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Fibred Amalgamation, Descent Data, and Van Kampen Squares in Topoi

Uwe Wolter, Department of Informatics,
University of Bergen, Norway
H. König, University of Applied Science FHDW
Hannover, Germany

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Department of Informatics

UNIVERSITY OF BERGEN

Bergen, Norway

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Contents

Abstract

Reliable semantics for software systems has to follow the semantics-as-instance principal (fibred semantics) rather than the semantics-as-interpretation principal (indexed semantics). While amalgamation of interpretations is simple and nearly always possible, amalgamation of instances is very much involved and not possible in many cases.

The paper presents for presheaves, i.e. for functor categories $SET^{\mathscr{S}}$, a condition when two compatible instances, i.e. a span of pullbacks, are amalgamable. Based on this individual condition we prove further a total condition for amalgamation in presheaves, i.e. a necessary and sufficient condition for pushouts to be Van Kampen squares.

As a necessary and adequate basis to achieve these results we provide a full revision and adaption of the theory of descent data in topoi for applications in diagrammatic specifications including graph transformations. Especially, we characterize Van Kampen squares in arbitrary topoi by pullbacks of categories of descent data.

1 Introduction

Explaining formal arrangements like type diagrams, entity-relationship (ER) diagrams, class diagrams, state charts, process models or metamodels is challenging. The artefacts of a computer scientist are syntactic constructs. While, in most cases, we have accurate means available to specify correct syntax, like contextfree grammars or metamodels, for example, the description of the semantics, i.e. of the intended meaning, of our artefacts still remains intuitive and approximate. Basically, one can distinguish between static semantics, which explains the artefacts' meaning by means of static structures like sets and functions, for example, and operational semantics, which provides meaning in terms of behaviour. This paper, is mainly motivated by problems related to static semantics of *diagrammatic specifications* [7, 30, 33]. The presented concepts and results, however, are probably also useful for operational semantics.

A software engineer questions the meaning of an *element* x of a syntactic structure S. The syntactic shape of x may vary. In diagrammatic specifications, it may be a rectangle, a point, an arrow, a line, or some other graphical template. A (static) *interpretation* i of S may assign to elements of S a set or a map between sets. If the semantics of x is a set, each element of $i(x)$ appears, in this context, as *indexed* by x. If S is a set, the resulting family of sets is often denoted by $(i(x))_{x \in S}$ and called an *indexed set*. If S is a category (compare Section 4.2), an interpretation of S becomes a functor from S or S^{op}, respectively, into SET. More generally, interpretations may assign to syntactic elements categories and functors instead of sets and functions respectively. This gives rise then to interpretations as functors $i : S \to CAT$ or $i : S^{op} \to CAT$ also called *indexed categories*. The common codomain of a certian kind of interpretations, as the categories SET or CAT, for example, is called a *semantic universe* in [33] because it contains all necessary correctness knowledge. Bearing in mind the special cases of indexed sets and indexed categories, respectively, we will refer to any incarnation of the *semantics-as-interpretation* pattern as *indexed semantics*.

Software engineering is a discipline which has to understand and to deal also with the relation between different specifications S and S' and their potential co-evolution: S might be a graphical class model which represents classes of an object-oriented program, S' might be an entity relationship diagram describing how these entities are stored in a database.

Relations between specifications are usually formalized by specification morphisms turning the collection of all specifications into a category $\mathscr{C}$. The collection of all interpretations of a fixed object S in $\mathscr{C}$ together with natural transformations between them constitute then a category $Mod(S)$. In this indexed setting, any specification morphism $m : S \to S'$ gives rise, by simple pre-composition with m, to a (forgetful) functor $Mod(m) : Mod(S') \to Mod(S)$. Thus we obtain, on the global level, a functor $Mod : \mathscr{C}^{op} \to CAT$.

A paradigmatic example of indexed semantics are algebraic specifications [9, 10], where S is a *signature* (containing sorts and operation symbols) or a *specification*, in which axioms like equations or conditional equations [3, 28, 32] are added to a signature. Algebras interpret sort symbols by sets and operation symbols by functions, respectively. Homomorphisms between algebras are families of functions, indexed by sort symbols, satisfying a homomorphism/naturality condition for each operation symbol. If we denote the category of all algebras and homomorphisms for a given S by $Alg(S)$, any specification morphism $m : S \to S'$ induces a forgetful functor from $Alg(S')$ to $Alg(S)$, where specification morphisms are given by compatible and axiom preserving pairs of functions between sets of sort symbols and sets of operation symbols, respectively.

Equations and conditional equations can also be formalized diagrammatically by product and limit *sketches*, respectively [1]. An extension of this framework are *generalized* sketches [6, 25, 7]. They are an appropriate underpinning for *diagrammatic specifications* in model driven engineering [7, 29, 30]: Diagrammatic specifications contain (data) types, directed associations between them, and probably distinguished and labeled subdiagrams to express constraints. Sketch morphisms relate different specifications. The original definitions of sketch semantics [1, 5, 4, 25] are "indexed". In most applications interpretations of types are sets and arrows are interpreted as functions between sets.

Compositionality is an important and well-known concept in theoretical computer science [12]. It is a method to uniquely and correctly compose (overlapping) semantics of components of an already composed specification. The composition of specifications is usually carried out with the help of *colimits*. I.e. the category $\mathscr{C}$ of specifications and specification morphisms is assumed to be cocomplete. E.g. in the left diagram of Figure 1 components A and R are related via the common part L whose role as substructure of A and R is formalized with specification morphisms a and r, resp. Syntactic composition is carried out by constructing the pushout of a and r.

Assume there are interpretations τ, γ, and β of the components A, L, and R resp., which are related to each other according to the action of the functor Mod, i.e. $Mod(a)(\tau) = \gamma = Mod(r)(\beta)$. The *compositionality problem* is formulated in [12], p. 10, as follows: *Can these interpretations for the component specifications uniquely be composed in the same way as the specifications, such that the composed interpretation is correct w.r.t. the composed specification?* Of course, such an implication is something we know from all over the mathematical world: It is very desirable to infer global correctness from local correctness, because proofs can then be carried out locally. A prominent examples are *sheaves* over topological spaces [27], where e.g. local properties like (analytic) continuity ensure the corresponding global property.

Compositionality can shortly be circumscribed by the equation

$$Mod(Colim(\mathcal{D})) = Lim(Mod \circ \mathcal{D}^{op})$$

where $\mathcal{D} : \Sigma \to \mathscr{C}$ is an arbitrary diagram (in Figure 1 Σ is the pattern $\cdot \longleftarrow \cdot \longrightarrow \cdot$) and $Colim$ and Lim denote colimit and limit of diagrams. I.e. compositionality is continuity of Mod.

As an example consider the Amalgamation Lemma of [9], cf. Figure 1. It states that (2) is a pullback in the category CAT of categories if (1) is a pushout of specifications. Here V_m denotes the above mentioned forgetful

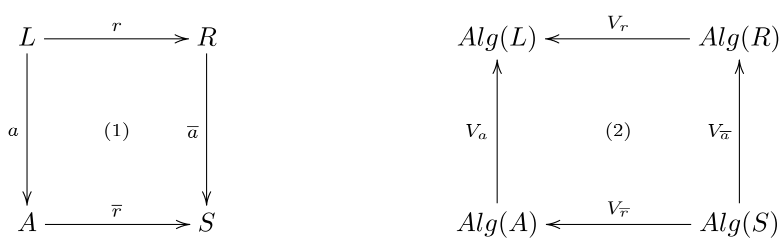


Figure 1: Amalgamation Lemma (Indexed version)

functor along a specification morphism m.

Note that the "philosophy" of semantic universes and of semantics-as-interpretation implies two important facts: On the one hand, elements of a set (objects) can be multiply interpreted (typed) (if e.g. $i(t_1) \cap i(t_2) \neq \emptyset$ for two different elements t_1, t_2 of specification S). On the other hand, i is a fixed assignment which maps an element t to all objects that are t-typed.

Reliable semantics for model-driven structures, however, has to drop the "philosophy" of semantic universes, because in software environments each object possesses exactly one type and it should not be possible to determine the set of t-typed objects. The second requirement is important since it enables external extensions of software systems using inheritance and specialization techniques.

This mismatch between indexed semantics and software engineering requirements calls for a shift of paradigm. First, we have to give up the strict separation of syntax and semantics. Instead of the syntax, on one side, and a semantic universe, on the other side, we have to allow that syntactic and semantic entities live all together in the same category $\mathscr{C}$, and that, in addition, certain entities may play, at the same time, a syntactic and a semantic role. Second, we have to switch to fibred semantics [7, 33]. Fibred semantics means *semantics-as-instance*, i.e., the semantics of specifications are *instances* and are formalized by objects of the slice categories $\mathscr{C}\downarrow A$, $\mathscr{C}\downarrow L$, $\mathscr{C}\downarrow R$, and $\mathscr{C}\downarrow S$. Mod gets traded for a functor $Inst$ assigning to a specification S the category $\mathscr{C}\downarrow S$ or a subcategory of it. Moreover $Inst$ assigns to each specification morphism $m : S \to S'$ the "pulling back"-functor along m (e.g. the functor r^* which constructs the pullback of r and $\beta \in \mathscr{C}\downarrow R$, this is the back face in Figure 2). Since the "meaning" of an element x of a specification S in terms of an instance $\iota \in \mathscr{C}\downarrow S$ is now given by $\iota^{-1}(x)$, i.e. the fibre of ι over x, this approach is called *fibred semantics*.

A crucial question arises whether compositionality smoothly carries over to the fibred setting, i.e.

$$Inst(Colim(\mathcal{D})) \stackrel{?}{=} Lim(Inst \circ \mathcal{D}^{op}). \qquad (1)$$

Unfortunately and surprisingly it turns out, that this is not the case. We encounter two serious problems. First, $Inst : \mathscr{C}^{op} \to CAT$ is only a pseudofunctor, i.e. composition is only unique up to isomorphism.

The second problem can be uncovered already in the special case of amalgamation: Given two instances $\tau \in \mathscr{C}\downarrow A$ and $\beta \in \mathscr{C}\downarrow R$ with common part γ, i.e. $r^*\beta = \gamma = a^*\tau$, one wants to prove that the syntactic composition (pushout of a and r) is reflected on the instance level by a unique construction. The counterpart for correctness is the requirement to obtain an S-instance of $\mathscr{C}\downarrow S$, such that its pullbacks along $\overline{a}$ and $\overline{r}$ yield β and τ, resp., cf. Figure 2. This question, however, is closely related to the following concept:

Definition 1 (Van Kampen Square) *A pushout as in the bottom of Figure 2 is called a* Van Kampen square *if for all commutative cubes as in Figure 2 (without question marks) with two pullbacks as rear faces the following equivalence holds: The top face is a pushout if and only if the front and right faces are pullbacks.*

In indexed semantics we have amalgamation for arbitrary pushouts. In contrast, we know that already in the category SET there are pushouts that are not Van Kampen squares (see [8] and the forthcoming examples). In these cases there are rear pullback spans for which amalgamation fails as well as spans that can successfully be amalgamated. Thus fibred amalgamation may or may not fail, if a and r together with their pushout do not enjoy Van Kampen exactness.

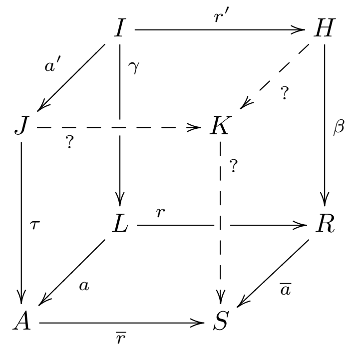


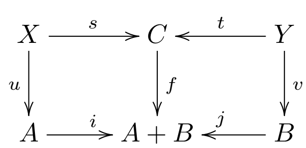
Figure 2: Fibred Amalgamation of τ and β with common part γ

One way out of this dilemma is to restrict ourself to pushouts with a special property ensuring Van Kampen exactness in most of the relevant application areas. A variant of this restrictive policy is the concept of *adhesive categories* [31] where it is required that pushouts along monomorphisms are Van Kampen squares. This means that if either a and r are monic, we can construct the pushout of a' and r' and know that front and right face are pullbacks as desired. Topoi are adhesive [21] thus this restriction works fine for a wide spectrum of relevant categories, as e.g. SET and $GRAPH$. It is well-known, however, that already in SET there are many more Van Kampen squares than the ones where one participating morphism is monic. A precise characterization of *all* Van Kampen squares in our categories of interest remained an open problem which is as well theoretical as practical relevant.

Another way out of the dilemma could be to restrict the quantification in Definition 1 to the "good" pullback spans. The crucial observation is that for those pullback spans that arise by applying a corresponding *Grothendieck construction* to coherent pairs of interpretations the equivalence in Definition 1 is satisfied [33]. A feasible characterization of those "good" pullback spans is practical relevant and would give us, at the same time, a better understanding of the essential differences between indexed and fibred semantics.

A category possesses all finite colimits if there are pushouts and an initial object. Elementary conditions for amalgamation of spans (by pushouts), if there are any, can easily be carried over to all coequalizers, because a coequalizer of $f, g : A \to B$ is given by one arrow in the cocone of the pushout of $[f, g] : A + A \to B$ and $[id, id] : A + A \to A$.

If the underlying category is extensive, i.e. if in the diagram



C is the coproduct of X and Y if and only if both squares are pullbacks (which is the case in our most general setting, as we will see soon), these conditions can directly be used to describe compositional colimits, because any colimit is constructed with a coequalizer and a coproduct. Hence understanding (the limits and conditions for) fibred amalgamability means to understand the circumstances for fibred compositionality in general.

In this paper we want to draw and to describe an exact line which marks this border for amalgamation. We distinguish between a *total* and an *individual* view. The total view addresses the problem when a diagram of specifications yields amalgamation for all instance constellations, i.e. under which conditions (1) holds for pushouts (or equivalently the resulting diagram is a Van Kampen square). The goal is to find a feasible condition which is well-applicable for corresponding situations in software engineering (see e.g. Example 4 below).

The individual view is a precursor of the total view: It shall produce a similarly useful condition for successful amalgamation of a *fixed* rear pullback span (especially, if the bottom square is not Van Kampen). This condition should come in terms of the structures of the two rear pullbacks only. A request for such a condition was first raised in [33].

To sum this up, the goal of this paper is to find answers for the following two questions:

Question 1 *Can we find necessary and sufficient conditions for a pushout to be a Van Kampen square in terms of the span (a, r) only?*

Question 2 *Can we find feasable conditions which characterize successful amalgamation of a fixed pullback span in terms of the structure of the span (even in the case that the bottom square is not a Van Kampen square)?*

We are interested to perform a comprehensive investigation of these questions in a more general categorical setting that subsumes our main motivating application areas as graph transformations and generalized sketches. The category $GRAPH$ of multi graphs, as many other categories of graph structures, can be described as *presheaves*, i.e., as functor categories $SET^{\mathscr{S}}$ with $\mathscr{S}$ a finite "meta schema" category (see subsection 4.2). Also the category of generalized multi sketches becomes a presheaf as long as the underlying base category is a presheaf [25]. In such a way, presheaves appear as an adequate abstraction level to answer the above questions.

Presheaves are *topoi* [14], which are thus a good common biotop to develop and to present the theoretical foundation for our investigations. A topos is a category with finite limits, which is cartesian closed, and where the subobject functor is representable. We mention some further properties of a topos $\mathscr{C}$ which will be used frequently throughout the paper:

1. $\mathscr{C}$ has all finite colimits, [14], 4.3.
2. $\mathscr{C}$ is extensive [14].
3. If an arrow is monic and epic, it is an isomorphism [14], 5.1.
4. Pullbacks preserve epimorphisms [14], 5.3.
5. Pushouts preserve monomorphisms and, moreover, pushouts with a monomorphism involved are also pullbacks [14], 13.3.
6. The pullback functor preserves colimits [14], 15.3.
7. Epimorphisms are regular epimorphisms [26], Lemma 16.18.

Topoi are adhesive [21], i.e. pushouts along monomorphisms are Van Kampen squares. Thus the above questions are relevant especially for the case where both a and r in Figure 2 have non-trivial kernel relations.

The paper is organized as follows. In section 2 we give examples which show the subtleties which may destroy successful compositionality and how difficult it may be to decide whether a square is Van Kampen or whether a given pullback span can be amalgamated. In order to show that the problems may frequently occur in practice, we include an example from software engineering. Section 3 discusses related work.

Descent Theory [15] is a good tool for quantifying the interrelation of kernels on a common domain and it turns out that its theoretical results unfold their power in categories which are "essentially the same as sets" [22]. As one of the main contributions of the paper subsection 4.1 presents a full revision and adaption of the theory of descent data in topoi for applications in diagrammatic specifications and, especially, in graph transformations. We point out the two main facets of descent data: On the one hand, it describes algebraic structures, on the other hand, it codes lifted equivalence relations in pullback squares. Because a careful investigation of the relation of these algebraic structures and *all possible* pullbacks (even differentiating isomorphic pairs) is necessary, a well-known result on *effective decent morphisms* [27] is slightly sharpened (Proposition 15).

In subsection 4.2 we investigate, in more detail, descent data in presheaves. We show that there is a bijective correspondence between abstract objects of the category of descent data and families of bijections on the fibres over related elements (Proposition 16). Moreover, we emphasize the role of descent data as information that codes how to construct pullback complements in the spirit of [18].

In Section 5 we introduce the precise notions of *amalgamability* of pullback spans and of *coherence* of pairs of algebraic structures. Algebraic structures are coherent if they are reducts of a uniquely determined larger algebraic structure. Pullback spans are amalgamable, if they are simultaneous projections of an essentially unique "larger" pullback. We investigate the strong relationship between these two concepts.

We introduce several useful functorial relations on the level of pullbacks (where amalgamation is carried out) as well as on the level of descent data (where the question of coherence arises) and interrelate them according to the results of Section 4. The central technical result is Lemma 22 which is the precondition for the validity of Propositions 25 and 27, one of the main contributions of this paper. In these propositions, we basically prove that amalgamability is equivalent to coherence. Although still in an abstract context, this already provides and prepares a more feasable answer to Question 2.

There is also an easy consequence of this result (Theorem 28) which provides a fibred amalgamation lemma in terms of descent data. Theorem 28 gives us a nice (but still unpractical) equivalent characterization of Van

Kampen squares in terms of the limit (pullback) of certain categories (cf. (1)). However, the theorem provides the prerequisite to gain a practical answer to Question 1 for presheaves.

Section 6 investigates amalgamation and coherence in presheaves. A general analysis in the context of the results of subsection 4.2 is performed. It turns out that the aforemetioned border between failure of and successful amalgamation can be characterized with the help of so-called *domain cycles*, i.e. conglomerates of elements in carrier sets that let the kernels of morphisms interact too much. Based on this definition, Theorem 30 provides a full answer to Question 2 whereas Proposition 32 yields the missing link to the total view.

It can be observed here that the answer for Question 2 indeed was a precondition to find a full answer to Question 1 in Theorem 33 which is the desired equivalent characterization of Van Kampen squares in terms of the interacting kernels of a and r. In order to show the effects of the results, the examples from section 2 are revisited.

