]. Dually, the existence of cofree coalgebras is ensured if the functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ is ω^{op}-continuous [7, 11, 13, 17]. Fortunately, extended polynomial functors are ω^{op}-continous thus we have cofree coalgebras for a wide range of applications.

Remark 2 (Final Coalgebra)**.** *An $\mathcal{F}$-coalgebra (A, α) and a coloring $c : A \to C$ define an $\mathcal{F} \times C$-coalgebra $(A, \langle \alpha, c\rangle : A \to \mathcal{F}(A) \times C)$ where the functor $\mathcal{F} \times C : \mathbf{Set} \to \mathbf{Set}$ is given by $(\mathcal{F} \times C)(A) \stackrel{def}{=} \mathcal{F}(A) \times C$ for all sets A. The statement that $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ together with $\varepsilon_{\mathcal{F},C} : P_{\mathcal{F}}(C) \to C$ is cofree over C w.r.t. $U^{c}_{\mathcal{F}}$ becomes, in such a way, equivalent to the statement that $(P_{\mathcal{F}}(C), \langle \phi_{\mathcal{F},C}, \varepsilon_{\mathcal{F},C}\rangle)$ is a* final $\mathcal{F} \times C$-coalgebra*. This makes evident that colors play the rôle of additional "observations" in the definition of processes. That is, colors allow for additional "distinctions" between processes. This is dual to the rôle of variables as additional "generators".*

Straightforward categorical reasoning shows that the structure maps in final coalgebras are isomorphisms [10, 13]. In such a way the $\mathcal{F}$-coalgebra cofree over C w.r.t. $U^{c}_{\mathcal{F}}$ provides an isomorphism $\langle \phi_{\mathcal{F},C}, \varepsilon_{\mathcal{F},C}\rangle : P_{\mathcal{F}}(C) \to \mathcal{F}(P_{\mathcal{F}}(C)) \times C$.

Example 2. *The functor $\mathcal{D}_I : \mathbf{Set} \to \mathbf{Set}$ is polynomial thus there exists for any C an $\mathcal{D}_I$-coalgebra $(P_{\mathcal{D}_I}(C), \phi_{\mathcal{D}_I})$ cofree over C. Due to Remark 2 we have an isomorphism*

$$\langle \phi_{\mathcal{D}_I,C}, \varepsilon_{\mathcal{D}_I,C}\rangle : P_{\mathcal{D}_I}(C) \to [I \Rightarrow \mathbf{1} + P_{\mathcal{D}_I}(C)] \times C$$

thus it becomes evident that the elements of $P_{\mathcal{D}_I}(C)$ are, possibly infinite, trees with elements from C at the nodes and "input symbols" from I at the branching edges. There can be finite and infinite branches where the finite branches are caused by the partiality of the state transition. $\varepsilon_{\mathcal{D}_I,C} : P_{\mathcal{D}_I}(C) \to C$ provides the color at the root and $\phi_{\mathcal{D}_I,C} : P_{\mathcal{D}_I}(C) \to [I \Rightarrow \mathbf{1} + P_{\mathcal{D}_I}(C)]$ is defined for $p \in P_{\mathcal{D}_I}(C)$ and $i \in I$ iff there is an edge from the root labelled by i and $\phi_{\mathcal{D}_I,C}(p)(i)$ provides than the subtree at this branch.

For a coloring $c : A \to C$ of the states of an automaton $(A, \alpha : A \to [I \Rightarrow \mathbf{1} + A])$ the unique $\mathcal{D}^{c}_{I}$-extension $c^{\mathcal{D}_I} : A \to P_{\mathcal{D}_I}(C)$ assigns to each state $a \in A$ the process starting in this state, i.e., a tree with the colors of all the states, reachable from a via α, at the nodes and with input symbols from I at the edges, according to the state transition α.

For $C = \mathbf{1}$ a singleton set $P_{\mathcal{D}_I}(\mathbf{1})$ is equivalent to the set of all deterministic CSP processes over the alphabet I, and $(P_{\mathcal{D}_I}(\mathbf{1}), \phi_{\mathcal{D}_I,\mathbf{1}})$ is, of course, the final $\mathcal{D}_I$-coalgebra [17].

A Σ-equation over a set X of variables is a pair of Σ-terms and can be represented, in such a way, as a pair $l, r : \mathbf{1} \to T_{\Sigma}(X)$ of maps. A thouroughly categorical analysis [16] shows that a reasonable dualization is given by

Definition 4 (Coequations)**.** *Let $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ be a functor and let C be a set of colors such that there exists an $\mathcal{F}$-coalgebra $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ cofree over C. An $\mathcal{F}$*-coequation *on C is a pair $l, r : P_{\mathcal{F}}(C) \to \mathbf{2}$ of maps where $\mathbf{2} = \{0, 1\}$.*

For a co-syntactical functor $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ we denote by $Ceq(\mathcal{F})$ the set of all $\mathcal{F}$-coequations on arbitrary sets C.

Remark 3 (Coequation vs. subcoalgebra)**.** *Any Σ-equation gives rise to (or can be represented by) a quotient Σ-algebra of the Σ-algebra of Σ-terms over X, i.e., of the Σ-algebra freely generated by X [11].*

Dually, any $\mathcal{F}$-coequation $ce = (l, r : P_{\mathcal{F}}(C) \to \mathbf{2})$ defines an $\mathcal{F}$-subcoalgebra (P_{ce}, ϕ_{ce}) of the $\mathcal{F}$-coalgebra $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ cofree over C. (P_{ce}, ϕ_{ce}) is the greatest $\mathcal{F}$-subcoalgebra such that P_{ce} is contained in the equalizer of the pair $l, r : P_{\mathcal{F}}(C) \to \mathbf{2}$ of maps [13]. P_{ce} can be seen as the behaviour required or specified by the coequation ce (compare [11, 13] where subcoalgebras have been proposed as a synonym for coequational specifications).