Finally, section 7 discusses open problems and outlines interesting directions for applications and future research. Moreover, minor auxiliary results and proofs can be found in an Appendix.

2 The Problem of Successful Amalgamation

In contrast to indexed amalgamation, there are intrinsic difficulties in the fibred setting, because the given rear pullback span can be located over a non-Van Kampen square. In other words, a reasonable construction on the instance level fails if and only if the pullback span is not *amalgamable*. This is demonstrated in

Example 2 *In Figure 3, objects are denoted $i:t$, instances map objects to their types. a and r map according to the letters. $i:t, j:s \in I$ are connected via dashed lines if $r'(i:t) = r'(j:s)$. Dotted lines depict the kernel of a'. It can easily be computed that the two rear squares establish a pullback span in SET.*

However, the span can not successfully be amalgamated: On the one hand, pullback complements for the right and the front face with sets over S containing two elements will always yield a non-commutative top face. On the other hand, the pushout on the top face creates a $\mathscr{C}\downarrow S$-object (the mediator out of the pushout), whose domain is a singleton set. But pulling back this instance along $\overline{r}$ and $\overline{a}$ does not yield τ and β, resp.

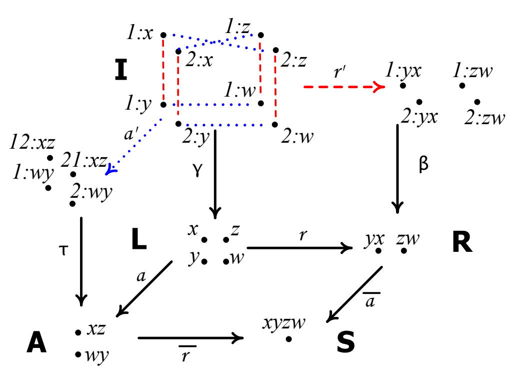


Figure 3: Amalgamation fails

These effects can not occur in the indexed setting because multiple typing was allowed. The transition from indexed to fibred semantics, however, entails the production of copies. E.g. in the indexed setting, it would be sufficient to let γ map each element of L to the set $\{1, 2\}$, whereas the transformation to the fibred setting produces 4 copies of this 2-element set (yielding the set I in Figure 3).

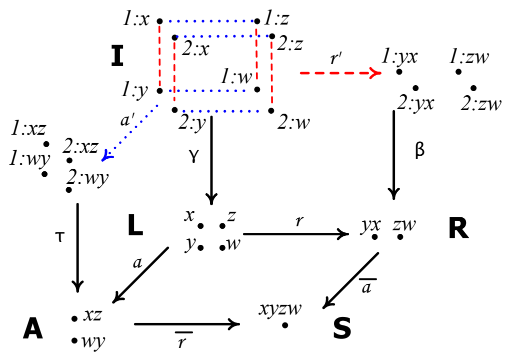


Figure 4: Amalgamation is successful

As pointed out before, we can consider specifications as categories and each interpretation becomes a functor to CAT (probably only reaching discrete categories, i.e. sets). It is well-known that these indexed categories are related to fibrations via the Grothendieck construction [1, 33]. However, since the image of this construction is the category of *split* fibrations, all produced copies behave in a uniform way.

Example 3 *In the pullback span in Figure 4 fibres are lifted in a uniform way. The pullback span can now be amalgamated. The instance over S is $\sigma : \{1{:}xyzw, 2{:}xyzw\} \to S$.*

But if pullback spans are not results of the Grothendieck construction, we suffer from the enlarged degree of freedom for defining the relationship between fibres, i.e. the equivalence relations of a' and r' may chaotically be intertwined as in Example 2. Moreover, instances must no longer be fibrations. Intuitively the set of possible instances for a specification is much larger than the set of interpretations. The problems immediately occur in practical environments as demonstrated by the next example:

Example 4 *In this example we consider a (parametric) specification with data types* Person *and* Business Partner*, both of which shall possess two different kinds of* contact information *(abbreviated* $\text{cInfo}_{1/2}$*) as in Figure 5. Specification morphism a replaces the formal parameters by strings, which is reasonable because both types of information may be textual. Moreover, the architects of the system will simplify matters by identifying* Person *and* Business Partner *(formalized by $r : L \to R$). The combined entity type will be called* "Individual"*. This is reasonable, especially because* Person *and* Business Partner *possessed the same attributes.*

The specification morphisms together with its pushout are shown in Figure 5. The pushout object is the result of passing an actual parameter into the restructuring procedure r. The question arises whether each compatible pair (τ, β) of instances of A and R, resp. can be amalgamated. We will give an answer in Section 6.

3 Related Work

To our best knowledge this is the first journal paper elucidating the close connection between Van Kampen squares and fibred amalgamation. This connection was presented and discussed the first time by the first author at WADT 2008 and some results, based on descent data, have been presented at ACCAT 2012. The present paper extends essentially the paper [20] published in the post-proceedings of ACCAT 2012 in two ways. First, the foundational part on descent data, amalgamation, coherence, and Van Kampen is completely revised and essentially extended. Second, the characterization of amalgamable pushout spans (and thus of Van Kampen squares) has been revised and generalized to arbitrary presheaves instead of SET only.

The result, that "topoi are adhesive" [21] provides the equivalence of amalgamability and coherence for the simple cases of pushouts with, at least, one monomorphism involved (see Lemma 26) and is integrated in the proof of Proposition 27. [21] uses also some results of descent theory and some similar auxiliary results from topos theory. Besides a full revision and adaption of descent theory the present paper exceeds [21] also by addressing amalgamability for arbitrary pushouts.

[17] shows that being a Van Kampen square in a category $\mathscr{C}$ is equivalent to saying that its embedding into a certain span category over $\mathscr{C}$ is a pushout. In contrast to this abstract reformulation of Van Kampen exactness by

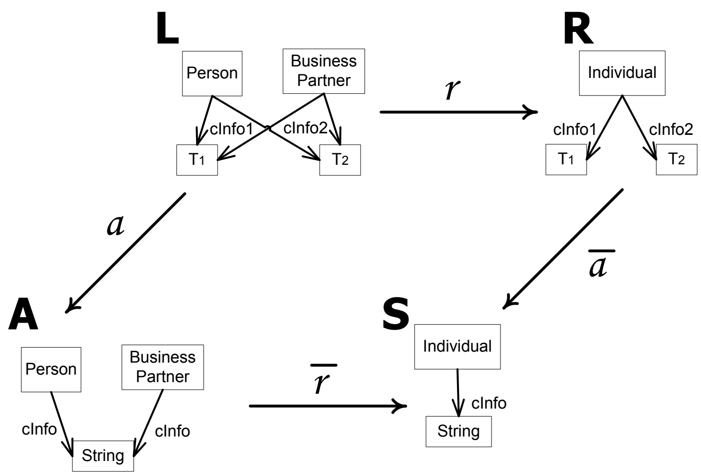


Figure 5: Software system reengineering with simultaneous parameter passing

means of higher level categorical structures, we are looking here for an elementary and feasable characterization of Van Kampen squares locally within $\mathscr{C}$.

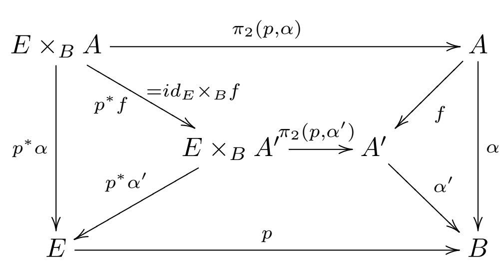
Finally, we have to mention that the crucial technical observation about domain cycles and Van Kampen squares goes back to Michael Löwe and has been elaborated in SET (without any references to descent data) in [23].

4 Descent Theory

In this section the underlying category $\mathscr{C}$ is a topos. We will use the following notations: $Ob_{\mathscr{C}}$, $Mor_{\mathscr{C}}$ denote objects and arrows of $\mathscr{C}$, resp. $x \in \mathscr{C}$ means $x \in Ob_{\mathscr{C}}$. The application of a functor $\mathcal{F}$ to an object or an arrow x will usually be denoted without parenthesis: $\mathcal{F}x$. The composition of functors $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ and $\mathcal{G} : \mathscr{D} \to \mathscr{S}$ is denoted by $\mathcal{G} \circ \mathcal{F}$. $\rightarrowtail$, $\twoheadrightarrow$, $\rightarrowtail\!\!\!\twoheadrightarrow$ denote monomorphism, epimorphism, isomorphism, resp in every category.

For an arrow $p : E \to B$ of $\mathscr{C}$ we sometimes want pullbacks along p to be uniquely determined. Thus we work with *chosen* pullbacks. It's well-known that any fixed choice of pullbacks gives rise to a pullback functor $p^* : \mathscr{C} \downarrow B \to \mathscr{C} \downarrow E$ where $(p^*\alpha, \pi_2(p, \alpha))$ denotes the chosen pullback of (α, p). The image p^*f of an arrow $f \in Mor_{\mathscr{C}\downarrow B}(\alpha, \alpha')$ is given by the unique arrow $p^*f := id_E \times_B f : E \times_B A \to E \times_B A'$ of $\mathscr{C}$ such that

$$p^*\alpha' \circ (id_E \times_B f) = p^*\alpha \quad \text{and} \quad \pi_2(p, \alpha') \circ (id_E \times_B f) = f \circ \pi_2(p, \alpha). \tag{2}$$



Whenever we use notations "$p^*\alpha$" or "$\pi_2(p, \alpha)$" (or shortly "π_2" if p and α are fixed), we rely on a fixed choice of pullbacks. Sometimes we use the analogical notation for the "first projection" of the pullback: $\pi_1(p, \alpha) := p^*\alpha$ (or shortly "π_1" if p and α are fixed).

It is an old observation that pullback functors establish adjoint situations:

Lemma 5 *Let $\mathscr{C}$ be a category with pullbacks and $E \xrightarrow{p} B$. Let $p^* : \mathscr{C}\downarrow B \to \mathscr{C}\downarrow E$ be the pullback functor for a fixed choice of pullbacks and $p_* : \mathscr{C}\downarrow E \to \mathscr{C}\downarrow B$ be the post composing functor, which sends an arrow $h \in Mor_{\mathscr{C}\downarrow E}(\gamma, \gamma')$ to $h : p\circ\gamma \to p\circ\gamma'$, an arrow of $Mor_{\mathscr{C}\downarrow B}$. Then p_* is left-adjoint to p^*, $p_* \dashv p^*$ in symbols, with unit $\eta^p : id_{\mathscr{C}\downarrow E} \Rightarrow p^*\circ p_*$ and co-unit $\varepsilon^p : p_*\circ p^* \Rightarrow id_{\mathscr{C}\downarrow B}$ where*

- *$\varepsilon^p_\alpha := \pi_2(p, \alpha)$ for each $\alpha \in \mathscr{C}\downarrow B$, and*
- *$\eta^p_\gamma := \langle\gamma, id_C\rangle$ for each $\gamma \in \mathscr{C}\downarrow E$, i.e., η^p_γ is the unique arrow such that*

$$\varepsilon^p_{p_*\gamma}\circ\eta^p_\gamma = id_C \quad \text{and} \quad p^*p_*\gamma\circ\eta^p_\gamma = \gamma. \tag{3}$$

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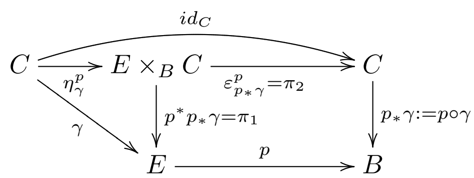


A proof can be found in Section 8.1.

The monad arising from the adjunction $p_* \dashv p^*$ is denoted by $(\mathcal{T}^p, \eta^p, \mu^p)$, i.e. $\mathcal{T}^p := p^*\circ p_* : \mathscr{C}\downarrow E \to \mathscr{C}\downarrow E$ with natural transformations $\eta^p : id_{\mathscr{C}\downarrow E} \Rightarrow \mathcal{T}^p$ and $\mu^p := p^*\varepsilon_{p_*} : (\mathcal{T}^p)^2 \Rightarrow \mathcal{T}^p$ (see [2] for a quite comprehensive investigation on this subject).

4.1 Descent Data in Topoi

In this first subsection, *monadic* descent theory in the spirit of [15, 16] is introduced. The second subsection deals with a more practical view on descent data in presheaves, i.e., in functor categories $\mathscr{C} = SET^{\mathscr{S}}$.

Let $p : E \to B$ be an arrow in the topos $\mathscr{C}$. Descent theory was originally invented by Grothendieck in order to reason about structures in $\mathscr{C}\downarrow B$ (which may be difficult) by reasoning about monadic algebraic structures over $\mathscr{C}\downarrow E$, thus in a sense "descending" along p.

This subsection revises, adapts and extends results in [18] to facilitate our intended characterization of amalgamation in terms of descent data. The main result will be Proposition 15 which extends a theorem on *effective descent* in [18] in that it avoids a certain degree of freedom when passing from $\mathscr{C}\downarrow B$ to algebraic structures: By exchanging $\mathscr{C}\downarrow B$ with the category of *all possible* pullbacks along p we obtain a more precise description of the relationship between these algebraic structures and pullbacks.

Definition 6 (Descent Data) *Let $C \xrightarrow{\gamma} E \xrightarrow{p} B$ be given and $(\mathcal{T}^p, \eta^p, \mu^p)$ be the monad on $\mathscr{C}\downarrow E$ arising from the adjunction $p_* \dashv p^*$.* Descent data *for γ relative to p is an arrow*

$$\xi : \mathcal{T}^p\gamma \to \gamma \quad \text{of } \mathscr{C}\downarrow E \text{ with } \quad \xi\circ\eta^p_\gamma = id_C \text{ and } \xi\circ\mathcal{T}^p\xi = \xi\circ\mu^p_\gamma. \tag{4}$$

The situation is as in Figure 6. Besides the $\mathscr{C}\downarrow B$-arrow $\pi_2 := \pi_2(p, p\circ\gamma)$, the right-hand side shows objects and the arrow ξ after applying the left-adjoint p_* only.

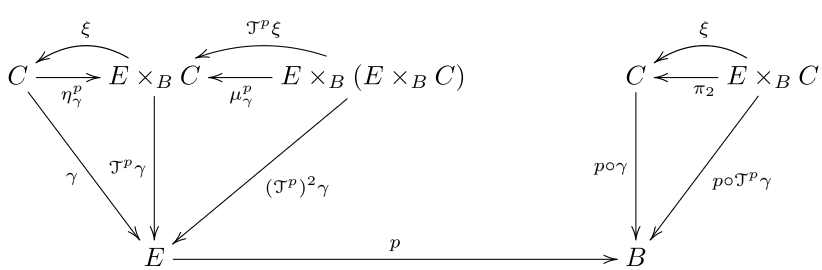


Figure 6: Monadic Descent Data

According to the definition of the pullback monad and Lemma 5 we have

$$\pi_2\circ\eta^p_\gamma = id_C \quad \text{and} \quad \mu^p_\gamma = p^*\pi_2. \tag{5}$$

Note that, for some γ and p, an arrow ξ as in Definition 6 may not exist or may be not unique, in case it exists. For future reference, we note that ξ can be reconstructed from the $E \times_B C$-endomorphism $\overline{\xi} := \langle \gamma \circ \pi_2, \xi \rangle$, i.e. the uniquely determined $\overline{\xi}$ for which

$$\mathcal{T}^p\gamma \circ \overline{\xi} = \gamma \circ \pi_2 \quad \text{and} \quad \pi_2 \circ \overline{\xi} = \xi. \tag{6}$$

[18] gives a detailed investigation on that topic. It is also shown that

$$\overline{\xi} \circ \overline{\xi} = id_{E \times_B C}. \tag{7}$$

Definition 7 (Category of Descent Data) *The category $des(p)$ of all descent data relative to an arrow $p : E \to B$ in $\mathscr{C}$ has objects (γ, ξ) with the properties of Definition 6 and arrows $h : (\gamma, \xi) \to (\gamma', \xi')$ the morphisms $h : \gamma \to \gamma'$ of $\mathscr{C} \downarrow E$ with $\xi' \circ \mathcal{T}^p h = h \circ \xi$.*

Let $|\_|_p : des(p) \to \mathscr{C} \downarrow E$ be the "carrier" functor, i.e. $|(\gamma, \xi)|_p = \gamma$ and $|h|_p = h$.

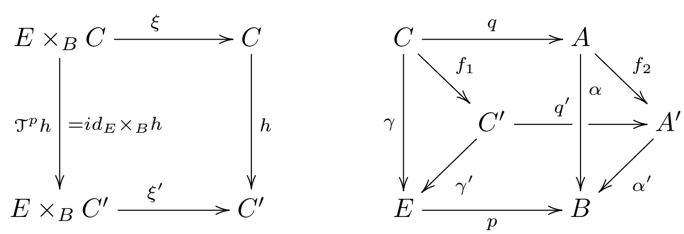


Figure 7: The categories $des(p)$ and $pb(p)$

Now we consider not only chosen but arbitrary pullbacks along p.

Definition 8 (Category of Pullbacks) *For any arrow $p : E \to B$ in $\mathscr{C}$ let $pb(p)$ denote the category with objects (γ, q, α) commutative diagrams of arbitrary pullbacks along p together with morphism pairs $(f_1, f_2) \in Mor_{\mathscr{C}\downarrow E} \times Mor_{\mathscr{C}\downarrow B}$ such that the rear square in the right diagram in Figure 7 commutes. Note that (by the decomposition property of pullbacks) the rear face is a pullback, too.*

Let $\Lambda_p : pb(p) \to \mathscr{C} \downarrow E$ be the (left) projection $\Lambda_p(\gamma, q, \alpha) = \gamma$, $\Lambda_p(f_1, f_2) = f_1$.

Our goal is now to establish the relationship between $pb(p)$ and $des(p)$ with the help of suitable functors in both directions. Since the monoidal conditions (4) (neutrality and associativity) imply that $des(p)$ is the *Eilenberg-Moore Category* associated with the monad $\mathcal{T}^p$, there is the comparison functor $\Phi^p : \mathscr{C} \downarrow B \to des(p)$ [2]. Because $pb(p)$ is equivalent to $\mathscr{C} \downarrow B$ via chosen pullbacks, the composition of this equivalence and the comparison functor seems to be a good choice for the direction from $pb(p)$ to $des(p)$. For our purposes, however, we omit the stopover $\mathscr{C} \downarrow B$ and construct directly a functor

$$\Phi^p : pb(p) \to des(p) \quad \text{with} \quad |\Phi^p|_p = \Lambda_p \tag{8}$$

(which, in fact, yields the above mentioned composed functor up to natural isomorphism, such that we still use the name Φ^p for this functor).

For this let us consider an arbitrary pullback (γ, q, α) along p in $\mathscr{C}$, i.e., an arbitrary pullback complement (q, α) of (γ, p), cf. the square (pb) in Fig. 8. We denote by $\xi^{(q,\alpha)}$ the unique arrow for which

$$q \circ \xi^{(q,\alpha)} = q \circ \pi_2 \quad \text{and} \quad \gamma \circ \xi^{(q,\alpha)} = \mathcal{T}^p\gamma, \tag{9}$$

(the dashed arrow in Fig. 8) which establishes, in such a way, also an arrow $\xi^{(q,\alpha)} : \mathcal{T}^p\gamma \to \gamma$. From (5), (9), and the uniqueness of mediating morphisms for the original pullback one easily deduces

$$\xi^{(q,\alpha)} \circ \eta^p_\gamma = id_C.$$

Moreover, mapping the commutative diagram (of $\mathscr{C} \downarrow B$) $q \circ \xi^{(q,\alpha)} = q \circ \pi_2$ by p^* yields

$$\xi^{(q,\alpha)} \circ \mathcal{T}^p\xi^{(q,\alpha)} = \xi^{(q,\alpha)} \circ \mu^p_\gamma$$

by the second equation of (5) (cf. Fig. 8). Hence $\xi^{(q,\alpha)}$ fulfills (4) such that the functor $\Phi^p : pb(p) \to des(p)$ can now be defined by

$$\Phi^p(\gamma, q, \alpha) := (\gamma, \xi^{(q,\alpha)})$$

on objects. Mapping the rear square in Fig. 7 (which also lives in $\mathscr{C} \downarrow B$) by p^* yields $\xi^{(q',\alpha')} \circ \mathfrak{T}^p f_1 = f_1 \circ \xi^{(q,\alpha)}$ for any $(f_1, f_2) \in Mor_{pb(p)}$, hence Φ^p extends to arrows:

$$\Phi^p(f_1, f_2) := f_1.$$

This implies especially

$$\Phi^p(\gamma, q', \alpha') = \Phi^p(\gamma, q, \alpha) \quad \text{for all} \quad (id, f) : (\gamma, q, \alpha) \to (\gamma, q', \alpha') \quad \text{in} \quad pb(p). \tag{10}$$

Moreover, we have $|\Phi^p|_p = \Lambda_p$.

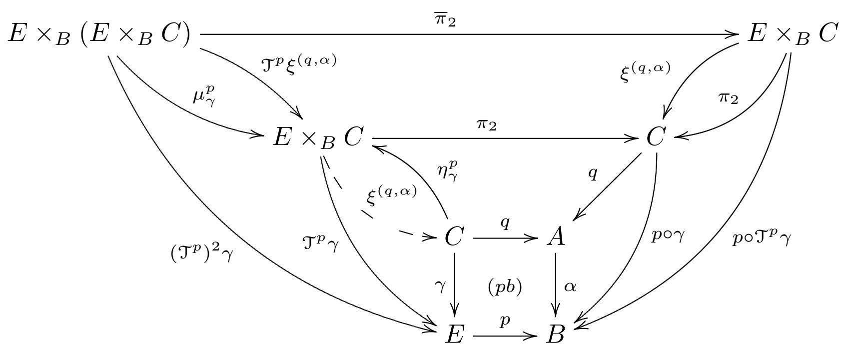


Figure 8: The assignment $(\gamma, q, \alpha) \mapsto (\gamma, \xi^{(q,\alpha)})$ of Φ^p

Now we are looking for a functor in the opposite direction

$$\Psi^p : des(p) \to pb(p) \quad \text{with} \quad \Lambda_p \circ \Psi^p = |\_|_p. \tag{11}$$

For each $(\gamma, \xi) \in Ob_{des(p)}$ we have $p \circ \gamma \circ \pi_2 = p \circ \mathfrak{T}^p \gamma = p \circ \gamma \circ \xi$ since $\xi : \mathfrak{T}^p\gamma \to \gamma$ is an arrow in $\mathscr{C} \downarrow E$ and due to the definition of $\mathfrak{T}^p$ (cf. the diagram in Lemma 5). Hence there is an assignment $\Psi^p : Ob_{des(p)} \to Ob_{comm(p)}$ which maps (γ, ξ) to the commutative square $(\gamma, c^\xi, \alpha^\xi)$ along p where α^ξ is the unique arrow, which mediates $p \circ \gamma$ and a (fixed choice of) coequalizer c^ξ of π_2 and ξ (cf. Fig. 9). Ψ^p extends to a functor because any

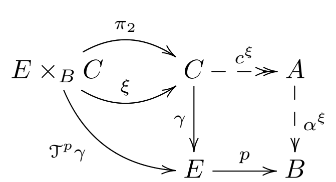


Figure 9: The assignment $(\gamma, \xi) \mapsto (\gamma, c^\xi, \alpha^\xi)$ of Ψ^p

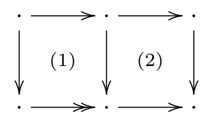
$h \in Mor_{des(p)}((\gamma, \xi), (\gamma', \xi'))$ not only results in a commutative square $h \circ \xi = \xi' \circ \mathfrak{T}^p h$ but also yields by (2)

$$h \circ \pi_2(p, p \circ \gamma) = \pi_2(p, p \circ \gamma') \circ \mathfrak{T}^p h.$$

Thus there is a unique arrow $\hat{h}$ which mediates $c^{\xi'} \circ h$ out of the coequalizer c^ξ . The uniqueness of mediating morphisms out of c^ξ entails $\alpha^{\xi'} \circ \hat{h} = \alpha^\xi$ thus $\hat{h} : \alpha^\xi \to \alpha^{\xi'}$ establishes an arrow in $\mathscr{C} \downarrow B$. We define $\Psi^p h := (h, \hat{h})$.

It remains to show that $(\gamma, c^\xi, \alpha^\xi)$ is not only a commutative but a pullback square along p. Although this can be deduced from considerations on discrete fibrations in topoi ([27], Remark VIII, 2.7.), we give here a short proof introducing, along the way, some concepts and results we will need later on. We need an auxiliary result (cf. [27], Chapter VIII):

Lemma 9 *Let $\mathscr{C}$ be a topos and a commutative diagram be given with an epimorphism as indicated. If $(1) + (2)$ and (1) are pullbacks, then (2) is a pullback, too.* □



By pullback decomposition the span $(\xi^{(q,\alpha)}, \pi_2)$ in Figure 8 becomes a pullback of the cospan (q, q) which means that $\langle \xi^{(q,\alpha)}, \pi_2 \rangle$ is the kernel pair of q and thus an equivalence.

Definition 10 (Equivalence Relation) *An* equivalence relation *on $A \in Ob_{\mathscr{C}}$ is a pair of arrows $a, b : U \to A$, such that $U \xrightarrow{\langle a,b \rangle} A \times A$ is a monomorphism, and which is*

1. *reflexive: $\exists r : A \to U : a \circ r = b \circ r = id$,*
2. *symmetric: $\exists s : U \to U : a \circ s = b, b \circ s = a$, and*
3. *transitive: If $(p : P \to U, q : P \to U)$ is the pullback of (a, b) (especially $b \circ p = a \circ q$), there is $t : P \to U$, such that $a \circ t = a \circ p$ and $b \circ t = b \circ q$.*

Not only "canonical" descent data $\xi^{(q,\alpha)}$ but arbitrary descent data provide equivalences:

Lemma 11 $E \times_B C \xrightarrow{\langle \xi, \pi_2 \rangle} C \times C$ *establishes an equivalence relation for any descent data $\xi : \mathfrak{T}^p\gamma \to \gamma$.*

Proof: Since for any pullback $\langle \mathfrak{T}^p\gamma, \pi_2 \rangle$ is monic, the equation $\gamma \circ \xi = \mathfrak{T}^p\gamma$ implies that $\langle \xi, \pi_2 \rangle$ is monic as well. For reflexivity, let $r := \eta^p_\gamma$ and use (4) and (5). Symmetry follows with $s := \overline{\xi}$, (6), and (7). Transitivity can be established via $t := \mu^p_\gamma$ using both commuting top squares in Fig. 8 (with ξ instead of $\xi^{(q,\alpha)}$), that are also pullbacks by decomposition, and (4). □

The crucial consequence of Lemma 11 is that $\langle \xi, \pi_2 \rangle$ is the kernel pair of its coequalizer, because in topoi, equivalence relations are effective (see [19], A 2.4.1.). Consider now the above introduced coequalizer construction for Ψ^p.

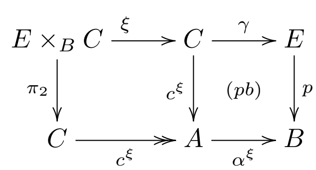


Figure 10: Coequalizer construction

Lemma 12 *The right square in Figure 10 is a pullback.*

Proof: By Lemma 11 and because equivalence relations are the kernel pair of their coequalizer, the left square in Figure 10 is a pullback. Because $\gamma \circ \xi = \mathfrak{T}^p\gamma$, by assumption $\xi : \mathfrak{T}^p\gamma \to \gamma$, and $\alpha^\xi \circ c^\xi = p \circ \gamma$, by definition of α^ξ, the outer rectangle in Figure 10 is the chosen pullback of $p \circ \gamma$ and p. Since c^ξ is epic, the result follows from Lemma 9. □

After having proved that Ψ^p is indeed a functor with codomain $pb(p)$ and $\Lambda_p \circ \Psi^p = |\_|_p$ we are going to show now that both functors establish an adjunction $\Psi^p \dashv \Phi^p$ between the categories $des(p)$ and $pb(p)$.

It is well-known [24] that two functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ are adjoint, $F \dashv G$ in symbols, iff there exist natural transformations $\eta : id_{\mathscr{C}} \Rightarrow G \circ F$, the unit of the adjunction, and $\varepsilon : F \circ G \Rightarrow id_{\mathscr{D}}$, the co-unit, such that the following two equations hold for the corresponding horizontal compositions of functors and natural transformations:

$$G\varepsilon \circ \eta G = id_G \quad \text{and} \quad \varepsilon F \circ F\eta = id_F. \tag{12}$$

$$G \xRightarrow{\eta G} GFG \xRightarrow{G\varepsilon} G \qquad\qquad F \xRightarrow{F\eta} FGF \xRightarrow{\varepsilon F} F$$

In our case of descent data and pullbacks the unit $\eta : id_{des(p)} \Rightarrow \Phi^p \circ \Psi^p$ is the identity:

Lemma 13 $\Phi^p \circ \Psi^p = id_{des(p)}$.

Proof: By the definition of Ψ^p and Φ^p, respectively, we have $(\Phi^p \circ \Psi^p)(\gamma, \xi) = (\gamma, \xi^{(c^\xi, \alpha^\xi)})$ for all objects (γ, ξ) in $des(p)$. Both arrows $\xi, \xi^{(c^\xi, \alpha^\xi)} : E \times_B C \to C$ satisfy the defining equations (9), with $q = c^\xi$ the coequalizer of ξ and π_2, thus $\xi = \xi^{(c^\xi, \alpha^\xi)}$ by uniqueness. For morphisms we have trivially $(\Phi^p \circ \Psi^p)(h) = \Phi^p(h, \hat{h}) = h$. □

The co-unit $\varepsilon : \Psi^p \circ \Phi^p \Rightarrow id_{pb(p)}$ is provided by epi-mono-factorization. For this we need the following Lemma, which is proven in Section 8.2.

Lemma 14 *Let $f : A \to B$ be an arrow in a topos $\mathscr{C}$ and let c be a coequalizer of the (chosen) kernel pair (p_1, p_2) of f. Then the mediator m of f out of a coequalizer c is monic. This epi-mono-factorization $f = m \circ c$ is unique up to a unique isomorphism, cf. Figure 11.* □

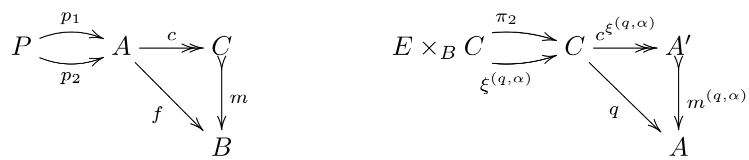


Figure 11: Epi-mono factorization

Given an object (γ, q, α) in $pb(p)$ we consider a coequalizer $c^{\xi^{(q,\alpha)}}$ of the kernel pair $(\pi_2, \xi^{(q,\alpha)})$ of q. The monic mediator according to Lemma 14 is denoted by $m^{(q,\alpha)}$ and will establish – for a fixed choice of coequalizer – the co-unit for (γ, q, α), cf. Fig. 11:

$$\varepsilon_{(\gamma,q,\alpha)} := (id_C, m^{(q,\alpha)}) : (\gamma, c^{\xi^{(q,\alpha)}}, \alpha^{\xi^{(q,\alpha)}}) \to (\gamma, q, \alpha). \tag{13}$$

We assume $q = id_A \circ q$ to be the epi-mono-factorization of epic q, i.e. $\varepsilon_{(\gamma,q,\alpha)} = id_{(\gamma,q,\alpha)}$ whenever q is epic.

$\alpha \circ m^{(q,\alpha)} = \alpha^{\xi^{(q,\alpha)}}$ and naturality for the co-unit $\varepsilon_{(\gamma,q,\alpha)}$ is insured by the uniqueness of mediators out of the coequalizers $c^{\xi^{(q,\alpha)}}$, cf. the definition of Ψ^p.

It remains to show adjointness. By Lemma 13 the two equations in (12) reduce, in our case, to the requirements

$$\Phi^p \varepsilon = id_{\Phi^p} \quad \text{and} \quad \varepsilon \Psi^p = id_{\Psi^p}. \tag{14}$$

The first equation follows immediately from (10) and the definition of ε and Φ^p, respectively. The second equation is an immediate consequence of Lemma 13. Since – in topoi – epimorphisms are preserved under pullbacks, we have shown

Proposition 15 (Adjunction between Pullbacks and Descent Data) *For any topos $\mathscr{C}$ and any fixed choice of pullbacks and coequalizers in $\mathscr{C}$, respectively, we have for any arrow $p : E \to B$ in $\mathscr{C}$ functors $\Psi^p : des(p) \to pb(p)$ and $\Phi^p : pb(p) \to des(p)$ such that*

a) $\Psi^p \dashv \Phi^p$ *with* $\Phi^p \circ \Psi^p = id_{des(p)}$.

b) If p is an epimorphism, $\Psi^p \dashv \Phi^p$ becomes an equivalence of categories. □

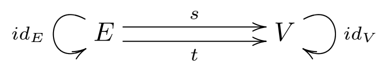
Note that b) outlines the well-known fact that in topoi the class of epimorphisms is precisely the class of all effective descent morphisms, i.e. those morphisms for which the comparison functor becomes an equivalence [18]. The novelty, however, is that we showed this result in terms of an equivalence involving the category of pullbacks along p.

4.2 Descent Data in Presheaves

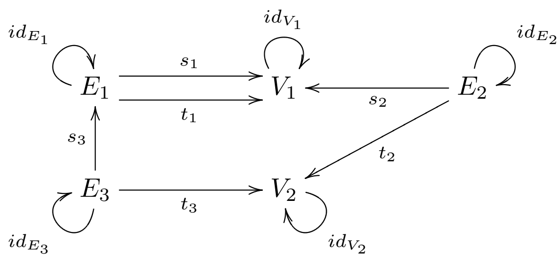
4.2.1 Diagrammatic Specifications and Presheaves

As pointed out in the introduction, the paper is devoted to investigate amalgamation for fibred semantics of diagrammatic specifications as it has been defined, for instance, in the *Diagram Predicate Framwork (DPF)* [7, 30, 29]. Thereby "diagrammatic" is not meant as a synonym for "visual" but rather for "graph-based". DPF is generic in the sense that it can be instantiated for a wide range of graph-based structures. A specification in DPF is a generalized sketch in the sense of [25].

The kind of underlying graph-based structure, we use in a certain application area, can be described, in general, by a “meta schema”, i.e. by a small (schema) category. As a simple meta schema we can consider, for instance, the following schema category for *multi graphs*:

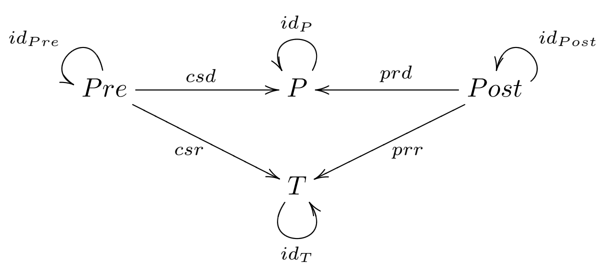


Another example are attributed graphs [11]. They are based on so-called E-graphs which conform to the schema category



where edges (E_1) connect complex vertices (V_1) and attributes (E_2) link complex and primitive vertices (V_2). Moreover edges may be annotated (E_3) with primitive types (e.g. to specify list indices).

A third prominent example is the schema category for Petri-nets:



(pre- and postconditions specify places that are consumed/produced by a consumer/producer transition).

A structure which conforms to such a schema category $\mathscr{S}$ is a functor $F : \mathscr{S} \to SET$. Structure morphisms are given by natural transformations between them. Hence *presheaves*, i.e. functor categories $\mathscr{C} = SET^{\mathscr{S}}$ with a small schema category $\mathscr{S}$ are an appropriate level of abstraction for our intended applications. This is underlined by the fact that the category of generalized multi sketches over a chosen category of underlying structures is equivalent to a presheaf as long as the category of underlying structures is equivalent to a presheaf as well [25].

An object of $\mathscr{S}$ will sometimes be called a *sort*, an arrow of $\mathscr{S}$ will be called a (necessarily unary) *operation symbol*. Accordingly, we also use the more intuitive spelling for the application of a functor $F \in SET^{\mathscr{S}}$ to an object/sort X:

$$F_X := FX$$

and to an arrow $op : X \to Y$:

$$op^F := F(op) : F_X \to F_Y.$$

In natural transformations $n : F \Rightarrow G$, $n = (n_X)_{X \in Ob_{\mathscr{S}}}$, we often omit the index X and write $n : F_X \to G_X$ if there is no danger of confusion.

For any functor $F : \mathscr{S} \to SET$ and $X \in Ob_{\mathscr{C}}$ we call the set F_X the *carrier set* of sort X. E.g. for Petri-nets there are four sorts $Pre, Post, P, T$ and four non-identity operation symbols specifying the causal producer-consumer relation.

Our previous theoretical results apply to presheaves since any presheaf $SET^{\mathscr{S}}$ is a topos ([14], Section 9.3).

In fibred semantics, restriction of instances along some specification morphism is pullback. By Proposition 15 the cleavage kernels generated by the pullback procedure can precisely be described with descent data. Compositionality problems (see the examples in Section 2) arise from uncoordinated kernel generation of two morphisms on a common domain. The following subsections shall foster a better understanding of descent data in presheaves in terms of its action on carrier sets and shall significantly support the understanding of these effects.

First, we analyse descent data in SET (Section 4.2.2). Since limits and colimits in a presheaf are constructed componentwise by limits and colimits in SET, respectively, our analysis extends then smoothly to presheaves $SET^{\mathscr{S}}$ in Section 4.2.3.