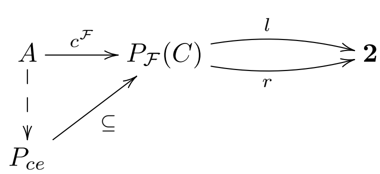
Example 3. *To what extend can we consider CSP as a coequational specification formalism? Any CSP process (expression) Q over an alphabet I determines an element in the final $\mathcal{D}_I$-coalgebra $(P_{\mathcal{D}_I}(\mathbf{1}), \phi_{\mathcal{D}_I,\mathbf{1}})$. There is a minimal $\mathcal{D}_I$-subcoalgebra (P_Q, ϕ_Q) of $(P_{\mathcal{D}_I}(\mathbf{1}), \phi_{\mathcal{D}_I,\mathbf{1}})$ containing this element [13]. And, of course we can define a $\mathcal{D}_I$-coequation $ce_Q = (l_Q, r_Q : P_{\mathcal{D}_I}(\mathbf{1}) \to \mathbf{2})$ with $l_Q(p) = 1$ for all $p \in P_{\mathcal{D}_I}(\mathbf{1})$ and with $r_Q(p) = 1$ for all $p \in P_Q$ and $r_Q(p) = 0$ for all $p \in P_{\mathcal{D}_I}(\mathbf{1}) \setminus P_Q$ such that $(P_{ce_Q}, \phi_{ce_Q}) = (P_Q, \phi_Q)$.*

A Σ-equation $l, r : \mathbf{1} \to T_\Sigma(X)$ is satisfied in a Σ-algebra with carrier A for a variable assignment $\gamma : X \to A$ iff the corresponding unique evaluation $\overline{\gamma} : T_\Sigma(X) \to A$ of Σ-terms equalizes l and r, i.e., if we have $\overline{\gamma} \circ l = \overline{\gamma} \circ r$. Dually we define

Definition 5 (Validity)**.** *An $\mathcal{F}$-coalgebra (A, α)* satisfies *an $\mathcal{F}$-coequation $ce = (l, r : P_{\mathcal{F}}(C) \to \mathbf{2})$ for a coloring $c : A \to C$, $(A, \alpha), c \models_{\mathcal{F}} ce$ in symbols, iff $l \circ c^{\mathcal{F}} = r \circ c^{\mathcal{F}}$ for the unique $\mathcal{F}^c$-extension $c^{\mathcal{F}} : (A, \alpha) \to (P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ of c.*

The $\mathcal{F}$-coequation ce is valid in (A, α), *$(A, \alpha) \models_{\mathcal{F}} ce$ in symbols, iff $(A, \alpha), c \models_{\mathcal{F}} ce$ for all colorings $c : A \to C$.*

Remark 4 (Satisfaction vs. specified behaviour)**.** *According to the definition of the $\mathcal{F}$-subcoalgebra (P_{ce}, ϕ_{ce}) in Remark 3 the statement $(A, \alpha), c \models_{\mathcal{F}} ce$ can be reformulated by the requirement $c^{\mathcal{F}}(A) \subseteq P_{ce}$, i.e., $c^{\mathcal{F}}$ has to factorize through P_{ce}*



In other words, the $\mathcal{F}$-coalgebra (A, α) realizes or implements indeed the behaviour specified by ce.

Remark 5 (Coequation vs. excluded behaviour)**.** *The concept of coequational specification proposed in [7] is based on the idea to exclude behaviour: That is, a coequation is an element $e \in P_{\mathcal{F}}(C)$ and $(A, \alpha), c \models_{\mathcal{F}} e$ iff $e \notin c^{\mathcal{F}}(A)$. We can define, of course, an $\mathcal{F}$-coequation $c_e = (l_e, r_e : P_{\mathcal{F}}(C) \to \mathbf{2})$ with $l_e(p) = 1$ for all $p \in P_{\mathcal{F}}(C)$ and with $r_e(e) = 0$ and $r_e(p) = 1$ for all $p \in P_{\mathcal{F}}(C) \setminus \{e\}$. In such a way (P_{c_e}, ϕ_{c_e}) will be the greatest $\mathcal{F}$-subcoalgebra of $(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})$ such that P_{c_e} is contained $P_{\mathcal{F}}(C) \setminus \{e\}$ (see Remark 3). This shows that the concept of coequation in [7] is also covered by our definition.*

Remark 6 (Many-sorted Coequations)**.** *For a many-sorted signature Σ with a set $S = \{s_1, \ldots, s_n\}$ of sorts an S-set of variables is given by an n-tuple $X = (X_1, \ldots, X_n) \in |\mathbf{Set}^n|$ of sets and the corresponding S-set of Σ-terms over X is an n-tuple $T_\Sigma(X) = (T_\Sigma(X)_{s_1}, \ldots, T_\Sigma(X)_{s_n}) \in |\mathbf{Set}^n|$ of sets of Σ-terms of sort s_i, $1 \leq i \leq n$. A Σ-equation has a sort s_i and is given by a pair $l, r : \mathbf{1} \to T_\Sigma(X)_{s_i}$ of maps. Those pairs of maps can be presented, equivalently, by pairs $\overline{l}, \overline{r} : \mathbf{1}^n_{s_i} \to T_\Sigma(X)$ of morphisms in $\mathbf{Set}^n$, where $\mathbf{1}^n_{s_i} = (\emptyset, \ldots, \mathbf{1}, \ldots, \emptyset)$ is an n-tuple of empty sets except a singleton set at position i, and $\overline{l}, \overline{r}$ are n-tuples of inclusions of the empty set except the map l or r, respectively, at position i.*

Dually, it is reasonable to assume that also many-sorted coequations are sorted, i.e., put requirements on single "system components": Given a co-syntactical functor $\mathcal{F} : \mathbf{Set}^n \to \mathbf{Set}^n$ an $\mathcal{F}$-coequation of sort s_i on an S-set C of colors is a pair $l, r : P_{\mathcal{F}}(C)_{s_i} \to \mathbf{2}$ of maps. Equivalently, those pairs can be represented by pairs $\bar{l}, \bar{r} : P_{\mathcal{F}}(C) \to \mathbf{2}^n_{s_i}$ of morphisms in $\mathbf{Set}^n$, where $\mathbf{2}^n_{s_i} = (\mathbf{1}, \ldots, \mathbf{2}, \ldots, \mathbf{1})$ is an n-tuple of singleton sets except the set $\mathbf{2}$ at position i, and $\bar{l}, \bar{r}$ are n-tuples of constant maps into $\mathbf{1}$ except the map l or r, respectively, at position i. Note that, the empty set $\emptyset$ is the initial object in $\mathbf{Set}$, and that, dually, $\mathbf{1}$ is the terminal object in $\mathbf{Set}$.