4.2.2 Descent Data in SET

If $\mathscr{C} = SET$, $p : E \to B$, and $\gamma : C \to E$, we choose as canonical pullbacks

$$\begin{aligned} E \times_B C &= \{(e, c) \in E \times C \mid p(e) = p(\gamma(c))\} \text{ and} \\ E \times_B (E \times_B C) &= \{(e'', (e', c)) \in E \times (E \times C) \mid p(e'') = p(e') = p(\gamma(c))\}. \end{aligned}$$

A purely set-oriented description of descent data (γ, ξ) emerges from the following observations: Because products in comma categories are pullbacks in the underlying category, $p \circ \mathcal{T}^p\gamma = p \circ \gamma \circ \pi_2(p, p \circ \gamma) : E \times_B C \to B$ in Definition 6 provides a product $p \times (p \circ \gamma)$ in $\mathscr{C} \downarrow B$ with projections $\mathcal{T}^p\gamma$ and $\pi_2(p, p \circ \gamma)$, respectively. Moreover, any $\xi : E \times_B C \to C$ in $\mathscr{C}$ which is an arrow $\xi : \mathcal{T}^p\gamma \to \gamma$ in $\mathscr{C} \downarrow E$ establishes also an arrow $\xi : p \times (p \circ \gamma) \to p \circ \gamma$ in $\mathscr{C} \downarrow B$. $\mathscr{C} \downarrow B$, however, is also a topos by the fundamental theorem of Freyd [13] and thus, espcially cartesian closed, such that

$$Hom_{\mathscr{C}\downarrow B}(p \times (p \circ \gamma), p \circ \gamma) \cong Hom_{\mathscr{C}\downarrow B}(p, (p \circ \gamma)^{p \circ \gamma}).$$

From Lemma 37 in Section 8.3 we know that the exponent $(p \circ \gamma)^{p \circ \gamma}$ is defined on the set of all endomaps $\{k : (p \circ \gamma)^{-1}(b) \to (p \circ \gamma)^{-1}(b) \mid b \in B\}$ and each k is mapped to its base point b. In such a way, ξ can be regarded as a map that assigns to any element $e' \in E$ an endomap $\xi(e', \_)$ of the fibre of $p \circ \gamma$ over $p(e')$.

Let us make this intuition more precise: For any $e' \in E$ we have $p^{-1}(p(e')) = [e']_{ker(p)}$.[1] Thus the fibre of $p \circ \gamma$ over $p(e')$ is the set $\gamma^{-1}([e']_{ker(p)})$ and can be described, in such a way, as the union of all pairwise disjoint fibres $\gamma^{-1}(e)$ with $(e, e') \in ker(p)$. We let $\xi_{e,e'}$ be the map $\xi(e', \_)$ restricted to $\gamma^{-1}(e)$. If $c \in \gamma^{-1}(e)$ we obtain $\gamma(\xi(e', c)) = e'$ since $\mathcal{T}^p\gamma = \pi_1(p, p \circ \gamma)$. Thus the codomain of $\xi_{e,e'}$ is $\gamma^{-1}(e')$ and ξ represents a family

$$(\xi_{e,e'} : \gamma^{-1}(e) \to \gamma^{-1}(e'))_{(e,e') \in ker(p)} \tag{15}$$

for which by definition

$$\xi(e', c) = \xi_{e,e'}(c) \text{ whenever } \gamma(c) = e. \tag{16}$$

Let us now investigate the influence of neutrality and associativity (4) to this family. From Lemma 5 and (2) we obtain

$$\eta^p_\gamma(c) = (\gamma(c), c),\ \mu^p_\gamma(e'', (e', c)) = (e'', c),\ \mathcal{T}^p\xi(e'', (e', c)) = (e'', \xi(e', c)). \tag{17}$$

Thus (4) and the first equation in (17) yield

$$\xi_{e,e}(c) = c \text{ for all } c \in \gamma^{-1}(e), \text{ i.e., } \xi_{e,e} = id_{\gamma^{-1}(e)}, \tag{18}$$

whereas (4) (applied to a triple $(e'', (e', c))$) and the second and third equation of (17) imply

$$\xi_{e',e''}(\xi_{e,e'}(c)) = \xi_{e,e''}(c) \text{ for all } c \in \gamma^{-1}(e), \text{ i.e., } \xi_{e',e''} \circ \xi_{e,e'} = \xi_{e,e''} \tag{19}$$

for all $(e, e'), (e', e'') \in ker(p)$. By choosing $e'' = e$ both properties force each $\xi_{e,e'}$ to be bijective.

By reversing the whole argumentation, it is easy to see that a familiy

$$(\xi_{e,e'} : \gamma^{-1}(e) \to \gamma^{-1}(e') \mid (e, e') \in ker(p))$$

which satisfies (18) and (19) yields a descent data $\xi : E \times_B C \to C$ of γ relative to p in $\mathscr{C} = SET$ by defining ξ as in (16) for $e := \gamma(c)$.

4.2.3 Descent Data in $SET^{\mathscr{S}}$

Pullbacks in $SET^{\mathscr{S}}$ are constructed componentwise by pullbacks in SET. Applied to our situation, this means that the pullback of two morphisms $p : E \to B$ and $p \circ \gamma : C \to B$ in $SET^{\mathscr{S}}$ is given by the functor $E \times_B C : \mathscr{S} \to SET$ defined by

$$(E \times_B C)_X := E_X \times_{B_X} C_X \quad \text{on objects } X \in Ob_{\mathscr{S}} \text{ and} \tag{20}$$

$$op^{E \times_B C} := op^E \times_{op^B} op^C \quad \text{on arrows } op : X \to Y \tag{21}$$

[1] $[\_]_\equiv$ denotes the canonical map from a set A to $A/_\equiv$ for any equivalence relation $\equiv$ on A.

(see Fig. 12) together with the natural transformation $\pi_1 : E \times_B C \to E$, $\pi_2 : E \times_B C \to C$ defined by ordinary set projections

$$(\pi_1)_X := \pi_1 : (E \times_B C)_X \to E_X \quad \text{and} \quad (\pi_2)_X := \pi_2 : (E \times_B C)_X \to C_X. \tag{22}$$

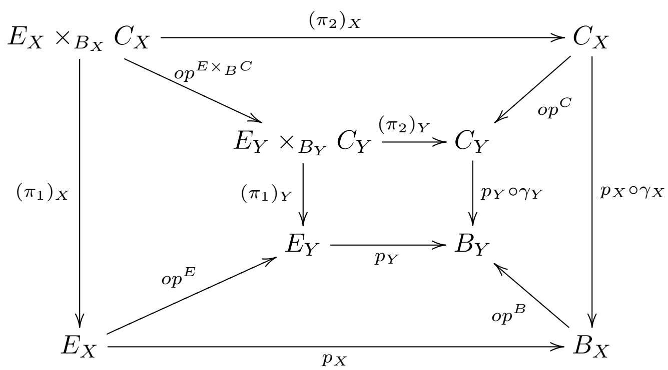


Figure 12: Componentwise construction of pullbacks

We rely on the same canonical choice of pullbacks as in Section 4.2.2 pointwise for each $X \in Ob_{\mathscr{S}}$. In such a way, for each descent data $\xi : E \times_B C \to C$ in $\mathscr{C} = SET^{\mathscr{S}}$ the component $\xi_X : (E \times_B C)_X \to C_X$ is uniquely described by a family $((\xi_X)_{e,e'})_{(e,e')\in ker(p_X)}$ of bijections satisfying neutrality (18) and associativity (19).

It remains to show that compatibility with operation symbols, that is the naturality of ξ, can be expressed equivalently on the level of pre-images of fibres: Given any $op : X \to Y \in Mor_{\mathscr{S}}$, naturality of γ ensures that $op^C : C_X \to C_Y$ restricts to a map $op^C_e : \gamma_X^{-1}(e) \to \gamma_Y^{-1}(op^E(e))$ for each $e \in E$.

Naturality of ξ means, according to (21) and our choice of pullbacks, that we have $\xi_Y(op^E(e'), op^C(c)) = op^C(\xi_X(e', c))$ for all $e' \in E_X$ and $c \in C_X$ where $p_X(e') = p_X(\gamma_X(c))$ (see Fig. 13). By (16) this is equivalent to the requirement

$$(\xi_Y)_{op^E(e),op^E(e')}(op^C_e(c)) = op^C_{e'}((\xi_X)_{e,e'}(c))$$

for all $(e, e') \in ker(p_X)$ and $c \in \gamma_X^{-1}(e)$.

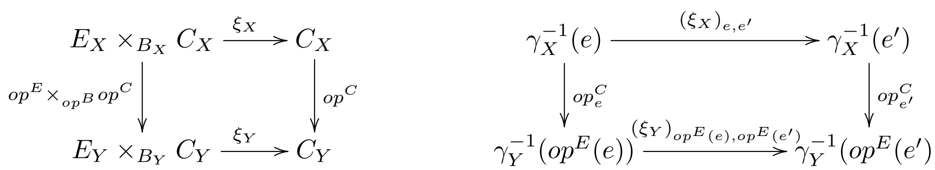


Figure 13: Naturality of descent data in $SET^{\mathscr{S}}$

Altogether we obtain the following statement which subsumes the monoidal nature of descent data in $SET^{\mathscr{S}}$.

Proposition 16 (Descent Data in $SET^{\mathscr{S}}$) *Let $\mathscr{C} = SET^{\mathscr{S}}$, and $p \in Mor_{\mathscr{C}}$. There is a bijective correspondence between objects (γ, ξ) of $des(p)$ and families*

$$((\xi_X)_{e,e'} : \gamma_X^{-1}(e) \to \gamma_X^{-1}(e'))_{X\in Ob_{\mathscr{S}},(e,e')\in ker(p_X)}$$

of bijections which satisfy

$$(\xi_X)_{e,e} = id_{\gamma_X^{-1}(e)} \quad , (\xi_X)_{e,e''} = (\xi_X)_{e',e''} \circ (\xi_X)_{e,e'}$$

for all $X \in Ob_{\mathscr{S}}$, $(e, e'), (e', e'') \in ker(p_X)$, and

$$(\xi_Y)_{op^E(e),op^E(e')} \circ op^C_e = op^C_{e'} \circ (\xi_X)_{e,e'}$$

for all $op : X \to Y \in Mor_{\mathscr{S}}$, $(e, e') \in ker(p_X)$. □

We want to close this section with descent data's role as congruence relation (i.e. equivalence relation compatible with operation symbols). Recall that the functor $\Psi^p : des(p) \to pb(p)$ created a pullback with the help of the coequalizer c of ξ and π_2, cf. Figure 9, where (ξ, π_2) is the kernel pair of c.

For $X \in Ob_{\mathscr{S}}$ any pair $(x, x') \in ker(c_X)$ gives rise to the existence of a unique $(e, x'') \in E_X \times_{B_X} C_X$ with $\pi_2(e, x'') = x$ and $\xi_X(e, x'') = x'$, because kernel pairs are pullbacks. Hence $x'' = x$ such that $(x, x') = (x, \xi_X(e, x)) = (x, (\xi_X)_{\gamma_X(x),e}(x))$ with $p_X(e) = p_X(\gamma_X(x))$ by (16). Since we know that $y := (\xi_X)_{\gamma_X(x),e}(x)$ is in the fibre over e, we obtain

$$ker(c_X) = \{(x, (\xi_X)_{\gamma_X(x),\gamma_X(y)}(x)) \mid x, y \in C_X,\, (\gamma_X(x), \gamma_X(y)) \in ker(p_X)\} \tag{23}$$

for each $X \in Ob_{\mathscr{S}}$. By Lemma 12 and Figure 10, the résumé can be stated as follows:

Each $(\gamma, \xi) \in des(p)$ represents a congruence relation $\equiv_\xi$ on C which highlights one pullback complement $C \twoheadrightarrow A = C/_{\equiv_\xi} \longrightarrow B$ of γ and p.

5 Amalgamation, Van Kampen, and Coherence in Topoi

As pointed out in the introduction, amalgamation of instances provides the basis for compositionality. The goal of this section is to characterize those pullback spans in arbitrary topoi that can be successfully amalgamated, cf. Fig. 14.

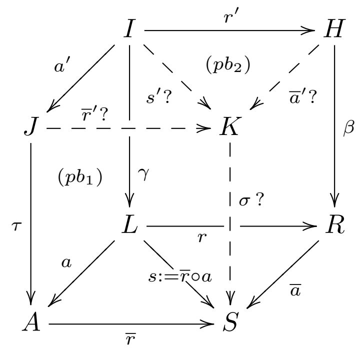


Figure 14: Is amalgamation possible for the rear pullback span (pb_1, pb_2)?

We will demonstrate that the close relationship between descent data and pullbacks in Proposition 15 provides such a theorem in terms of the degree of coherence of descent data of the two pullbacks. Surprisingly, this result (Propositions 25 and 27) leads to an analogon of the classical amalgamation lemma for indexed semantics (see Theorem 28). It will turn out in Section 6 that Theorem 28 prepares a more feasible and easily checkable criterion for amalgamability in typical contexts of computer science, i.e. presheaves.

5.1 Amalgamation and Van Kampen Squares

To lift our following discussion of amalgamation to a more structural level we consider the coslice category $L{\downarrow}\mathscr{C}$. Taking this abstract viewpoint, the assignments $f \mapsto pb(f)$ for morphisms $f : L \to F$ in $\mathscr{C}$ define a map from the objects in $L{\downarrow}\mathscr{C}$ to the objects in CAT. We show now that these assignments can be extended to morphisms in $L{\downarrow}\mathscr{C}$.

Let objects $f : L \to F$, $g : L \to G$ and a morphism $h : f \to g$ in $L{\downarrow}\mathscr{C}$ be given, i.e. a morphism $h : F \to G$ in $\mathscr{C}$ with $h \circ f = g$, cf. Figure 15. We can decompose any pullback in $pb(g)$ into a pulback in $pb(f)$ and a pullback along h:

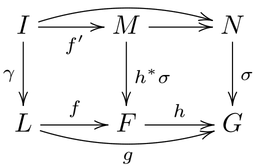


Figure 15: Pullback decomposition along $h : f \to g$

First, we calculate the chosen pullback $h^*\sigma$. Second, the pullback $(\gamma, f', h^*\sigma)$ in $pb(f)$ arises uniquely from the mediator f' from the outer pullback into the right chosen pullback. The assignment of the outer pullback to the left pullback extends to a functor $\Delta^h : pb(g) \to pb(f)$ (the notation Δ is meant to remind of "decomposition").

Since chosen pullbacks do not necessarily compose to chosen pullbacks, we will, in general, not have $\Delta^{h_2 \circ h_1} = \Delta^{h_1} \circ \Delta^{h_2}$ for arbitrary morphisms $h_1 : f_1 \to f_2$, $h_2 : f_2 \to f_3$ in $L\downarrow\mathscr{C}$ but only that $\Delta^{h_2\circ h_1}$ and $\Delta^{h_1} \circ \Delta^{h_2}$ are natural isomorphic. In such a way, the assignments $f \mapsto pb(f)$ and $h \mapsto \Delta^h$ do not define, on the global level, a functor but only a contravariant *pullback decomposition* pseudo functor

$$\mathcal{PD}_L : (L\downarrow\mathscr{C})^{op} \to CAT. \tag{24}$$

For a commuting square, as in the bottom of Figure 14, we consider the pairing

$$\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle : pb(s) \to pb(a) \times_{\mathscr{C}\downarrow L} pb(r).$$

where $pb(a)\times_{\mathscr{C}\downarrow L}pb(r)$ is the category of all pullback spans over (a, r) together with morphism triples (m_1, m_2, m_3), i.e. $pb(a) \times_{\mathscr{C}\downarrow L} pb(r)$ is given by a standard construction of pullbacks in CAT, cf. Figure 16 and the figure in Definition 17. [2]

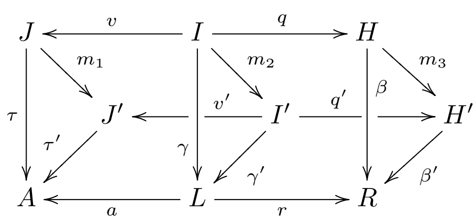
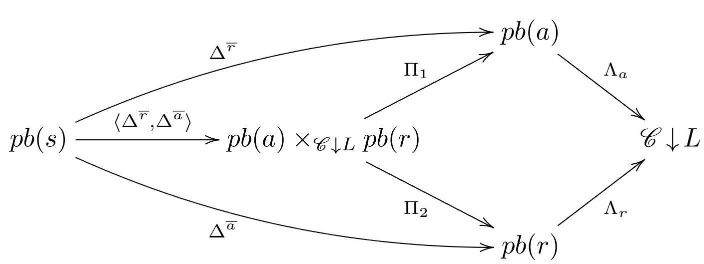


Figure 16: Objects and morphisms in the category $pb(a) \times_{\mathscr{C}\downarrow L} pb(r)$

Definition 17 (Amalgamability) *A pullback span in $pb(a) \times_{\mathscr{C}\downarrow L} pb(r)$ is* amalgamable*, if the span is in the image of $\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle$ up to a $pb(a) \times_{\mathscr{C}\downarrow L} pb(r)$-isomorphism of the form (m_1, id_γ, m_3).*



That is, we require that the pairing of the decomposition procedure (based on chosen pullbacks) "almost reaches" the given pullback span: The instances τ and β may isomorphically be distorted while γ is reached exactly. We have to work with "reachable up to isomorphism" since we decided to consider pb as a functorial construction, based on chosen pullbacks, and not just as a relation.

For future reference, we note that any arrow (m_1, m_2, m_3) between two amalgamable spans has a preimage under the pairing functor:

$$\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle (m_2, \hat{m}) = (m_1, m_2, m_3) \tag{25}$$

[2] A similar construction with $pb(s)$ replaced by (the equivalent category) $\mathscr{C}\downarrow S$ has been considered in different investigations on *adhesive* categories, cf. [31, 21].

where $\hat{m}$ is the mediator between the two resulting top pushouts during amalgamation (recall that pushouts are stable under pullbacks in topoi).

Finally, we mention that universal amalgamability is equivalent to the Van Kampen property [31]:

Proposition 18 *A pushout square, as in the bottom face of Figure 14, is a Van Kampen square if and only if each rear pullback span over this square is amalgamable.* □

5.2 Coherence in Topoi

We now investigate conditions for amalgamability in terms of the descent data of the two back face pullbacks in Figure 14 by applying the methodology of Section 4. Again, a more structured view is achieved by extending the assignment $f \mapsto des(f)$ to a functor from $L{\downarrow}\mathscr{C}$ into CAT.

Let $f : L \to F$, $g : L \to G$ and $h : f \to g$ in $L{\downarrow}\mathscr{C}$ be given as in Section 5.1. We consider the pullbacks $f^*(f \circ \gamma)$ and $g^*(g \circ \gamma)$ as in Figure 6 (with $C := I$, $E := L$, and $p : E \to B$ replaced by $f : L \to F$, $g : L \to G$, resp.), cf. Figure 17.

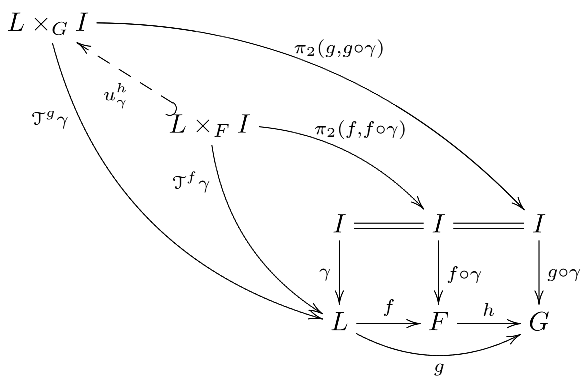


Figure 17: Monad embedding

Because $g = h \circ f$, we obtain $g \circ \gamma \circ \pi_2(f, f \circ \gamma) = g \circ \mathfrak{T}^f\gamma$. Hence there is a *unique* $u^h_\gamma : L \times_F I \to L \times_G I$ such that

$$\pi_2(g, g \circ \gamma) \circ u^h_\gamma = \pi_2(f, f \circ \gamma) \quad \text{and} \quad \mathfrak{T}^g\gamma \circ u^h_\gamma = \mathfrak{T}^f\gamma. \tag{26}$$

Note, that in $SET^{\mathscr{S}}$, $\mathfrak{T}^f\gamma, \pi_2(f, f \circ \gamma)$ and $\mathfrak{T}^g\gamma, \pi_2(g, g \circ \gamma)$ are componentwise first and second projections (of sets), which actually makes u^h_γ invariant under projections: Indeed $L \times_F I = \{(l, i) \,|\, f(\gamma(i)) = f(l)\} \subseteq \{(l, i) \,|\, g(\gamma(i)) = g(l)\} = L \times_G I$ for each carrier, where the inclusion is u^h_γ. This justifies the use of the hooked arrow in Figure 17.

The uniqueness of mediators ensures that the u^h_γ establish a natural transformation $u^h : \mathfrak{T}^f \Rightarrow \mathfrak{T}^g$. Moreover, this natural transformation provides a perfect embedding of a "small" into a "larger" monad: The proof of the following lemma consists of a collection of routine calculations. It can be found in Section 8.4 in the Appendix.

Lemma 19 *For any morphism $h : f \to g$ in $L{\downarrow}\mathscr{C}$ the natural transformation $u^h : \mathfrak{T}^f \Rightarrow \mathfrak{T}^g$ is monic and defines a monad morphism from $(\mathfrak{T}^f, \eta^f, \mu^f)$ to $(\mathfrak{T}^g, \eta^g, \mu^g)$, i.e. the following laws are satisfied*

$$u^h \circ \eta^f = \eta^g \quad \textit{and} \quad u^h \circ \mu^f = \mu^g \circ (u^h)^2$$

where $(u^h)^2 : (\mathfrak{T}^f)^2 \to (\mathfrak{T}^g)^2$ is the horizontal composition of u^h with itself. □

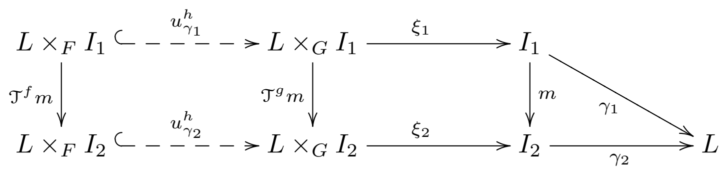
Monad morphisms give rise, by simple pre-composition, to "forgetful functors" between the corresponding categories of descent data:

Lemma 20 *For any morphism $h : f \to g$ in $L{\downarrow}\mathscr{C}$ we can define a functor $U^h : des(g) \to des(f)$ by*

$$U^h(\gamma, \xi) := (\gamma, \xi \circ u^h_\gamma)$$

on objects and on arrows by

$$U^h(\ (\gamma_1,\xi_1) \xrightarrow{m} (\gamma_2,\xi_2)\ ) := (\ (\gamma_1,\xi_1\circ u^h_{\gamma_1}) \xrightarrow{m} (\gamma_2,\xi_2\circ u^h_{\gamma_2})\ ).$$



Proof: Since $des(p)$ is the category of Eilenberg-Moore-Algebras associated with $\mathcal{T}^p$ for $p \in \{f,g\}$, the result follows from Lemma 19 and the proof of a theorem of Barr and Wells ([2], Theorem 6.3 in Chapter 3). □

The intutition that descent data abstract from the monic part of a morphism can be grasped, in a formal way, by the statement that each monic h establishes an isomorphism between categories of descent data.

Corollary 21 *The functor $U^h : des(g) \to des(f)$ is an isomorphism for all morphisms $h : f \to g$ in $L{\downarrow}\mathscr{C}$ with h monic in $\mathscr{C}$.*

Proof: In case h monic, the right square in Figure 17 is already a pullback, such that its composition with the pullback of f and $f \circ \gamma$ already yields the outer pullback (of g and $g \circ \gamma$) up to a canonical isomorphism. By (26) this isomorphism must be u^h_γ. We denote the inverse of u^h_γ by v^h_γ.

The v^h_γ establish a natural transformation (isomorphism) $v^h : \mathcal{T}^g \Rightarrow \mathcal{T}^f$ with $v^h \circ u^h = id_{\mathcal{T}^f}$ and $u^h \circ v^h = id_{\mathcal{T}^g}$. In such a way, $u^h \circ \eta^f = \eta^g$ and $u^h \circ \mu^f = \mu^g \circ (u^h)^2$ entail $\eta^f = v^h \circ \eta^g$ and $\mu^f \circ (v^h)^2 = v^h \circ \mu^g$, respectively, thus v^h defines a monad morphism from $(\mathcal{T}^g, \eta^g, \mu^g)$ to $(\mathcal{T}^f, \eta^f, \mu^f)$. By the same construction as in Lemma 20 we obtain, finally, the required functor $V^h : des(f) \to des(g)$ with $V^h \circ U^h = id_{des(g)}$ and $U^h \circ V^h = id_{des(f)}$. □

By uniqueness of mediators we have $u^{id_f} = id_{\mathcal{T}^f}$ and $u^{h_2\circ h_1} = u^{h_2} \circ u^{h_1}$. According to the construction of our forgetful functors this entails $U^{id_f} = id_{des(f)}$ and $U^{h_2\circ h_1} = U^{h_1} \circ U^{h_2}$. In such a way, the assignments $f \mapsto des(f)$ and $h \mapsto U^h$ define, on the global level, for any object L in $\mathscr{C}$ a contravariant *descent data* functor

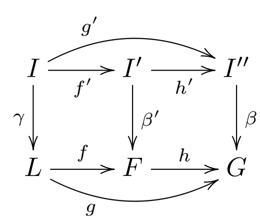
$$\mathcal{DD}_L : (L{\downarrow}\mathscr{C})^{op} \to CAT. \tag{27}$$

We are now able to settle a close interrelation between the assignments $\mathcal{DD}_L h = U^h$ and $\mathcal{PD}_L h = \Delta^h$ which will allow us to replace pseudoriality (of $\mathcal{PD}_L$) by compositionality "on the nose" (of $\mathcal{DD}_L$):

Lemma 22 *For any morphism $h : f \to g$ in $L{\downarrow}\mathscr{C}$ we have*

$$U^h \circ \Phi^g = \Phi^f \circ \Delta^h : pb(g) \to des(f).$$

Proof: Take any pullback $(\gamma, g', \beta) \in pb(g)$ and decompose it by Δ^h:



Due to the definition of U^h, Φ^g, Φ^f, and Δ^h we have to show that $\xi^{(g',\beta)} \circ u^h_\gamma = \xi^{(f',\beta')}$: We obtain

$$\begin{aligned} h' \circ f' \circ \xi^{(g',\beta)} \circ u^h_\gamma &= h' \circ f' \circ \pi_2(g, g\circ\gamma) \circ u^h_\gamma && \text{By (9)} \\ &= h' \circ f' \circ \pi_2(f, f\circ\gamma) && \text{By (26)} \end{aligned}$$

and

$$
\begin{aligned}
\beta' \circ f' \circ \xi^{(g',\beta)} \circ u^h_\gamma &= f \circ \gamma \circ \xi^{(g',\beta)} \circ u^h_\gamma && \text{Left square commutes} \\
&= f \circ \mathfrak{T}^g \gamma \circ u^h_\gamma && \xi^{(g',\beta)} : \mathfrak{T}^h\gamma \to \gamma \\
&= f \circ \mathfrak{T}^f \gamma && \text{By (26)} \\
&= f \circ \gamma \circ \pi_2(f, f \circ \gamma) && \text{Definition of } \mathfrak{T}^f\gamma \\
&= \beta' \circ f' \circ \pi_2(f, f \circ \gamma) && \text{Left square commutes}
\end{aligned}
$$

Since h' and β' are jointly monic, we obtain $f' \circ \xi^{(g',\beta)} \circ u^h_\gamma = f' \circ \pi_2(f, f\circ\gamma)$. By $\xi^{(g',\beta)} : \mathfrak{T}^h\gamma \to \gamma$ and (26) we have, moreover, $\gamma \circ \xi^{(g',\beta)} \circ u^h_\gamma = \mathfrak{T}^f\gamma$ thus the desired equality follows from the fact that $\xi^{(f',\beta')}$ is unique with these two properties (cf. (9)). □

A commutative square as in the bottom of Figure 14 can be seen, equivalently, as a commutative square of morphisms $\overline{a} : r \to s$, $\overline{r} : a \to s$, $a : id_L \to a$, $r : id_L \to r$ in $L{\downarrow}\mathscr{C}$ giving rise, in such a way, to a commutative diagram of forgetful functors as in the top of Figure 18. $des(id_L)$ represents the "common parts" of objects in the categories $des(a)$ and $des(r)$, respectively. $des(id_L)$ is isomorphic to $\mathscr{C} \downarrow L$ and if we assume pullbacks to be chosen such that $id_L^*\gamma = \gamma, \pi_2(id_L, id_L \circ \gamma) = id_\gamma$, we have by (4) and (5)

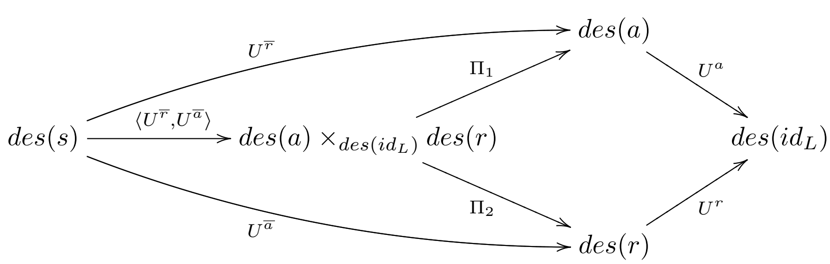
$$Ob_{des_{id_L}} = ((\gamma, id_\gamma))_{\gamma \in \mathscr{C}\downarrow L}.$$

Now we can define *coherence* by adapting the same methodology used to define amalgamability. First, we construct the pullback of U^a and U^r (see the diagram in Definition 23) which yields the category $des(a) \times_{des(id_L)} des(r)$. It has objects pairs $((\gamma, \xi_1), (\gamma, \xi_2))$ of descent data and morphisms are given by those morphisms $h : \gamma \to \gamma'$ in $\mathscr{C} \downarrow L$ providing a morphism $h : (\gamma, \xi_1) \to (\gamma', \xi_1')$ in $des(a)$ as well as a morphism $h : (\gamma, \xi_2) \to (\gamma', \xi_2')$ in $des(r)$. Second, we consider the pairing

$$\langle U^{\overline{r}}, U^{\overline{a}} \rangle : des(s) \to des(a) \times_{des(id_L)} des(r)$$

which gives rise to

Definition 23 (Coherence) *Descent data $(\gamma, \xi_1) \in des(a)$ and $(\gamma, \xi_2) \in des(r)$ are called* coherent*, if the pair $((\gamma, \xi_1), (\gamma, \xi_2))$ is in the image of $\langle U^{\overline{r}}, U^{\overline{a}}\rangle$, i.e. there exists $(\gamma, \xi) \in des(s)$ such that $U^{\overline{r}}(\gamma, \xi) = (\gamma, \xi_1)$ and $U^{\overline{a}}(\gamma, \xi) = (\gamma, \xi_2)$. Any $(\gamma, \xi) \in des(s)$ with this property will be called a* coherence witness *(for (γ, ξ_1) and (γ, ξ_2)).*



This means that two algebraic structures (γ, ξ_1) and (γ, ξ_2) are coherent if they can be combined faithfully into a single algebraic structure over γ relative to s. In other words, (γ, ξ_1) and (γ, ξ_2) are "independent/orthogonal" in the sense that an interaction of both algebraic effects does not distort any of the two structures.

In accordance with the fact that "topoi are adhesive" [21], coherence is trivially satisfied for pushouts with at least one monomorphism involved (compare Theorem 28).

Corollary 24 (Pushout along mono implies Coherence) *For any pushout square as in the bottom of Figure 14 with a or r monic $\langle U^{\overline{r}}, U^{\overline{a}}\rangle$ becomes an isomorphism between categories.*

Proof: In case a (or r) monic, we have also $\overline{a}$ (or $\overline{r}$) monic since pushouts preserve monomorphisms. According to Corollary 21 this means that U^a and $U^{\overline{a}}$ (or U^r and $U^{\overline{r}}$) are isomorphisms thus the outer diamond in the figure in Definition 23 becomes a pullback. Thus $\langle U^{\overline{r}}, U^{\overline{a}}\rangle$ is the canonical isomorphism. □

5.3 Amalgamation vs. Coherence in Topoi

The complete picture so far is depicted in Figure 18, in which all squares commute except front and left face involving Ψ. According to Proposition 15 these squares commute only up to the co-unit of the adjunction $\Psi^a \dashv \Phi^a$ and $\Psi^r \dashv \Phi^r$, respectively.

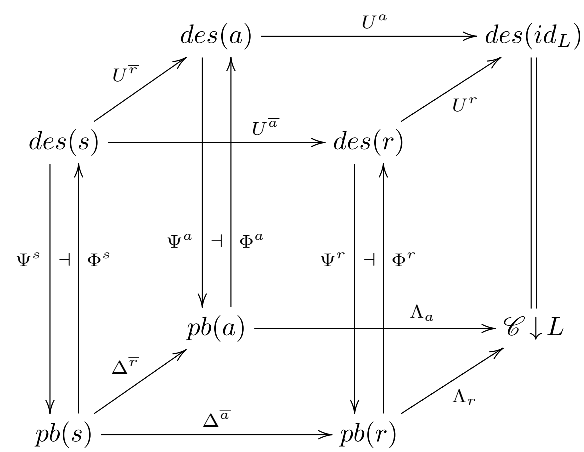


Figure 18: Amalgamation and Coherence

We fix the rear pullback span $((\gamma, a', \tau), (\gamma, r', \beta))$ in Figure 14 and consider the associated descent data:

$$(\gamma, \xi^{(a',\tau)}) = \Phi^a(\gamma, a', \tau) \quad \text{and} \quad (\gamma, \xi^{(r',\beta)}) = \Phi^r(\gamma, r', \beta).$$

First, we show that amalgamability entails coherence for arbitrary commutative squares.

Proposition 25 (Amalgamation entails Coherence) *Let in a topos $\mathscr{C}$ a diagram be given like the solid arrows in Figure 14 where the bottom square is commutative. If a span $((\gamma, a', \tau), (\gamma, r', \beta))$ of pullbacks is amalgamable then $(\gamma, \xi^{(a',\tau)})$ and $(\gamma, \xi^{(r',\beta)})$ are coherent.*

Proof: By assumption there exists a pullback $(\gamma, s', \sigma) \in pb(s)$ with

$$\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle (\gamma, s', \sigma) \cong ((\gamma, a', \tau), (\gamma, r', \beta))$$

(this is Figure 14 without question marks). The isomorphism from $\Delta^{\overline{r}}(\gamma, s', \sigma)$ to (γ, a', τ) has the form (id_γ, i) such that $\Phi^a(\gamma, a', \tau) = (\Phi^a \circ \Delta^{\overline{r}})(\gamma, s', \sigma)$ by (10) and the definition of Φ^a. Analogously, we obtain $\Phi^r(\gamma, r', \beta) = (\Phi^r \circ \Delta^{\overline{a}})(\gamma, s', \sigma)$. Hence, by Lemma 22 with $\overline{a} : r \to s$ and $\overline{r} : a \to s$

$$((\gamma, \xi^{(a',\tau)}), (\gamma, \xi^{(r',\beta)})) = \langle U^{\overline{r}}, U^{\overline{a}} \rangle \Phi^s(\gamma, s', \sigma) = \langle U^{\overline{r}}, U^{\overline{a}} \rangle (\gamma, \xi^{(s',\sigma)})$$

which is coherence with witness $(\gamma, \xi^{(s',\sigma)})$. □

To ensure that – vice versa – coherence entails amalgamation we have to require not only a commutative but a pushout square. Descent data represent (up to isomorphism) only the epi-part of a morphism and due to Corollary 24 coherence concerns only the interaction between the epi-parts of two morphisms. To put this observation into work and to take advantage of the fact that "topoi are adhesive" [21], we first state an auxiliary result which allows us to decompose any pushout square into epi-mono tiles.

Lemma 26 *Let $\mathscr{C}$ be a topos. Let $a = a_m \circ a_e$ and $r = r_m \circ r_e$ be epi-mono-factorized (cf. Lemma 14). Any pushout of a and r can be divided into pushout quarters, as shown in Figure 19. In this way each involved arrow of the original pushout is factorized. Moreover, the diagonal arrows s_m and s_e are an epi-mono-factorization of $s = \overline{a} \circ r = \overline{r} \circ a$.*

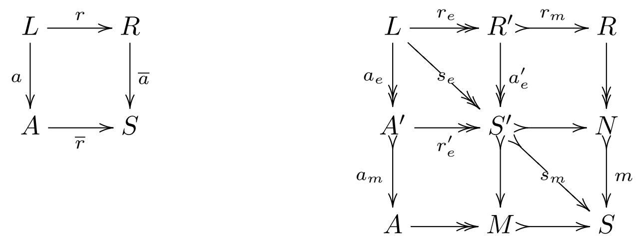


Figure 19: Division into epi-mono tiles

Proof: One starts with the pushout of a_e and r together with the mediator $m : N \to S$ from this inner square into the outer pushout. m is then part of a second rectangle which is a pushout by the decomposition property of pushouts. The two resulting rectangles can both be split up in the same way yielding four squares, in which each newly created arrow opposite to an epimorphism / a monomorphism is an epimorphism / a monomorphism again, because pushouts preserve epis and monos. Hence, $s = s_m \circ s_e$ is an epi-mono-factorization of s. ☐

Proposition 27 (Coherence entails Amalgamation) *Let in a topos $\mathscr{C}$ a diagram be given as the solid arrows in Figure 14 where the bottom square is a pushout. Then any pullback span $((\gamma, a', \tau), (\gamma, r', \beta))$ is amalgamable if $(\gamma, \xi^{(a',\tau)}) = \Phi^a(\gamma, a', \tau)$ and $(\gamma, \xi^{(r',\beta)}) = \Phi^r(\gamma, r', \beta)$ are coherent.*

Moreover, the coherence witness for $(\gamma, \xi^{(a',\tau)})$ and $(\gamma, \xi^{(r',\beta)})$ is unique.