4 Coinstitutions

Up to now we have only considered single, isolated signatures. But within a stepwise and modular system design we have also to take into account relations between signatures. The comprehensive studies on structured and modular specifications within the area of Algebraic Specifications have shown the the concept of *institution* provides a well-structured, abstract scheme for presenting logics and specification formalims in a uniform way. This concept was introduced by GOGUEN and BURSTALL [6] and allowed them to reformulate and to generalize, independent of the underlying logic, the work they had done in the 70's on structuring (equational) specifications. A similar proposal of an abstract concept of a logic had been given already by BARWISE [2]. By interchanging the rôle of syntax and semantics we obtain the following

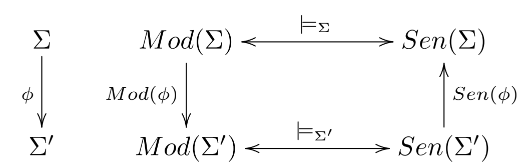
Definition 6 (Coinstitution)**.** *A* coinstitution $\mathcal{I}^c = (\mathbf{Sign}, Mod, Sen, \models)$ *consists of*

- *a category* **Sign** *of (abstract) signatures;*
- *a model functor* $Mod : \mathbf{Sign} \to \mathbf{Cat}$*;*
- *a syntax functor* $Sen : \mathbf{Sign}^{op} \to \mathbf{Set}$*;*
- *and an indexed family of satisfaction relations* $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$*,* $\Sigma \in |\mathbf{Sign}|$

such that the following satisfaction condition

$$M \models_\Sigma Sen(\phi)(\varphi') \quad \textbf{iff} \quad Mod(\phi)(M) \models_{\Sigma'} \varphi'$$

holds for each $\phi : \Sigma \to \Sigma'$ *in* **Sign***,* $M \in |Mod(\Sigma)|$*, and* $\varphi' \in Sen(\Sigma')$*.*



Coinstitutions are dual to institutions in the trivial sense that coinstitutions are based on covariant model functors and on contravariant syntax functors, where institutions, instead, are based on contravariant model functors and covariant syntax functors, respectively. More formally expressed: Any coinstitution $\mathcal{I}^c = (\mathbf{Sign}, Mod, Sen, \models)$ defines an institution $\mathcal{I} = (\mathbf{Sign}^{op}, Mod, Sen, \models)$ and, vice versa, any institution institution $\mathcal{I} = (\mathbf{Sign}, Mod, Sen, \models)$ defines a coinstitution $\mathcal{I}^c = (\mathbf{Sign}^{op}, Mod, Sen, \models)$. That is, in principle there is no need for a new concept. But as a matter of taste and to emphasize the duality between algebras and coalgebras, we prefer to coin the dual concept of coinstitution. That the present categorical approach to Universal Coalgebra leads quite naturally to structures establishing coinstitutions will be shown in the following subsections.

4.1 Signature Morphisms

In our unsorted categorical approach to Universal Coalgebra "signatures" are given by functors, thus the canonical choice for "signature morphisms" will be, of course, natural transformations between functors.

Definition 7. *A category* **Sign** *of* (unsorted) abstract signatures *is an arbitrary, but fixed subcategory of the functor category* $\mathbf{Func}(\mathbf{Set}, \mathbf{Set})$*, i.e., abstract signatures are given by functors* $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$ *and abstract signature morphisms are given by natural transformations* $\tau : \mathcal{F} \Rightarrow \mathcal{G} : \mathbf{Set} \to \mathbf{Set}$. **Sign** *is a category of* abstract co-syntactical signatures *iff all abstract signatures* $\mathcal{F} \in |\mathbf{Sign}|$ *are co-syntactical (see Corollary 2).*

Example 4. *In case of CSP it is natural to choose sets* I *of input symbols as signatures and maps* $\phi : I \to J$ *as signature morphisms. How does this approach fits into the general categorical scheme of Definition 7? For any* I *we have the functor* $\mathcal{D}_I : \mathbf{Set} \to \mathbf{Set}$ *(see Example 1) as the corresponding abstract signature, and any translation* $\phi : I \to J$ *of input symbols gives rise to a natural transformation in the opposite direction* $\phi^{\mathcal{D}} : \mathcal{D}_J \Rightarrow \mathcal{D}_I$ *where the components*

$$\phi^{\mathcal{D}}_A \stackrel{def}{=} (\_ \circ \phi) : [J \Rightarrow \mathbf{1} + A] \to [I \Rightarrow \mathbf{1} + A]$$

are simply given by pre-composition. These natural transformations are the corresponding abstract signature morphisms. In other words: The assignments $I \mapsto \mathcal{D}_I$ *and* $(\phi : I \to J) \mapsto (\phi^{\mathcal{D}} : \mathcal{D}_J \Rightarrow \mathcal{D}_I)$ *define a contravariant embedding* $\mathcal{D} : \mathbf{Set}^{op} \to \mathbf{Func}(\mathbf{Set}, \mathbf{Set})$. *And instead of* **Set** *(or* $\mathbf{Set}^{op}$*) we take as* **Sign** *the subcategory of* $\mathbf{Func}(\mathbf{Set}, \mathbf{Set})$ *given by the image of* $\mathcal{D}$.

Remark 7 (Many-sorted Signature Morphisms)**.** *For many-sorted coalgebras the situation will be more involved since different signatures may have different numbers of sorts: For two abstract many-sorted signatures* $\mathcal{F} : \mathbf{Set}^n \to \mathbf{Set}^n$ *and* $\mathcal{G} : \mathbf{Set}^m \to \mathbf{Set}^m$ *an abstract many-sorted signature morphism* $(\mathcal{V}, \tau) : \mathcal{F} \to \mathcal{G}$ *will be given by a functor* $\mathcal{V} : \mathbf{Set}^n \to \mathbf{Set}^m$ *and a natural transformation* $\tau : \mathcal{V} \circ \mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{V} : \mathbf{Set}^m \to \mathbf{Set}^m$.