Proof: Let $(\gamma, \xi) \in des(s)$ be a coherence witness. Assume first, we already knew the result for arbitrary pushouts of epimorphisms. We consider then the decomposition of the pushout into tiles according to Lemma 26, and decompose both pullbacks (γ, a', τ) and (γ, r', β). We obtain two "inner" pullbacks $\Delta^{a_m}(\gamma, a', \tau)$ and $\Delta^{r_m}(\gamma, r', \beta)$ along a_e and r_e, respectively, and for the corresponding descent data $\Phi^{a_e}(\Delta^{a_m}(\gamma, a', \tau))$ and $\Phi^{r_e}(\Delta^{r_m}(\gamma, r', \beta))$ we have

$$
\begin{aligned}
&(\Phi^{a_e}(\Delta^{a_m}(\gamma, a', \tau)), \Phi^{r_e}(\Delta^{r_m}(\gamma, r', \beta))) \\
&= ((U^{a_m} \circ \Phi^a)(\gamma, a', \tau), (U^{r_m} \circ \Phi^r)(\gamma, r', \beta)) && \text{Lemma 22} \\
&= (U^{a_m}(\gamma, \xi^{(a',\tau)}), U^{r_m}(\gamma, \xi^{(r',\beta)})) && \text{Definition of } \Phi^p \\
&= ((U^{a_m} \circ U^{\overline{r}})(\gamma, \xi), (U^{r_m} \circ U^{\overline{a}})(\gamma, \xi)) && \text{Coherence assumption} \\
&= (U^{\overline{r} \circ a_m}(\gamma, \xi), U^{\overline{a} \circ r_m}(\gamma, \xi)) && (27) \\
&= (U^{s_m \circ r'_e}(\gamma, \xi), U^{s_m \circ a'_e}(\gamma, \xi)) && \text{Lemma 26} \\
&= \langle U^{r'_e}, U^{a'_e} \rangle (U^{s_m}(\gamma, \xi)) && (27)
\end{aligned}
$$

thus the descent data of these "inner" pullbacks are also coherent with coherence witness $U^{s_m}(\gamma, \xi)$. By assumption, the inner pullback span $(\Delta^{a_m}(\gamma, a', \tau), \Delta^{r_m}(\gamma, r', \beta))$ is amalgamable. Since topoi are adhesive [21], this amalgamation can be continued along the other pairs of bottom arrows (of which either one or both are now monic) by constructing top face pushouts each time. All side face pullbacks compose to 4 side pullbacks of the whole cube, which amalgamates the original pullback span.

Now, we assume that r and a are epic. For the coherence witness $(\gamma, \xi) \in des(s)$ we can construct the pullback span $\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle (\Psi^s(\gamma, \xi))$ and we only need to show that this pullback span is related to the original one by an isomorphism of the form (m_1, id_γ, m_3) (cf. Definition 17): By coherence assumption, we have $(\gamma, \xi^{(a',\tau)}) = U^{\overline{r}}(\gamma, \xi)$ thus the definition of Φ^a, Proposition 15 (a) and Lemma 22 give us

$$\Phi^a(\gamma, a', \tau) = U^{\overline{r}}(\gamma, \xi) = (U^{\overline{r}} \circ \Phi^s \circ \Psi^s)(\gamma, \xi) = (\Phi^a \circ \Delta^{\overline{r}} \circ \Psi^s)(\gamma, \xi). \tag{28}$$

Since a is epic, the co-unit of the adjunction $\Psi^a \dashv \Phi^a$ is a natural isomorphism by Proposition 15 (b). Applying Ψ^a to (28) yields a composite isomorphism of the required form from $\Delta^{\overline{r}}(\Psi^s(\gamma, \xi))$ to (γ, a', τ):

$$(id_\gamma, m_1) := \varepsilon_{(\gamma, a', \tau)} \circ \varepsilon^{-1}_{\Delta^{\overline{r}}(\Psi^s(\gamma, \xi))}.$$

The isomorphism (id_γ, m_3) from $\Delta^{\overline{a}}(\Psi^s(\gamma, \xi))$ to (γ, r', β) is obtained analogously.

Uniqueness of witness: Assume there are two coherence witnesses $(\gamma, \xi), (\gamma, \xi')$. As shown above, $U^{s_m}(\gamma, \xi)$ and $U^{s_m}(\gamma, \xi')$ are then coherence witnesses for the pullback span reduced to the epi-parts a_e and r_e. U^{s_m} is, according to Corollary 21, an isomorphism thus it suffices to restrict again to the case where a and r are both epic.

From the construction above, we obtain two cubes each of which possess 4 pullbacks as side faces with diagonal pullbacks $\Psi^s(\gamma, \xi)$ and $\Psi^s(\gamma, \xi')$. The cubes possess the same arrows except for $\overline{a}', \overline{r}'$, and σ, cf. Figure 14. But the two variants of the arrows $\overline{a}', \overline{r}'$ both form a top pushout of a' and r' because, in topoi, pullbacks (along σ) preserve colimits. Moreover, the two variants of σ are mediators out of these pushouts with the appropriate universal property. The mediating isomorphism between the two pushouts can then be shown to mediate between the two variants of σ. Hence

$$\Psi^s(\gamma, \xi) \cong \Psi^s(\gamma, \xi')$$

with an isomorphism of the form (id_γ, i), such that $(\gamma, \xi) = (\Phi^s \circ \Psi^s)(\gamma, \xi) = (\Phi^s \circ \Psi^s)(\gamma, \xi') = (\gamma, \xi')$ by Proposition 15 (a) and (10). □

This proposition is a significant step towards an answer to Question 2. It characterizes, in terms of local data, those situations where instances can be amalgamated. It is, however, still too abstract in the sense that it does not provide an algorithm which tests for amalgamability. In the slightly more special setting of presheaves, however, we can provide such a procedure, see Section 6, where we will also recall the introductory examples from Section 2 to elucidate the practical importance of the theoretical results gained so far.

Question 1 addresses the *global universal* view on amalgamation. An answer to it could have been given in terms of the involved categories of pullbacks: Amalgamation is always possible and hence the bottom square in Figure 14 is Van Kampen iff the functor $\langle \Delta^{\overline{r}}, \Delta^{\overline{a}} \rangle$ establishes an equivalence between the categories $pb(s)$ and $pb(a) \times_{\mathscr{C}\downarrow L} pb(r)$. Although the inner diamond in the figure in Definition 17 is a pullback in CAT, this still does not mean that the outer diamond, which is also the bottom square in Figure 18, is a pullback. It is only a kind of a "pullback up to equivalence". Hence, in order to obtain a global view on amalgamation in terms of these pullback categories one would have to define such a concept formally which seems to be inappropriate or even impossible.

Considering, however, descent data instead of pullbacks, we get astonishingly "compositionality on the nose" as we know it from and use it intensively in traditional specification formalisms with indexed semantics [9, 10, 12]. Thus the following theorem provides a better answer to Question 1.

Theorem 28 (Descent Version of Fibred Amalgamation Lemma) *Let $\mathscr{C}$ be a topos. In Figure 20, the pushout (1) is a Van Kampen square if and only if (2) is a pullback in CAT.*

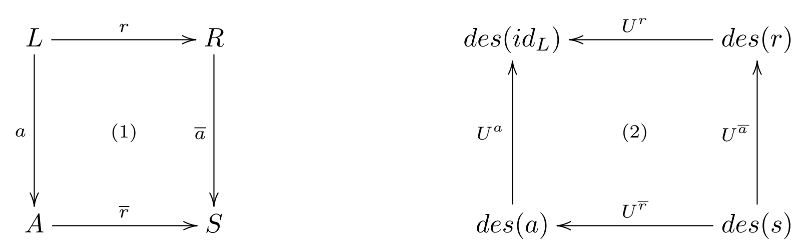


Figure 20: Amalgamation Lemma (Fibred setting)

Proof: Square (2) commutes by (27). Thus it is a pullback iff the functor $\langle U^{\overline{r}}, U^{\overline{a}} \rangle : des(s) \to des(a) \times_{des(id_L)} des(r)$ is an isomorphism, cf. the figure in Definition 23.

"⇒": Bijectivity of $\langle U^{\overline{r}}, U^{\overline{a}} \rangle$ on objects follows from coherence (by Propositions 18, 25), which is surjectivity, and uniqueness of coherence witness (Proposition 27), which is injectivity.

The definition of U induces injectivity on arrows (cf. Lemma 20) and it induces surjectivity, if we can show that any arrow h of $des(a) \times_{des(id_L)} des(r)$ is also an arrow of $des(s)$. But this follows from the fact that the arrow $(\Psi^a h, h, \Psi^r h)$ between two amalgamable pullback spans yields the existence of an arrow $(h, \hat{h})$ in $pb(s)$ by (25). Thus $h = \Phi^s(h, \hat{h}) \in Mor_{des(s)}$.

"⇐" $\langle U^{\overline{r}}, U^{\overline{a}} \rangle$ an isomorphism means that all the pairs in $des(a) \times_{des(id_L)} des(r)$ are coherent thus this direction follows directly from Propositions 27 and 18. □

6 Amalgamation and Van Kampen Squares in Presheaves

This section illustrates the use of Propositions 25/27 and develops a simply checkable characterization of amalgamable pullback spans as well as a new characterization of Van Kampen squares in presheaves. In the sequel $\mathscr{C} = SET^{\mathscr{S}}$ and, again, we use the more intuitive notations F_X and op^F for the application of a functor $F : \mathscr{S} \to SET$ to an object X and an operation symbol op, resp.

6.1 General analysis

In $SET^{\mathscr{S}}$ the monad embeddings $u^{\overline{a}} : \mathfrak{T}^r \to \mathfrak{T}^s$ and $u^{\overline{r}} : \mathfrak{T}^a \to \mathfrak{T}^s$ are componentwise inclusions, as discussed in Section 5.2. Hence coherence (cf. Definition 23) means the existence of descent data ξ relative to $s = \overline{a} \circ r = \overline{r} \circ a$ with

$$\forall (x, x') \in ker(r_X) : \xi_{x,x'} = \xi^{\beta}_{x,x'} \text{ and } \forall (y, y') \in ker(a_X) : \xi_{y,y'} = \xi^{\tau}_{y,y'} \tag{29}$$

for all $X \in Ob_{\mathscr{S}}$, where all mappings are understood as the components of the families of bijections from Proposition 16. Note, that we use throughout this section ξ^{β} instead of $\xi^{(r',\beta)}$ and ξ^{τ} instead of $\xi^{(a',\tau)}$, respectively.

We observe Propositions 25 and 27 at work: Recall the situation in Figure 3 (here $\mathscr{S} = 1$). By Proposition 16, the associated descent data ξ^{β} and ξ^{τ} map within I along the dashed and dotted lines, resp. E.g. $\xi^{\tau}_{x,z}(1\!:\!x) = 2\!:\!z, \xi^{\tau}_{x,z}(2\!:\!x) = 1\!:\!z$. Amalgamation is possible, if we can determine an equivalence relation $\langle \xi, \pi_2(s, s \circ \gamma) \rangle$ in which the restriction of ξ to the kernels of a' and r' coincide with ξ^{τ} and ξ^{β}, resp., and which is monadic. However, associativity can not be fulfilled since

$$\xi_{x,z} = \xi^{\tau}_{x,z} \neq \xi^{\beta}_{w,z} \circ \xi^{\tau}_{y,w} \circ \xi^{\beta}_{x,y} = \xi_{w,z} \circ \xi_{y,w} \circ \xi_{x,y}$$

on the fibre over x, see Figure 3. This shows that Proposition 25 provides indeed an a priori test for the failure of amalgamability.

Obviously, in the example, the kernels of r and a, are intertwined through the cycle (x, z),(z, w), (w, y), $(y, x) \in ker(s)$ and are thus not enough "separated".

Definition 29 (Separated Kernels) *Let $X \in Ob_{\mathscr{S}}$ and a and r be given as in Figure 14. A sequence $(x_i)_{i\in\{0,1,...,2k+1\}}$ of elements in L_X is called an X-*domain cycle *of a_X and r_X (or just* domain cycle *of a and r, if the carrier is fixed), if $k \in \mathbb{N}$ and the following conditions hold:*

1. $\forall j \in \{0, 1, \dots, 2k + 1\} : x_j \neq x_{j+1}$
2. $\forall i \in \{0, \dots, k\} : (x_{2i}, x_{2i+1}) \in ker(a_X)$
3. $\forall i \in \{0, \dots, k\} : (x_{2i+1}, x_{2i+2}) \in ker(r_X)$

where the sums are understood modulo $2k + 2$ (i.e. $x_{2k+2} = x_0$). We call $2k + 2$ the length *of the domain cycle. Moreover, a domain cycle is* proper *if we have for all $i, j \in \{0, 1, \dots, 2k + 1\}$ that $x_i \neq x_j$ if $i \neq j$.*

The pair a and r is said to have separated kernels*, if it has no domain cycle.*

We draw attention to the fact that "having separated kernels" is not sufficient but only necessary for "being jointly monic". Indeed, being jointly monic induces a domain cycle of length 2. But longer domain cycles occur for jointly monic a and r (see Figure 3).

Domain cycles are connected to coherence as follows:

Theorem 30 *Let a commutative square be given like the bottom square in Figure 14 and let the two rear faces be pullbacks with associated descent data ξ^{τ} and ξ^{β}, resp.*

ξ^{τ} and ξ^{β} are coherent iff for all proper domain cycles $(x_i)_{i\in\{0,1,...,2k+1\}}$ of a and r we have

$$\xi^{\beta}_{x_{2k+1},x_0} \circ \xi^{\tau}_{x_{2k},x_{2k+1}} \circ \cdots \circ \xi^{\tau}_{x_2,x_3} \circ \xi^{\beta}_{x_1,x_2} \circ \xi^{\tau}_{x_0,x_1} = id_{\gamma^{-1}(x_0)} \tag{30}$$

The statement is illustrated in Example 3, where coherence is now achieved by harmonizing the equivalences of a' and r' in the two copies of L that make up the domain of γ. Alternatively, we can use Proposition 27 to check amalgamability: A coherence witness can be constructed by taking the transitive closure of the union of the two equivalence relations arising from ξ^{τ} and ξ^{β} (cf. (23)).

Based on this observation, Theorem 30 provides a feasible criterion to check amalgamability *without* computing explicitly a coherence witness or trying to complete the cube with pullbacks and thus it is an answer to Question 2. Although the proof is a bit technical, we include it here, because it sheds light on the set-theoretical correlations of descent theory. We first need:

Definition 31 (Alternating Sequence) *Let $X \in Ob_{\mathscr{S}}$. We call a sequence $(y_i)_{i\in\{0,1,\ldots,m\}}$ of elements in L_X an (X-)*alternating sequence *(of a and r), if $m \in \mathbb{N}$ and the following conditions hold:*

a) for all even $i \in \{0,\ldots,m-1\} : (y_i, y_{i+1}) \in ker(p_X)$

b) for all odd $i \in \{0,\ldots,m-1\} : (y_i, y_{i+1}) \in ker(-p_X)$

where $p \in \{a,r\}$ and $-a = r$ and $-r = a$. A sequence is called proper *if $y_i \neq y_j$ for all $i \in \{0,1,\cdots,m\}$ and $j \in \{1,2,\cdots,m-1\}$ with $i \neq j$.*[3]

Proof of Theorem 30: ($\Rightarrow$) follows immediately from (29) and Proposition 16 applied to the coherence witness ξ.

($\Leftarrow$): By Proposition 16 the desired coherence witness ξ is given by a family

$$((\xi_X)_{x,x'} : \gamma^{-1}(x) \to \gamma^{-1}(x'))_{(x,x')\in ker(s)}$$

of bijections on fibres in each carrier set I_X. It is well-known that pushouts in $SET^{\mathscr{S}}$ are constructed componentwise by pushouts in SET. Thus $(x,x') \in ker(s_X)$ iff there exists an alternating sequence $\sigma = (y_i)_{i\in\{0,1,\ldots,m\}}$ with $y_0 = x$ and $y_m = x'$. Since ξ must obey the restriction property (29), we must define $\xi_{x,x'} := \xi_\sigma : \gamma^{-1}(x) \to \gamma^{-1}(x')$ by

$$\xi_\sigma := \xi^{-!}_{y_{m-1},y_m} \circ \cdots \circ \xi^{-p!}_{y_1,y_2} \circ \xi^{p!}_{y_0,y_1} \tag{31}$$

(where $a! = \tau$ and $r! = \beta$). For $m = 0$ this reduces to $\xi_\sigma = id_{\gamma^{-1}(y_0)}$.

Note, that any domain cycle $c = (x_i)_{i\in\{0,1,\ldots,2k+1\}}$ gives rise to two alternating sequences connecting x_0 with x_0, namely $\sigma_c = (x_0, x_1, \ldots, x_{2k+1}, x_0)$ and $\sigma'_c = (x_0)$, thus condition (30) in Theorem 30 can be rewritten as $\xi_{\sigma_c} = \xi_{\sigma'_c}$.

In generalizing this observation, it can be shown by assumption and induction that condition (30) is equivalent to the requirement that $\xi_\sigma = \xi_{\sigma'}$ for arbitrary x and x' and any two alternating sequences σ and σ' connecting x and x' , see Lemma 39 in the appendix. This makes definition (31) independent of the choice of sequence.

It remains to show validity of neutrality, associativity as well as compatibility with operation symbols (cf. Proposition 16). Neutrality follows from (31) for $m = 0$. To show associativity we define the composition of two alternating sequences by

- $\sigma' \circ \sigma := (y_0, \ldots, y_m = z_0, \ldots, z_n)$ if $mn = 0$ or $m, n \geq 1$ and $(y_{m-1}, y_m) \in ker(p)$, $(z_0, z_1) \in ker(-p)$
- $\sigma' \circ \sigma := (y_0, \ldots, y_{m-1}, z_1, \ldots, z_n)$ if $m, n \geq 1$ and $(y_{m-1}, y_m) \in ker(p)$, $(z_0, z_1) \in ker(p)$

By independence of representative we obtain for each pair $(x,x'), (x',x'') \in ker(s_X)$ (with representing alternating sequences σ, σ'): $\xi_{\sigma'} \circ \xi_\sigma = \xi_{\sigma'\circ\sigma}$, hence associativity.