In analogy to Universal Algebra we could be even more specific about the functor $\mathcal{V}$*: In Universal Algebra a many-sorted signature morphism* $\phi : \Sigma \to \Sigma'$ *is based on a map* $\phi : S \to S'$ *between the corresponding sets* $S = \{s_1, \ldots, s_n\}$ *and* $S' = \{s'_1, \ldots, s'_m\}$ *of sorts. This map induces a functor* $\mathcal{V}_\phi : \mathbf{Set}^m \to \mathbf{Set}^n$ *with* $\mathcal{V}_\phi(A_{s'_1}, \ldots, A_{s'_m}) \stackrel{def}{=} (A_{\phi(s_1)}, \ldots, A_{\phi(s_n)})$. *Categorically spoken,* $\mathcal{V}_\phi$ *is essentially build up by the diagonal functor* $\Delta : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$ *(if* ϕ *is non-injective) and by the "delete" functor* $\delta : \mathbf{Set} \to \mathbf{1}$ *(if* ϕ *is non-surjective).* $\mathbf{1}$ *denotes here the category with only one object and with the identity on this object as the only morphism, i.e.,* $\mathbf{1}$ *is the empty product of categories and thus neutral w.r.t. formation of products of categories. In such a way, the signature morphism* $\phi : \Sigma \to \Sigma'$ *will be represented, finally, on the categorical level by the functor* $\mathcal{V}_\phi : \mathbf{Set}^m \to \mathbf{Set}^n$ *and natural transformation* $\tau_\phi : \mathcal{F}_\Sigma \circ \mathcal{V}_\phi \Rightarrow \mathcal{V}_\phi \circ \mathcal{G}_{\Sigma'} : \mathbf{Set}^n \to \mathbf{Set}^n$ *for the functors* $\mathcal{F}_\Sigma : \mathbf{Set}^n \to \mathbf{Set}^n$ *and* $\mathcal{G}_{\Sigma'} : \mathbf{Set}^m \to \mathbf{Set}^m$ *coding the signatures* Σ *and* Σ'*, respectively.*

Dually, we consider for coalgebras a map $\psi : \Sigma' \to \Sigma$ *with* $S = \{s_1, \ldots, s_n\}$ *and* $S' = \{s'_1, \ldots, s'_m\}$ *being the sets of sorts connected to the functors* $\mathcal{F} : \mathbf{Set}^n \to \mathbf{Set}^n$ *and* $\mathcal{G} : \mathbf{Set}^m \to \mathbf{Set}^m$*, respectively.* ψ *induces, as above, a functor* $\mathcal{V}_\psi : \mathbf{Set}^n \to \mathbf{Set}^m$*, build up by the diagonal functor* Δ *and by the "delete" functor* δ*, i.e., with* $\mathcal{V}_\psi(A_{s_1}, \ldots, A_{s_n}) \stackrel{def}{=} (A_{\psi(s'_1)}, \ldots, A_{\psi(s_m)})$ *for all* $(A_{s_1}, \ldots, A_{s_n}) \in |\mathbf{Set}^n|$. *Abstract signature morphisms* $(\mathcal{V}_\psi, \tau) : \mathcal{F} \to \mathcal{G}$ *induced by maps* $\psi : \Sigma' \to \Sigma$ *between sets of sorts will be called* sort based.

4.2 Covariant Model Functor

In Universal Algebra any signature morphism $\phi : \Sigma \to \Sigma'$ gives rise to a forgetful functor (in the opposite direction) from the category of all Σ'-algebras into the category of all Σ-algebras. Dually, we obtain for any unsorted abstract signature morphism a coforgetful functor

Definition 8 (Coforgetful Functor). *Any natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G} : \mathbf{Set} \to \mathbf{Set}$ gives rise to a functor $U^c_\tau : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{G})$ defined for any $\mathcal{F}$-coalgebra (A, α) and any $\mathcal{F}^c$-homomorphism $f : (A, \alpha) \to (B, \beta)$ by $U^c_\tau(A, \alpha) \stackrel{def}{=} (A, \tau_A \circ \alpha)$ and $U^c_\tau(f) \stackrel{def}{=} f$.*

$$
\begin{array}{ccccc}
A & \xrightarrow{\alpha} & \mathcal{F}(A) & \xrightarrow{\tau_A} & \mathcal{G}(A) \\
\downarrow f & & \downarrow \mathcal{F}(f) & & \downarrow \mathcal{G}(f) \\
B & \xrightarrow{\beta} & \mathcal{F}(B) & \xrightarrow{\tau_B} & \mathcal{G}(B)
\end{array}
$$

According to the definition of natural identities and of (vertical) composition of natural transformations we have immediately from Definition 8

Proposition 1 ((Covariant) Model Functor). *For any category* **Sign** *of (unsorted) abstract signatures (for coalgebras) the assignments $\mathcal{F} \mapsto \mathbf{Alg}^c(\mathcal{F})$ and $(\tau : \mathcal{F} \Rightarrow \mathcal{G}) \mapsto (U^c_\tau : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{G}))$ define a (covariant) model functor $Alg^c : \mathbf{Sign} \to \mathbf{Cat}$.*

Proof. According to Definition 8 the coforgetful functors U^c_τ are identities on morphisms or, for being more precise, on the underlying maps, thus we have only to prove something for coalgebras:
Identities: For any natural identity $\iota_\mathcal{F} : \mathcal{F} \Rightarrow \mathcal{F}$ and any $\mathcal{F}$-coalgebra (A, α) we obtain according to Definition 8 and the definition of natural identities: $U^c_{\iota_\mathcal{F}}(A, \alpha) = (A, \iota_{\mathcal{F},A} \circ \alpha) = (A, id_{\mathcal{F}(A)} \circ \alpha) = (A, \alpha)$. That is, $U^c_{\iota_\mathcal{F}}$ is indeed the identical functor on $\mathbf{Alg}^c(\mathcal{F})$.
Composition: For any natural transformations $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ and $\kappa : \mathcal{G} \Rightarrow \mathcal{H}$ we have to show that $U^c_\kappa \circ U^c_\tau = U^c_{\kappa\circ\tau} : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{H})$: For any $\mathcal{F}$-coalgebra (A, α) we obtain according to Definition 8 and according to the definition of composition for functors and natural transformations: $U^c_\kappa \circ U^c_\tau(A, \alpha) = U^c_\kappa(U^c_\tau(A, \alpha)) = U^c_\kappa(A, \tau_A \circ \alpha) = (A, \kappa_A \circ \tau_A \circ \alpha) = (A, (\kappa \circ \tau)_A \circ \alpha) = U^c_{\kappa\circ\tau}(A, \alpha)$. □ □

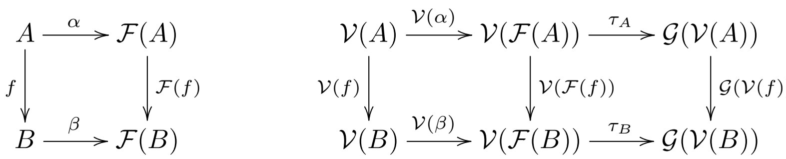
Example 5. *Given a map $\phi : I \to J$ the corresponding natural transformation $\phi^\mathcal{D} : \mathcal{D}_J \Rightarrow \mathcal{D}_I$ provides a functor $U^c_{\phi^\mathcal{D}} : \mathbf{Alg}^c(\mathcal{D}_J) \to \mathbf{Alg}^c(\mathcal{D}_I)$ with $U^c_{\phi^\mathcal{D}}(A, \alpha) = (A, \alpha(\_) \circ \phi)$ for any $\mathcal{D}_J$-coalgebra (A, α).*

$$
\begin{array}{ccccc}
A & \xrightarrow{\alpha} & [J \Rightarrow \mathbf{1} + A] & \xrightarrow{(\_\circ\phi)} & [I \Rightarrow \mathbf{1} + A] \\
\downarrow f & & \downarrow (id_\mathbf{1}+f)\circ\_ & & \downarrow (id_\mathbf{1}+f)\circ\_ \\
B & \xrightarrow{\beta} & [J \Rightarrow \mathbf{1} + B] & \xrightarrow{(\_\circ\phi)} & [I \Rightarrow \mathbf{1} + B]
\end{array}
$$

That is, we delete all transitions in the partial automaton (A, α) labelled by elements in $J \setminus \phi(I)$, and the other transitions are multiplied, if ϕ is non-injective, and the labels are renamed according to ϕ.

Remark 8 (Many-sorted Coforgetful Functors). *Given an abstract many-sorted signature morphism $(\mathcal{V}, \tau) : \mathcal{F} \to \mathcal{G}$ as in Remark 7 we will obtain a functor $U^c_{(\mathcal{V},\tau)} : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{G})$ defined for any $\mathcal{F}$-coalgebra (A, α) and any $\mathcal{F}^c$-homomorphism $f : (A, \alpha) \to$*

(B,β) *by* $U^c_{(\mathcal{V},\tau)}(A,\alpha) \stackrel{def}{=} (\mathcal{V}(A), \tau_A \circ \mathcal{V}(\alpha))$ *and* $U^c_{(\mathcal{V},\tau)}(f) \stackrel{def}{=} \mathcal{V}(f)$.



For a sort based abstract many-sorted signature morphism $(\mathcal{V}_\psi, \tau) : \mathcal{F} \to \mathcal{G}$ *the coforgetful functor* $U^c_{(\mathcal{V}_\psi,\tau)} : \mathbf{Alg}^c(\mathcal{F}) \to \mathbf{Alg}^c(\mathcal{G})$ *may "destroy" some system components (if* $\psi : \Sigma' \to \Sigma$ *is non-surjective) and may duplicate other components (if* ψ *is non-injective).*

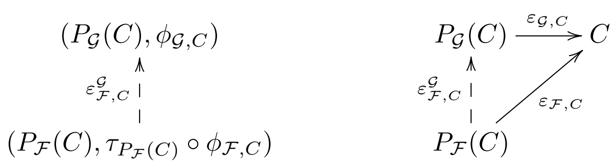
4.3 Contravariant Syntax Functor

In Universal Algebra, not only the evaluation of terms in algebras and thus satisfaction and validity of equations are based on the property "freely generated", but also the translation of terms and equations along signature morphisms. A thoroughly categorical analysis of this situation leads to the following dualization

Definition 9 (Translation of Coequations)**.** *Any natural transformation* $\tau : \mathcal{F} \Rightarrow \mathcal{G} : \mathbf{Set} \to \mathbf{Set}$ *between co-syntactical functors* $\mathcal{F}$ *and* $\mathcal{G}$ *gives rise to a map* $Ceq(\tau) : Ceq(\mathcal{G}) \to Ceq(\mathcal{F})$*: To any* $\mathcal{G}$*-coequation* $ce = (l, r : P_\mathcal{G}(C) \to \mathbf{2})$ *on a set* C *we assign the* $\mathcal{F}$*-coequation*

$$Ceq(\tau)(ce) \stackrel{def}{=} (l \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}, r \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} : P_\mathcal{F}(C) \to \mathbf{2})$$

on C*, where* $\varepsilon^{\mathcal{G}}_{\mathcal{F},C} : U^c_\tau(P_\mathcal{F}(C), \phi_{\mathcal{F},C}) \to (P_\mathcal{G}(C), \phi_{\mathcal{G},C})$ *is the unique* $\mathcal{G}^c$*-extension of the* $\mathcal{F}^c$*-counit* $\varepsilon_{\mathcal{F},C} : P_\mathcal{F}(C) \to C$.



Proposition 2 (Contravariant Syntax Functor)**.** *For any category* **Sign** *of (unsorted) abstract co-syntactical signatures the assignments* $\mathcal{F} \mapsto Ceq(\mathcal{F})$ *and* $(\tau : \mathcal{F} \Rightarrow \mathcal{G}) \mapsto (Ceq(\tau) : Ceq(\mathcal{G}) \to Ceq(\mathcal{F}))$ *define a contravariant syntax functor* $Ceq : \mathbf{Sign}^{op} \to \mathbf{Set}$.