In order to show compatibility with operation symbols, let $op : X \to Y$ be any arrow of $\mathscr{S}$. Let $(x,x') \in ker(s_X)$ and $\sigma = (y_i)_{i\in\{0,1,\ldots,m\}}$ with $y_0 = x$ and $y_m = x'$ be an associated alternating sequence. Clearly, $op^L(\sigma) := (op^L(y_i))_{i\in\{0,1,\ldots,m\}}$ is an alternating sequence for $(op^L(x), op^L(x')) \in ker(s_Y)$. Then from the definitions of $((\xi_X)_{x,x'})_{(x,x')\in ker(s_X)}$ and $((\xi_Y)_{y,y'})_{(y,y')\in ker(s_Y)}$ in (31) as well as the compatibility of ξ^τ and ξ^β (cf. Proposition 16) we have

$$\begin{aligned}
&(\xi_Y)_{op^L(x),op^L(x')} \circ op^I_x \\
&= (\xi_Y)_{op^L(\sigma)} \circ op^I_x \\
&= (\xi^{-!}_Y)_{op^L(y_{m-1}),op^L(y_m)} \circ \cdots \circ (\xi^{-p!}_Y)_{op^L(y_1),op^L(y_2)} \circ (\xi^{p!}_Y)_{op^L(y_0),op^L(y_1)} \circ op^I_x \\
&= op^I_{x'} \circ (\xi^{-!}_X)_{y_{m-1},y_m} \circ \cdots \circ (\xi^{-p!}_X)_{y_1,y_2} \circ (\xi^{p!}_X)_{y_0,y_1} \\
&= op^I_{x'} \circ (\xi_X)_{x,x'}
\end{aligned}$$

□

Theorem 30 also tells us that the descent data of all pullback spans are coherent, if there are no domain cycles, i.e. if a and r have separated kernels. If the bottom square is a pushout, this means that having separated kernels implies successful amalgamation by Proposition 27. In the rest of this section we show that the converse also holds.

Proposition 32 *Let $\mathscr{C} = SET^{\mathscr{S}}$ and a commutative square be given like the bottom square in Figure 14. If all pullback spans in the rear are amalgamable, then a and r have separated kernels.*

[3] Thus, the only allowed equality is $y_0 = y_m$.

Proof: Assume to the contrary that a and r possess an X-domain cycle $(x_i)_{i\in\{0,1,\ldots,2k+1\}}$ for some $k \in \mathbb{N}$. By Lemma 38 in the appendix, we may assume that the cycle is already proper. We can define then a pullback span where amalgamation fails by defining instances $\tau : I^A \to A$, $\gamma : I^L \to L$, and $\beta : I^R \to R$ as follows: For $M \in \{A, L, R\}$ let

$$I_Y^M = \begin{cases} M_Y & \text{if} \quad Y \neq X \\ M_X^1 + M_X^2 + M_X^3 & \text{if} \quad Y = X \end{cases}$$

where $(M_X^i = \{(x,i) \mid x \in M_X\})_{i\in\{1,2,3\}}$ are three copies of M_X. For each arrow $op : Z \to Z'$ in $\mathscr{S}$, we define $op^{I^M} : I_Z^M \to I_{Z'}^M$ by

$$\begin{array}{ll}
op^{I^M} = op^M & \text{if} \quad Z \neq X \text{ and } Z' \neq X, \\
op^{I^M} : \begin{cases} M_Z \;\; \to \;\; M_X^1 + M_X^2 + M_X^3 \\ x \;\; \mapsto \;\; (op^M(x), 3) \end{cases} & \text{if} \quad Z \neq X \text{ and } Z' = X, \\
op^{I^M} : \begin{cases} M_X^1 + M_X^2 + M_X^3 \;\; \to \;\; M_{Z'} \\ (x,i) \;\; \mapsto \;\; op^M(x) \end{cases} & \text{if} \quad Z = X \text{ and } Z' \neq X, \\
op^{I^M} : \begin{cases} M_X^1 + M_X^2 + M_X^3 \;\; \to \;\; M_X^1 + M_X^2 + M_X^3 \\ (x,i) \;\; \mapsto \;\; (op^M(x), 3) \end{cases} & \text{if} \quad Z = X \text{ and } Z' = X.
\end{array}$$

Then we let $\tau_Y := id_{I_Y^A}$, if $Y \neq X$ and $\tau_X(a,i) := a$ for each $i \in \{1,2,3\}$. In the same way we define $\gamma : I^L \to L$ and $\beta : I^R \to R$, i.e. they are identical on the carriers of sort $Y \neq X$ and forget indices on the carrier of X. A straightforward calculation shows that τ, γ, β are natural transformations.

Elementary arguments (using pointwise pullback construction as mentioned in Section 4.2.3) show that defining $r' : I^L \to I^R$ by $r'_Y = r_Y$ if $Y \neq X$ and $r'_X(x,i) = (r_X(x), i)$ defines a natural transformation establishing a pullback square over r. The definition of r' furthermore yields

$$r'_X(x,i) = r'_X(y,j) \iff (i = j \text{ and } (x,y) \in ker(r_X)). \tag{32}$$

To establish the left rear square, we have to define $a' : I^L \to I^A$. As before, $a'_Y := a_Y$ for $Y \neq X$, but in order to create a situation where amalgamation fails, the definition introduces a "twist" in I_X^L as follows: Recall that there was a domain cycle in L_X which occurs threefold in I_X^L. Because $k \geq 0$ and the cycle is proper, there are at least $x_0 \neq x_1$ in the cycle for which $a_X(x_0) = a_X(x_1)$. Let $a'_X : L_X^1 + L_X^2 + L_X^3 \to A_X^1 + A_X^2 + A_X^3$ be defined by

$$a'_X(x,i) = \begin{cases} (a_X(x), i) & \text{if } x \notin \{x_0, x_1\} \text{ or } i = 3 \\ (a_X(x), i) & \text{if } i \neq 3 \text{ and } x = x_0 \\ (a_X(x), 3-i) & \text{if } i \neq 3 \text{ and } x = x_1 \end{cases} \tag{33}$$

This means that a' maps according to a on the fibres but interchanges the positions of the images of x_1 in the first two copies:

$$a'_X(x_0,1) = a'_X(x_1,2) \text{ and } a'_X(x_0,2) = a'_X(x_1,1) \tag{34}$$

whereas $a'_X(x,i) = a'_X(y,j) \iff (i = j \text{ and } (x,y) \in ker(r_X))$ whenever $x \notin \{x_0, x_1\}$ or $y \notin \{x_0, x_1\}$ or $i = 3$. We give a detailed proof in Section 8.6, why this special definition of a' still yields a pullback square over a.

By assumption, the constructed pullback span is amalgamable, hence by Proposition 25, the associated descent datas ξ^τ and ξ^β are coherent such that by Theorem 30

$$\xi^\beta_{x_{2k+1},x_0} \circ \xi^\tau_{x_{2k},x_{2k+1}} \circ \cdots \circ \xi^\tau_{x_2,x_3} \circ \xi^\beta_{x_1,x_2} \circ \xi^\tau_{x_0,x_1} = id_{\gamma^{-1}(x_0)}. \tag{35}$$

(34), (23), and the definition of γ, however, yield

$$\xi^\tau_{x_0,x_1}(x_0,1) = (x_1,2)$$

and

$$\xi^\tau_{x,x'}(x,i) = (x',i)$$

whenever $(x,x') \in ker(a)$ and $x \notin \{x_0,x_1\}$ or $x' \notin \{x_0,x_1\}$. Similarly $\xi^\beta_{x,x'}(x,i) = (x',i)$ for all $(x,x') \in ker(r)$ by (32). Thus

$$(\xi^\beta_{x_{2k+1},x_0} \circ \xi^\tau_{x_{2k},x_{2k+1}} \circ \cdots \circ \xi^\tau_{x_2,x_3} \circ \xi^\beta_{x_1,x_2} \circ \xi^\tau_{x_0,x_1})(x_0,1) = (x_0,2)$$

which contradicts (35). □

While Theorem 30 was a statement to check successful amalgamation *individually* for one given pullback span, we can summarize the other results with the following *total* statement, which simultaneously provides an answer to Question 1, i.e. a practical criterion to check the Van Kampen property.

Theorem 33 *Let $\mathscr{C} = SET^{\mathscr{S}}$ and the bottom square in Figure 14 be a pushout. The following conditions are equivalent:*

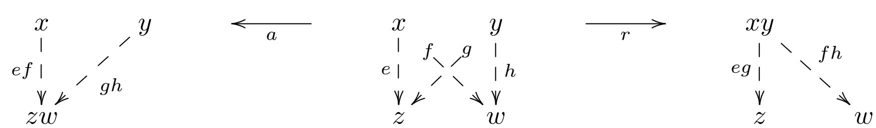
1. *The square is a Van Kampen square.*
2. *a and r do have separated kernels.*
3. *Each pullback span is amalgamable.*

Proof: 1 and 3 are equivalent by Proposition 18. 2⇒3 follows from Theorem 30 and 27. 3⇒2 is guaranteed by Proposition 32 □

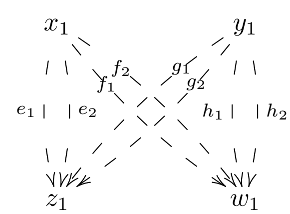
A computer scientist reasoning about the ability to merge instances of specifications along a common part may wish to encounter a Van Kampen square, because in this case, amalgamation is always possible. With the theorem above he can check this property in an algorithmic manner.

6.2 Examples (revisited)

Recall Example 4. The class diagram can simply be coded as a diagram of graphs:



There is the domain cycle $a(e) = a(f), r(f) = r(h), a(h) = a(g), r(g) = r(e)$ hence amalgamation may fail by Theorem 33. The proof of Proposition 32 gives a hint how to find an instance constellation where this happens: We just have to provide one object for each node and each time a three element set as fibre over an edge. In this example, in fact, two elements in each fibre over the edges are enough. The instance $\gamma : I \to L$ has domain



and a' may be defined to interchange edges: $a'(e_1) = a'(f_2), a'(e_2) = a'(f_1)$. If all other assignments respect indices, amalgamation fails.

Assume finally, identification of elements would be slightly different in Example 4. If (private) persons would possess only one type of contact info, i.e. if the specification is as in Fig. 21, the morphisms are still both non-injective with complicated entanglement, but it follows immediately from Theorem 33 that amalgamation is successful for each pullback span and that the square is a Van Kampen square.

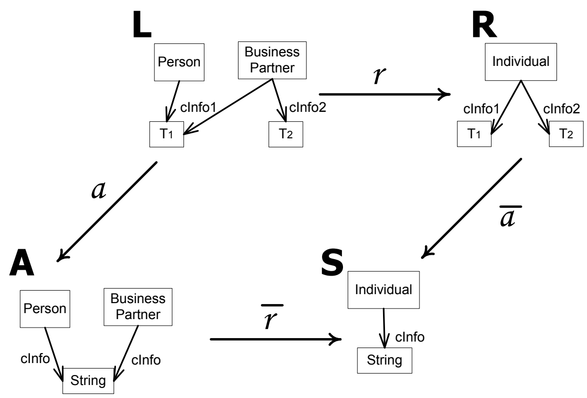


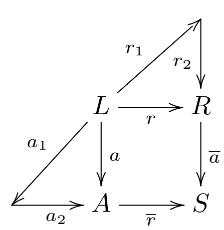
Figure 21: Compositionality holds

7 Outlook

The paper presents major outcomes of a comprehensive collaborative project based on [7, 23, 33] and addressing "compositionality of fibred semantics in topoi". There are several open problems and interesting topics for future research.

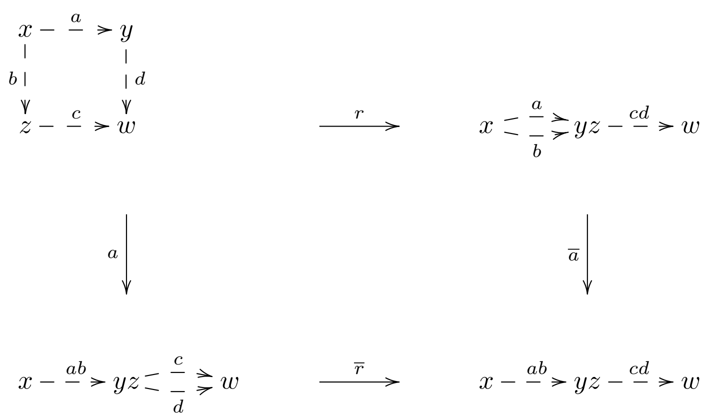
Categorical characterization of domain cycles In the light of Theorem 33 we can consider the Van Kampen property as a categorical characterization of separateness, i.e., of the absence of domain cycles. An open question is if there is a simpler and more feasable categorical characterization of domain cycles in terms of the span (a, r) only. We had a closer look at the following candidate for such a characterization: The pushout $(\overline{a}, \overline{r})$ of (a, r) is a Van Kampen square if for all factorizations $a = a_2 \circ a_1$ and $r = r_2 \circ r_1$ with a_1 and r_1 epic the following implicaton holds:

If $(\overline{a} \circ r_2, \overline{r} \circ a_2)$ is pushout of a_1 and r_1, then r_2, a_2 are isomorphisms.



The idea is that the square does not possess the VK property if the kernels of a and r are too much intertwined in L. So one tries to postpone a portion of the identification potential of both a and r in such a way that the intertwining is broken but still the pushout can be produced. One can show that this characterization works indeed for sets. But already for graphs it does not work since one has to be careful with identifications in the environment of domain cycles.

Example 34 *The pushout*



in the category $GRAPH$ is not a Van Kampen square (because nodes y and z constitute a domain cycle), but neither a nor r can be factorized as in the conjecture, because edge identification takes place in the vicinity of this cycle.

Conditional Van Kampen If we are not able to characterize the absence of domain cycles, in a feasable way, it may be worth to see if we can, instead, localize domain cycles categorically in arbitrary topoi.

A closer look at our results shows that Theorem 30 gives us actually for presheaves a kind of "conditional amalgamation": We consider the smallest subobject C of L containing all proper domain cycles and the corresponding inclusion morphism $i : C \to L$. Let $(\overline{r_C}, \overline{a_C})$ be the pushout of (a_C, r_C), with $a_C = a \circ i$ and $r_C = r \circ i$, and $(\overline{r}, \overline{a})$ be the pushout of (a, r) (see Fig. 22). By pulling back along i any pullback span over (a, r) reduces to a pullback span over (a_C, r_C) and Theorem 30 can be transformed into the statement: A pullback span over (a, r) is amalgamable, if the reduced pullback span over (a_C, r_C) is amalgamable. In case this implication is true for all pullback spans over (a, r), we say that the span (a, r) is "Van Kampen relative to C".

If there are no cycles, i.e., C is the (componentwise) empty set $\emptyset$ in $\mathscr{C} = SET^{\mathscr{S}}$, the reduced pullback span is always amalgamable, since $\mathscr{C} = SET^{\mathscr{S}}$ is extensive. That is, "Van Kampen relative to $\emptyset$" coincides with the traditional Van Kampen property. Any span (a, r) is trivially "Van Kampen relative to L" and the essence of Theorem 30 is that there exists for any span (a, r) in a presheaf a minimal object C of L such that (a, r) is "Van Kampen relative to C".

Our conjecture is that also in arbitrary topoi such a minimal condition C can be constructed for any span (a, r).

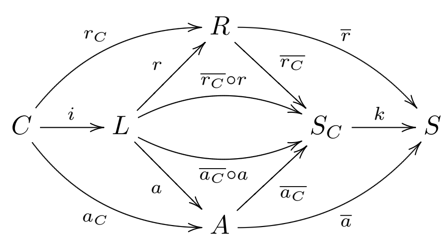


Figure 22: Conditional amalgamability

Full local compositionality As a matter of fact we can not assume "global identities" for our entities in programming and software engineering. This obstacle is reflected by the insight, presented in this paper, that compositionality of fibred semantics is only a local property relative to an arbitrary but fixed "context/base" L.

To reflect locality even better, we could describe the descent data functor $\mathcal{DD}_L$ as a functor from $(L \downarrow \mathscr{C})^{op}$ into $CAT \downarrow (\mathscr{C} \downarrow L)$. Taking such a viewpoint, Theorem 28 states that sums in $L \downarrow \mathscr{C}$ are mapped by the descent data functor into products in $CAT \downarrow (\mathscr{C} \downarrow L)$. With the machinery, developed in the paper, it should be possible to prove corresponding statements for arbitrary (finite) colimits in $L \downarrow \mathscr{C}$. We let this as a topic of future work.

Change of base To get a more complete picture of compositionality of fibred semantics we have to consider also the change of context/base. The situation for those changes should be the following: Any morphism $l : L' \to L$ induces, by pre-composition, a functor $\hat{l} : L \downarrow \mathscr{C} \to L' \downarrow \mathscr{C}$. Pulling back along l provides for each object $f : L \to F$ in $L \downarrow \mathscr{C}$ a functor from $des(f)$ into $des(\hat{l}(f)) = des(f \circ l)$, And, the collection of these functors constitutes a natural transformation from $\mathcal{DD}_L$ to $\mathcal{DD}_{L'} \circ \hat{l}$. In case, l monic, this natural transformation has local left-adjoints. We presume that these local left-adjoints will be useful to prove our conjecture concerning conditional Van Kampen in arbitrary topoi.

Sketches Finally, an interesting topic for future research are more general categories. We could, for example, extend our "meta-schema" categories to finite product sketches, to cover algebraic operations and equations, or even to finite limit sketches, to cover conditional equations.

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8 Appendix

This section lists additional technical calculations and proofs. They are either well-known (and are integrated due to completeness) or necessary routine calculations to ensure correctness of the results in the main part.

8.1 Adjointness of Pullback Functor

For the purpose of self-containedness, we include here a proof of Lemma 5 (one of the main results used throughout this report). Again, we use the adjointness characterization of [24] as in Section 4, i.e. $p_* \dashv p^*$, iff there exist natural transformations $\eta : id_{\mathscr{C}\downarrow E} \Rightarrow p^* \circ p_*$ (unit) and $\varepsilon : p_* \circ p^* \Rightarrow id_{\mathscr{C}\downarrow B}$ (co-unit), such that

$$p^*\varepsilon \circ \eta p^* = id_{p^*} \quad \text{and} \quad \varepsilon p_* \circ p_*\eta = id_{p_*}. \tag{36}$$

Proof of Lemma 5 (cf. [13], p. 16): If $h : (\,C \xrightarrow{\gamma} E\,) \to (\,C' \xrightarrow{\gamma'} E\,) \in Mor_{\mathscr{C}\downarrow E}$, we have $p^*p_*h = id_E \times_B h$ by (2). Thus naturality of $(\eta_\gamma)_{\gamma \in \mathscr{C}\downarrow E}$ follows from

$$(id_E \times_B h) \circ \langle \gamma, id_C \rangle = \langle \gamma', id_{C'} \rangle \circ h$$

(by uniqueness of mediators). Naturality of $(\varepsilon_\alpha)_{\alpha \in \mathscr{C}\downarrow B}$ is a consequence of

$$\pi_2(p, \alpha') \circ (id_E \times_B f) = f \circ \pi_2(p, \alpha)$$

for any $f \in Mor_{\mathscr{C}\downarrow B}(\alpha, \alpha')$ (by (2)) and the fact that also $id_E \times_B f = p_*p^*f$.