Proof. Identities: For any natural identity $\iota_\mathcal{F} : \mathcal{F} \Rightarrow \mathcal{F}$ we have $U^c_{\iota_\mathcal{F}} = Id_{\mathbf{Alg}^c(\mathcal{F})}$, according to Proposition 1, and, in such a way, also $Ceq(\iota_\mathcal{F})(ce) = ce$ for any $\mathcal{F}$-coequation $ce = (l, r : P_\mathcal{F}(C) \to \mathbf{2})$ since $\varepsilon^{\mathcal{F}}_{\mathcal{F},C} = id_{P_\mathcal{F}(C)}$ according to the uniqueness of $\mathcal{F}^c$-extensions.
Composition: For any natural transformations $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ and $\kappa : \mathcal{G} \Rightarrow \mathcal{H}$ we have to show that $Ceq(\kappa \circ \tau) = Ceq(\tau) \circ Ceq(\kappa)$. That is, for any $\mathcal{H}$-coequation $ce = (l, r : P_\mathcal{H}(C) \to \mathbf{2})$ we have to show

$$(l \circ \varepsilon^{\mathcal{H}}_{\mathcal{F},C}, r \circ \varepsilon^{\mathcal{H}}_{\mathcal{F},C} : P_\mathcal{F}(C) \to \mathbf{2}) = (l \circ \varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}, r \circ \varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} : P_\mathcal{F}(C) \to \mathbf{2}).$$

And this holds if the equation $\varepsilon^{\mathcal{H}}_{\mathcal{F},C} = \varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}$ holds in **Set**: According to Definition 3 we have for the $\mathcal{H}^c$-extension $\varepsilon^{\mathcal{H}}_{\mathcal{G},C} : U^c_\kappa(P_\mathcal{G}(C), \phi_{\mathcal{G},C}) \to (P_\mathcal{H}(C), \phi_{\mathcal{H},C})$ of the $\mathcal{G}^c$-counit $\varepsilon_{\mathcal{G},C} : P_\mathcal{G}(C) \to C$ the equation $\varepsilon_{\mathcal{H},C} \circ \varepsilon^{\mathcal{H}}_{\mathcal{G},C} = \varepsilon_{\mathcal{G},C}$ in **Set**, and for the $\mathcal{G}^c$-extension $\varepsilon^{\mathcal{G}}_{\mathcal{F},C} : U^c_\tau(P_\mathcal{F}(C), \phi_{\mathcal{F},C}) \to (P_\mathcal{G}(C), \phi_{\mathcal{G},C})$ of the $\mathcal{F}^c$-counit $\varepsilon_{\mathcal{F},C}$:

$P_{\mathcal{F}}(C) \to C$ we have the equation $\varepsilon_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} = \varepsilon_{\mathcal{F},C}$ in **Set**. But due to Definition 8, we have the same equation in **Set** for the translation $U^c_\kappa(\varepsilon^{\mathcal{G}}_{\mathcal{F},C}) = \varepsilon^{\mathcal{G}}_{\mathcal{F},C} : U^c_\kappa(U^c_\tau(P_{\mathcal{F}}(C), \phi_{\mathcal{F},C})) \to U^c_\kappa(P_{\mathcal{G}}(C), \phi_{\mathcal{G},C})$. Thus we get finally for the composition $\varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}$ in $\mathbf{Alg}^c(\mathcal{H})$ the equation $\varepsilon_{\mathcal{H},C} \circ \varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} = \varepsilon_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} = \varepsilon_{\mathcal{F},C}$ in **Set**, thus we have $\varepsilon^{\mathcal{H}}_{\mathcal{F},C} = \varepsilon^{\mathcal{H}}_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}$ due to the uniqueness of $\mathcal{H}^c$-extensions. □ □

Remark 9 (Translation). *At the moment we are not able to give a satisfactory interpretation of the translation of coequations in terms of translations of processes or in terms of transformations of subcoalgebras. But it seems likely that the requirement on functors of being "weak pullback preserving" [7, 11, 13] will be relevant also in this context.*

Remark 10 (Many-sorted Translation). *In Universal Algebra a many-sorted signature morphism $\phi : \Sigma \to \Sigma'$ as in Remark 7 induces a translation of Σ-equations into Σ'-equations. And this translation is based on a translation of S-sets of variables into S'-sets of variables. That is, to any S-sets $X = (X_{s_1}, \ldots, X_{s_n}) \in |\mathbf{Set}^n|$ we assign an S'-set $Y = (Y_{s'_1}, \ldots, Y_{s'_m}) \in |\mathbf{Set}^m|$, where the components are constructed as sums (disjoint unions): $Y_{s'_i} \stackrel{def}{=} \uplus\{X_{s_j} \mid \phi(s_j) = s'_i\}$, $1 \leq i \leq m$. Note, that $Y_{s'_i}$ will be the empty set, i.e., the initial object in **Set** if $s'_i \in S' \setminus \phi(S)$.*

Categorically spoken, these translations define a functor $\mathcal{V}^\#_\phi : \mathbf{Set}^n \to \mathbf{Set}^m$ left-adjoint to the functor $\mathcal{V}_\phi : \mathbf{Set}^m \to \mathbf{Set}^n$, i.e., $\mathcal{V}^\#_\phi$ is essentially build up by the coproduct functor $+ : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ (the functor left-adjoint to the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$) and by the "initial object functor" $\emptyset : \mathbf{1} \to \mathbf{Set}$ (the functor left-adjoint to the "delete" functor $\delta : \mathbf{Set} \to \mathbf{1}$). Note, that $\mathcal{V}^\#_\phi(\mathbf{1}^n_{s_i}) = \mathbf{1}^m_{\phi(s_i)}$ thus the translation of the many-sorted equations introduced in Remark 6 is indeed insured.

Dually, we can define a translation $Ceq(\mathcal{V}_\psi, \tau) : Ceq(\mathcal{G}) \to Ceq(\mathcal{F})$ of many-sorted coequations, as defined in Remark 6, for any sort-based abstract many-sorted signature morphism $(\mathcal{V}_\psi, \tau) : \mathcal{F} \to \mathcal{G}$ as in Remark 7: Since $\mathcal{V}_\psi : \mathbf{Set}^n \to \mathbf{Set}^m$ is build up by the diagonal functor and the "delete" functor, there exists a functor $\mathcal{V}^\bullet_\psi : \mathbf{Set}^m \to \mathbf{Set}^n$ right-adjoint to $\mathcal{V}_\psi$. $\mathcal{V}^\bullet_\psi$ will be essentially build up by the product functor $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ (the functor right-adjoint to the diagonal functor) and by the "terminal object functor" $\mathbf{1} : \mathbf{1} \to \mathbf{Set}$ (the functor right-adjoint to the "delete" functor), thus there exists for all $1 \leq i \leq m$ an isomorphism $\iota_i : \mathcal{V}^\bullet_\psi(\mathbf{2}^m_{s'_i}) \to \mathbf{2}^n_{\psi(s'_i)}$.

$\mathcal{V}^\bullet_\psi$ right-adjoint to $\mathcal{V}_\psi$ means that we have for any $A \in |\mathbf{Set}^n|$ and any $C \in |\mathbf{Set}^m|$ a bijection $b_{A,C} : \mathbf{Set}^m(\mathcal{V}_\psi(A), C) \to \mathbf{Set}^n(A, \mathcal{V}^\bullet_\psi(C))$ (compare the concept of "corresponding variable assignment" in [19]). This ensures that we can assign to any many-sorted $\mathcal{G}$-coequation $\overline{l'}, \overline{r'} : P_{\mathcal{G}}(C) \to \mathbf{2}^n_{s'_i}$ of sort $s'_i \in S'$ on an S'-set $C \in |\mathbf{Set}^m|$ of colors the $\mathcal{F}$-coequation $\overline{l}, \overline{r} : P_{\mathcal{F}}(\mathcal{V}^\bullet_\psi(C)) \to \mathbf{2}^m_{\psi(s'_i)}$ of sort $\psi(s'_i) \in S$ on the S-set $\mathcal{V}^\bullet_\psi(C) \in |\mathbf{Set}^n|$ of colors where $\overline{l} \stackrel{def}{=} \iota_i \circ b_{P_{\mathcal{F}}(\mathcal{V}^\bullet_\psi(C)),C}(\overline{l'} \circ (b^{-1}_{P_{\mathcal{F}}(\mathcal{V}^\bullet_\psi(C)),C}(\varepsilon_{\mathcal{F},\mathcal{V}^\bullet_\psi(C)}))^{\mathcal{G}})$ and $\overline{r} \stackrel{def}{=} \iota_i \circ b_{P_{\mathcal{F}}(\mathcal{V}^\bullet_\psi(C)),C}(\overline{r'} \circ (b^{-1}_{P_{\mathcal{F}}(\mathcal{V}^\bullet_\psi(C)),C}(\varepsilon_{\mathcal{F},\mathcal{V}^\bullet_\psi(C)}))^{\mathcal{G}})$.

4.4 Satisfaction Condition

The satisfaction condition is the last missing piece for our main result. That is, we have to show that "truth is invariant under change of notation" [6]:

Proposition 3 (Satisfaction Condition). *For any natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G} : \mathbf{Set} \to \mathbf{Set}$ between co-syntactical functors $\mathcal{F}$ and $\mathcal{G}$, any $\mathcal{F}$-coalgebra (A, α), and any $\mathcal{G}$-coequation $ce = (l, r : P_{\mathcal{G}}(C) \to \mathbf{2})$ on a set C the following satisfaction condition holds:*

$$(A, \alpha) \models_{\mathcal{F}} Ceq(\tau)(ce) \quad \text{iff} \quad U^c_\tau(A, \alpha) \models_{\mathcal{G}} ce$$

Proof. Due to the Definitions 5, 8, and 9 we have to show that for all colorings $c : A \to C$: $(A, \alpha), c \models_{\mathcal{F}} (l \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C}, r \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} : P_{\mathcal{F}}(C) \to \mathbf{2})$ iff $(A, \tau_A \circ \alpha), c \models_{\mathcal{G}} (l, r : P_{\mathcal{G}}(C) \to \mathbf{2})$. Due to Definition 5 this statement is equivalent to the statement: $l \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} \circ c^{\mathcal{F}} = r \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} \circ c^{\mathcal{F}}$ iff $l \circ c^{\mathcal{G}} = r \circ c^{\mathcal{G}}$ for all colorings $c : A \to C$. And this equivalence holds, finally, if we can prove $\varepsilon^{\mathcal{G}}_{\mathcal{F},C} \circ c^{\mathcal{F}} = c^{\mathcal{G}}$ for all colorings $c : A \to C$. But due to the definition of $\mathcal{F}^c$- and $\mathcal{G}^c$-extensions we have $\varepsilon_{\mathcal{F},C} \circ c^{\mathcal{F}} = c$ and $\varepsilon_{\mathcal{G},C} \circ \varepsilon^{\mathcal{G}}_{\mathcal{F},C} = \varepsilon_{\mathcal{F},C}$. This entails $\varepsilon_{\mathcal{G},C} \circ (\varepsilon^{\mathcal{G}}_{\mathcal{F},C} \circ c^{\mathcal{F}}) = c$ thus the uniqueness of $\mathcal{G}^c$-extensions insures $\varepsilon^{\mathcal{G}}_{\mathcal{F},C} \circ c^{\mathcal{F}} = c^{\mathcal{G}}$, as required. □ □

Remark 11 (Many-sorted Satisfaction Condition)**.** *The satisfaction condition holds also for the many-sorted case. We drop the tedious and long proof, and let it as an exercise for the interested reader. Additionally to the uniqueness of $\mathcal{F}^c$- and $\mathcal{G}^c$-extensions the proof is based on the fact the the ι_i are isomorphisms and that the $b_{A,C}$ are bijections (natural in A and C).*

Summarizing Definition 7 and the Propositions 1, 2, 3 we obtain the main result of our paper

Theorem 1 (Coinstitutions)**.** *Any category* **Sign** *of abstract co-syntactical signatures together with the corresponding model functor* $Alg^c : \mathbf{Sign} \to \mathbf{Cat}$*, the corresponding syntax functor* $Ceq : \mathbf{Sign}^{op} \to \mathbf{Set}$ *and the corresponding family* $\models_{\mathcal{F}} \subseteq |Alg^c(\mathcal{F})| \times Ceq(\mathcal{F})$*,* $\mathcal{F} \in |\mathbf{Sign}|$ *of satisfaction relations constitutes a coinstitution* $\mathcal{I}^c = (\mathbf{Sign}, Alg^c, Ceq, \models)$ *for coalgebras.*

Example 6 (Coinstitutions for CSP)**.** *Applied to our example Theorem 1 tells us that there is a coinstitution of "colored deterministic CSP processes". Since the non-deterministic processes in CSP ared also based on extended polynomial functors [17], i.e., since there is no "real" non-determinism in CSP, it seems to be possible to define also coinstitutions of "colored non-deterministic CSP processes". But a more detailed analysis will be necessary to varify this conjecture.*

Remark 12 (Un-colored Coequations)**.** *Theorem 1 is formulated for co-syntactical functors* $\mathcal{F} : \mathbf{Set} \to \mathbf{Set}$*, i.e., concerns "colored coequations". In analogy to "ground equations" in Universal Algebra we could, of course, restrict ourselves to "un-colored coequations", i.e., we could only require that there is a terminal $\mathcal{F}$-coalgebra. Since the definitions and results in the paper include this simple case, we could formulate corresponding variants of Definition 7 and of Theorem 1.*

5 Conclusions and further work

In the paper we have shown that the concept of coequation introduced in [16] allows to define a wide range of coinstitutions for coalgebras. This result gives, besides [1], a further indication that the approach to dualization developed in [16] can be seen as a well-structured and reasonable one.

As an application we have shown that the model-theoretic extension of CSP presented in [17] gives rise to coinstitutions.

Further we have outlined, in some detail, a generalization of the definitions and results to the many-sorted case. But a full exposition of the many-sorted case deserves a next paper for its own.

In view of the three step methodology, described in the introduction, there are, at the present stage, open questions mainly concerning the "interpretation step":

- What are many-sorted coalgebras (good for)?
- How can the translation of coequations be interpreted in terms of translation of processes or in terms of transformations of subcoalgebras?

- What are "colors" good for in applications like CSP, for example?

Structuring formalisms in Algebraic Specifications are based, on the model-theoretic level, on free functors and on amalgamation [4, 5, 3]. First results on cofree functors for coalgebras are already available [18]. A natural next question would be to investigate possibilities to dualize amalgamation. Especially, the relation between co-amalgamation and synchronization will be worth to investigate.

References

[1] F. Bartels, A. Sokolova, and E. de Vink. A hierarchy of probabilistic sytem types. *TCS*, 327:3–22, 2004. 5

[2] K. J. Barwise. Axioms for Abstract Model Theory. *Annals of Mathematical Logic*, 7:221–265, 1974. 4

[3] H. Ehrig, M. Große–Rhode, and U. Wolter. Applications of category theory to the area of algebraic specification in computer science. *Applied Categorical Structures*, 6(1):1–35, 1998. 5

[4] H. Ehrig and B. Mahr. *Fundamentals of Algebraic Specification 1: Equations and Initial Semantics*, volume 6 of *EATCS Monographs on Theoretical Computer Science*. Springer, Berlin, 1985. 5

[5] H. Ehrig and B. Mahr. *Fundamentals of Algebraic Specification 2: Module Specifications and Constraints*, volume 21 of *EATCS Monographs on Theoretical Computer Science*. Springer, Berlin, 1990. 5

[6] J. A. Goguen and R. M. Burstall. Institutions: Abstract Model Theory for Specification and Programming. *Journals of the ACM*, 39(1):95–146, January 1992. (document), 1, 4, 4.4

[7] H. P. Gumm. Equational and implicational classes of coalgebras. *TCS*, 260:57–69, 2001. 1, 3, 3, 5, 9

[8] C.A.R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, 1985. 1

[9] C.A.R. Hoare. Unification of Theories: A Challenge for Computing Science. In M. Haveraaen, O. Owe, and O.-J. Dahl, editors, *Recent Trends in Data Type Specification*, pages 49–57. 11th Workshop on Specification of Abstract Data Types, WADT11, Oslo Norway, September 1995, Springer, LNCS 1130, 1996. 1

[10] B. Jacobs and J. Rutten. A Tutorial on (Co)Algebras and (Co)Induction. *Bulletin of EATCS*, 62:222–259, June 1997. 1, 2, 3, 2

[11] A. Kurz. *Logics for Coalgebras and Applications to Computer Science*. PhD thesis, Ludwig-Maximilians-Universität München, Fakultät für Mathematik und Informatik, 2000. 1, 2, 3, 3, 3, 9

[12] H. Reichel. *Initial Computability, Algebraic Specifications, and Partial Algebras*. Oxford University Press, 1987. 1

[13] J.J.M.M. Rutten. Universal coalgebra: A theory of systems. *TCS*, 249:3–80, 2000. (document), 1, 2, 2, 3, 3, 2, 3, 3, 9

[14] M.B. Smyth and G.D. Plotkin. The category theoretic solution of recursive domain equations. *SIAM Journ. Comput.*, 11:761–783, 1982. 3

[15] U. Wolter. An Algebraic Approach to Deduction in Equational Partial Horn Theories. *J. Inf. Process. Cybern. EIK*, 27(2):85–128, 1990. 1

[16] U. Wolter. On Corelations, Cokernels, and Coequations. In H. Reichel, editor, *Third Workshop on Coalgebraic Methods in Computer Science (CMCS'2000), Berlin, Germany, Proceedings*, volume 33 of ENTCS, pages 347–366. Elsevier Science, 2000. (document), 1, 2, 3, 5

[17] U. Wolter. CSP, Partial Automata, and Coalgebras. *TCS*, 280:3–34, 2002. (document), 1, 1, 3, 3, 2, 6, 5

[18] U. Wolter. Cofree Coalgebras for Signature Morphisms. In H.-J. Kreowski, editor, *Formal Methods (Ehrig Festschrift)*, pages 275–290. Springer, LNCS 3393, 2005. 1, 5

[19] U. Wolter, M. Klar, R. Wessäly, and F. Cornelius. Four Institutions – A Unified Presentation of Logical Systems for Specification. Technical Report Bericht-Nr. 94-24, TU Berlin, Fachbereich Informatik, 1994. 10