The second statement in (36) follows directly from the definition of η_γ as unique mediator, whereas the first statement is true because

$$\begin{aligned}
p^*\varepsilon_\alpha \circ \eta_{p^*\alpha} &= (id_E \times_B \pi_2(p,\alpha)) \circ \langle p^*\alpha, id_{E\times_B A} \rangle && \text{By (2) and the def. of } \eta \text{ and } \varepsilon \\
&= (id_E \times_B \pi_2(p,\alpha)) \circ \langle \pi_1(p,\alpha), id_{E\times_B A} \rangle && \text{Remarks before Lemma 5} \\
&= \langle \pi_1(p,\alpha), \pi_2(p,\alpha) \rangle = id_{E\times_B A}
\end{aligned}$$

□

8.2 Epi-Mono-Factorizations

Altough the following result is well-known in topos theory we give a proof here because we need some further details in the context of the result (a similar proof using dual arguments can be found in [14]). It is based on a property, which we mentioned in the introduction: In topoi epimorphisms are regular, i.e. each epimorphism coequalizes some parallel pair.

Lemma 35 *Let $\mathscr{C}$ be a topos and $f : A \to B$. The coequalizer c of the kernel pair (p_1, p_2) of f is the largest quotient that factors through f, i.e. if m is the mediator of f out of the coequalizer and $f = u \circ e$ with epic e, then there is a unique (necessarily epic) t, such that $t \circ e = c$ and $m \circ t = u$, cf. Figure 23.*

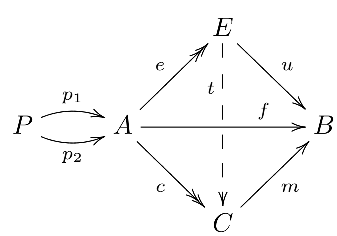


Figure 23: Towards epi-mono factorization

Proof: Let the epimorphism e coequalize the parallel pair (q_1, q_2). Hence we obtain $f \circ q_1 = f \circ q_2$. Because the kernel pair diagram for f is a pullback there is a unique h such that $p_1 \circ h = q_1$ and $p_2 \circ h = q_2$. But then

$$c \circ q_1 = c \circ q_2$$

which yields the required t out of the coequalizer e with $t \circ e = c$. Moreover, $m \circ t \circ e = m \circ c = f = u \circ e$ such that $m \circ t = u$ because e is epic. □

Lemma 36 *If $p_1 = p_2$ in Figure 23, then f is monic.*

Proof: If $f \circ x = f \circ y$, then there is a unique m such that $x = p_1 \circ m = p_2 \circ m = y$. □

Proof of Lemma 14: Let k_1, k_2 be the kernel pair of m. Then the coequalizer c' of this pair yields a factorization of $m = v \circ c'$ with epic c' as in Lemma 35. But then $f = m \circ c = v \circ (c' \circ c)$ is a factorization of f for which, by Lemma 35, there is a unique t such that $t \circ (c' \circ c) = c$, thus

$$t \circ c' = id. \tag{37}$$

Hence we obtain $k_1 = t \circ c' \circ k_1 = t \circ c' \circ k_2$ because c' was the coequalizer of k_1 and k_2. Using (37) again yields $k_1 = k_2$, hence m is monic by Lemma 36.

Given any epi-mono-factorization $f = u \circ e$ the mediator t from Lemma 35 is monic because $m \circ t = u$. In topoi, it is then an isomorphism [14]. Then uniqueness up to unique iso follows from the composition of two mediators t_1 and t_2 out of two epi-mono-factorizations. □

8.3 Towards a transition from monadic descent to families of fibre assignments

In this section we investigate exponents in $SET \downarrow I$ for some fixed set I. Let $f : A \to I$ and $g : B \to I$ be two objects in $SET \downarrow I$. For $i \in I$ define $A_i = f^{-1}(\{i\})$,

$$D_i = \{k : A_i \to B \mid g \circ k = i\}$$

where i is also the constant function from A_i to I mapping everything to i. Furthermore let

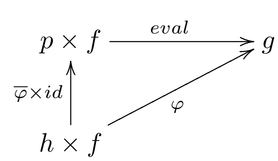
$$D := \biguplus_{i \in I} D_i$$

and $p : D \to I$ defined by $p(k : A_i \to B) = i$. We claim that p is the exponent of g and f, i.e. $p = g^f$.

A sketch of the proof of that claim is as follows: The diagonal arrow in the pullback of p and f is

$$p \times f : \{(k, a) \mid k : A_i \to B, f(a) = i\} \to I$$

for which e.g. $(p \times f)(k, a) = f(a)$. Since $eval(k, a) := k(a)$ is defined on the domain of $p \times f$, this yields $g \circ eval = p \times f$ such that $eval : p \times f \to g$ is an arrow of $SET \downarrow I$. It remains to show that for each $\varphi : h \times f \to g$, there is a unique $\overline{\varphi} : h \to p$ such that the diagram



commutes. But this is analogue to "currying" in SET: If $h : C \to I$ and $c \in C$, let $k : A_{h(c)} \to B$ be defined by

$$k(a) = \varphi(c, a).$$

The assignment $c \mapsto k$ induces a map $\overline{\varphi} : h \to p$ because, by definition, $p \circ \overline{\varphi}(c) = h(c)$. Moreover the definition of $eval$ easily yields that $\overline{\varphi}$ is the unique arrow making the above diagram commute.

We can summarize these considerations by saying that the domain of g^f is the set of all maps k defined on a fibre of f over some i for which $g \circ k$ is the constant i. g^f maps any such $k : A_i \to B$ to its base point i. Taking $g = f$, we obtain

Lemma 37 *For any $f \in SET \downarrow I$, the exponent f^f is defined on the set of all endomaps $k : f^{-1}(\{i\}) \to f^{-1}(\{i\})$. Each k is mapped to its base point i by f^f.* □

8.4 The Monad Embedding

In this section we give the proof details of Lemma 19. For simplicity we write u instead of u^h. There are several statements to prove:

1. Each u_γ is a monomorphism.
2. $u : \mathcal{T}^f \Rightarrow \mathcal{T}^g$ is a natural transformation.
3. u is compatible with units, i.e. $u \circ \eta^f = \eta^g$.
4. u is compatible with co-units, i.e. $\mu^g \circ u^2 = u \circ \mu^f$ where u^2 is the horizontal composition of u with itself.

1. To show that u_γ is monic for each γ, let $x, y : X \to L \times_B I$ with $u_\gamma \circ x = u_\gamma \circ y$ be given. By (26), one computes $\mathcal{T}^f\gamma \circ x = \mathcal{T}^f\gamma \circ y$ and $\pi_2(f, f \circ \gamma) \circ x = \pi_2(f, f \circ \gamma) \circ y$. Because $\mathcal{T}^f\gamma$ and $\pi_2(f, f \circ \gamma)$ are jointly monic (being a limit cone in a pullback square), we obtain $x = y$. In the sequel, the property of a pullback cone to be jointly monic will be used several times. We will do this without further reference.

2. Let $\gamma, \hat{\gamma} \in \mathscr{C}\downarrow L$ and

$$\varphi : \gamma \to \hat{\gamma}$$

be a $\mathscr{C}\downarrow L$-morphism. We let $\pi_2 := \pi_2(f, f \circ \gamma)$, $\pi_2' := \pi_2(g, g \circ \gamma)$, $\hat{\pi}_2 := \pi_2(f, f \circ \hat{\gamma})$, $\hat{\pi}_2' := \pi_2(g, g \circ \hat{\gamma})$ denote the second projections in the pullbacks involving γ and $\hat{\gamma}$.

Pulling back φ (as an arrow in $\mathscr{C}\downarrow F$ and as an arrow in $\mathscr{C}\downarrow G$) yields

$$\varphi \circ \pi_2 = \hat{\pi}_2 \circ \mathcal{T}^f\varphi \tag{38}$$

and

$$\varphi \circ \pi_2' = \hat{\pi}_2' \circ \mathcal{T}^g\varphi. \tag{39}$$

Let now $d_1 = \mathcal{T}^g\varphi \circ u_\gamma$ and $d_2 = u_{\hat{\gamma}} \circ \mathcal{T}^f\varphi$, which are both arrows from $\mathcal{T}^f\gamma$ to $\mathcal{T}^g\hat{\gamma}$. $d_1 = d_2$ (and thus the desired result) follows from

$$\begin{aligned}
\mathcal{T}^g\hat{\gamma} \circ d_1 &= \mathcal{T}^g(\hat{\gamma} \circ \varphi) \circ u_\gamma && \text{Definition of } d_1\\
&= \mathcal{T}^g\gamma \circ u_\gamma && \text{Because } \varphi : \gamma \to \hat{\gamma}\\
&= \mathcal{T}^f\gamma && \text{By (26)}\\
&= \mathcal{T}^f\hat{\gamma} \circ \mathcal{T}^f\varphi && \text{See two lines above}\\
&= \mathcal{T}^g\hat{\gamma} \circ u_{\hat{\gamma}} \circ \mathcal{T}^f\varphi && \text{By (26)}\\
&= \mathcal{T}^g\hat{\gamma} \circ d_2 && \text{Definition of } d_2
\end{aligned}$$

and

$$\begin{aligned}
\hat{\pi}_2' \circ d_1 &= \hat{\pi}_2' \circ \mathcal{T}^g\varphi \circ u_\gamma && \text{Definition of } d_1\\
&= \varphi \circ \pi_2' \circ u_\gamma && \text{By (39)}\\
&= \varphi \circ \pi_2 && \text{By (26)}\\
&= \hat{\pi}_2 \circ \mathcal{T}^f\varphi && \text{By (38)}\\
&= \hat{\pi}_2' \circ u_{\hat{\gamma}} \circ \mathcal{T}^f\varphi && \text{By (26)}\\
&= \hat{\pi}_2' \circ d_2 && \text{Definition of } d_2.
\end{aligned}$$

3. Compatibility with the units follows from $\pi_2' \circ u_\gamma \circ \eta^f_\gamma = \pi_2 \circ \eta^f_\gamma = id = \pi_2' \circ \eta^g_\gamma$ (apply (26) and (5) twice) and $\mathcal{T}^g\gamma \circ u_\gamma \circ \eta^f_\gamma = \mathcal{T}^f\gamma \circ \eta^f_\gamma = \gamma = \mathcal{T}^g\gamma \circ \eta^g_\gamma$ (again using (26) and the fact, that $\eta^p_\gamma : \mathcal{T}^p\gamma \to \gamma$ for $p \in \{f, g\}$).

4. Let $u^2 := u * u$ be the horizontal composition. By the definition of u^2 we have for each $\gamma \in \mathscr{C}\downarrow L$:

$$u^2_\gamma = u_{\mathcal{T}^g\gamma} \circ \mathcal{T}^f u_\gamma = \mathcal{T}^g u_\gamma \circ u_{\mathcal{T}^f\gamma}. \tag{40}$$

From Fig. 8, we get

$$\pi_2 \circ \overline{\pi}_2 = \pi_2 \circ \mu^p_\gamma \tag{41}$$

where $\mu^p_\gamma = p^*\pi_2$. In the sequel, we use this for $p := f$ and $p := g$. The diagrams

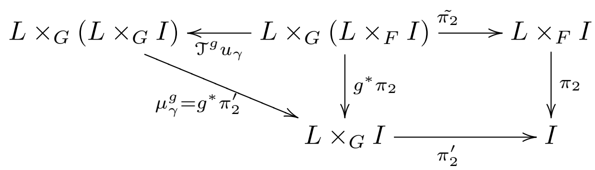


Figure 24: Compatibility with co-unit, part 1

and

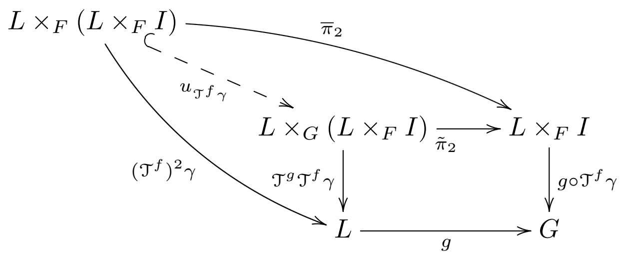


Figure 25: Compatibility with co-unit, part 2

commute: In the first diagram, the triangle commutes by applying g^* to (26) interpreted as diagram in $\mathscr{C}\downarrow G$. The square is just the pullback which arises from pulling back $\pi_2 : g \circ \mathcal{T}^f\gamma \to g \circ \gamma$ along g. We denote with $\tilde{\pi}_2$ the second projection in this case.

The second diagram is just Figure 17 taken at $\mathcal{T}^f\gamma$ instead of γ where the same $\tilde{\pi}_2$ occurs again. Thus

$$
\begin{aligned}
\pi_2' \circ \mu^g_\gamma \circ u^2_\gamma &= \pi_2' \circ \mu^g_\gamma \circ \mathcal{T}^g u_\gamma \circ u_{\mathcal{T}^f\gamma} && \text{By (40)}\\
&= \pi_2 \circ \tilde{\pi}_2 \circ u_{\mathcal{T}^f\gamma} && \text{Figure 24}\\
&= \pi_2 \circ \overline{\pi}_2 && \text{Figure 25}\\
&= \pi_2 \circ \mu^f_\gamma && \text{By (41)}\\
&= \pi_2' \circ u_\gamma \circ \mu^f_\gamma && \text{By (26)}
\end{aligned}
$$

On the other hand, by (26) and the fact that μ^f and μ^g are natural transformations from $(\mathcal{T}^f)^2\gamma$ to $(\mathcal{T}^f)\gamma$ and $(\mathcal{T}^g)^2\gamma$ to $(\mathcal{T}^g)\gamma$, resp., we obtain

$$\mathcal{T}^g\gamma \circ u_\gamma \circ \mu^f_\gamma = \mathcal{T}^f\gamma \circ \mu^f_\gamma = (\mathcal{T}^f)^2\gamma.$$

Since u^2 is a γ-indexed family of arrows from $(\mathcal{T}^f)^2\gamma$ to $(\mathcal{T}^g)^2\gamma$, we also have

$$\mathcal{T}^g\gamma \circ \mu^g_\gamma \circ u^2_\gamma = (\mathcal{T}^g)^2\gamma \circ u^2_\gamma = (\mathcal{T}^f)^2\gamma.$$

Because $\mathcal{T}^g\gamma = \pi_1'$ and π_2' are jointly monic, the proof is complete. □

8.5 Domain cycles and alternating sequences

In this section we show (1) that the existence of domain cycles entails the existence of proper cycles and (2) that the coherence condition (30) is inherited from proper cycles to all cycles (Lemma 38). The second part is restated in Lemma 39 and proven equivalent to the independence of the choice of alternating sequences from x to x' whenever $(x, x') \in ker(s)$ for the bottom diagonal s.

In the following lemmas we use the fact that a domain cycle $(x_0, x_1, \ldots, x_{2k+1})$ where the roles of a and r are interchanged also yields a domain cycle as in the original definition by shifting the numbering by 2 positions.

Lemma 38 (On proper cycles)

- *If a and r possess a domain cycle $(x_i)_{i\in\{0,\dots,2k+1\}}$ they also possess a proper subcycle $(y_i)_{i\in\{0,\dots,2k'+1\}}$ where $y_j \in \{x_0,\dots,x_{2k+1}\}$ and $k' \le k$.*

- *$\xi_{\sigma_c} = id_{\gamma^{-1}(x_0)}$ for all proper domain cycles implies $\xi_{\sigma_c} = id_{\gamma^{-1}(x_0)}$ for all domain cycles.*

Proof: The proof of both statements is by simultaneous induction over k. For $k = 0$ the pair (x_0, x_1) can only be a cycle if $x_0 \neq x_1$ and is proper. Hence the second statement also follows.

Let $k > 0$ and assume that the statement is true for each cycle of length smaller or equal $2k$. If it is not already proper, there are indices $0 \le i < j \le 2k+1$ such that $x_i = x_j$. By condition 1 in Definition 29 we even have $1 < j - i < 2k+1$. We have $p(x_i) = p(x_{i+1})$ for either $p = a$ or $p = r$.

If $p(x_{j-1}) = p(x_j)$ we have $j - i \equiv 1 \, mod \, 2$, hence $j - i \ge 3$, and we can delete $(x_{i+1}, \cdots, x_{j-1})$ (i.e. at least two positions) from the cycle. Because $p(x_i) = p(x_j)$, this yields a cycle of length less or equal $2k$.

For the second statement, observe that the deleted part is a cycle because $p(x_{j-1}) = p(x_j) = p(x_i) = p(x_{i+1})$ such that

$$\xi^{p!}_{x_j,x_{i+1}} \circ \xi^{p!}_{x_{j-1},x_j} \circ \cdots \circ \xi^{-p!}_{x_{i+1},x_{i+2}} = id_{\gamma^{-1}(x_{i+1})}$$

by induction hypothesis. Hence the composition in the defintion of ξ_σ becomes

$$\cdots \circ \xi^{-p!}_{x_j,x_{j+1}} \circ (\xi^{p!}_{x_j,x_{i+1}})^{-1} \circ \xi^{p!}_{x_i,x_{i+1}} \circ \xi^{-p!}_{x_{i-1},x_i} \cdots$$

in which the two middle factors reduce to id since $x_i = x_j$. This yields the definition of $\xi_{\sigma'_c} = id_{\gamma^{-1}(x_0)}$ along the reduced cycle c'.

If $-p(x_{j-1}) = -p(x_j)$, we delete $(x_{i+1}, \cdots, x_j)$ (again at least two positions) from the cycle. This time $p(x_i) = p(x_j) = p(x_{j+1})$ (where $j + 1 = 0 \; mod \; 2k + 2$ is possible), such that the reduced sequence is a domain cycle of length smaller or equal $2k$. In both cases, by induction hypotheses the remaining cycle contains a proper subcycle.

For the second statement, observe that $(x_i, \cdots, x_{j-1})$ is a domain cycle (because $-p(x_{j-1}) = -p(x_j) = -p(x_i)$). Thus the factors

$$\xi^{-p!}_{x_{j-1},x_j} \circ \cdots \circ \xi^{p!}_{x_i,x_{i+1}}$$

are identical by induction hypothesis. They can be removed from the definition of ξ_{σ_c}, the remaining term yields the definition of $\xi_{\sigma'_c}$ (the remaining cycle). □

Lemma 39 (Domain Cycles and Alternating Sequences) *The following conditions are equivalent:*

1. *$\xi_{\sigma_c} = id_{\gamma^{-1}(x_0)}$ for all proper domain cycles $c = (x_i)_{i\in\{0,1,\dots,2k+1\}}$.*

2. *$\xi_{\sigma_c} = id_{\gamma^{-1}(x_0)}$ for all domain cycles $c = (x_i)_{i\in\{0,1,\dots,2k+1\}}$.*

3. *$\xi_\sigma = \xi_{\sigma'}$ for all alternating sequences $\sigma = (y_i)_{i\in\{0,1,\dots,m\}}$ and $\sigma' = (z_i)_{i\in\{0,1,\dots,n\}}$ with σ proper and $y_0 = z_0$, $y_m = z_n$.*

4. *$\xi_\sigma = \xi_{\sigma'}$ for all alternating sequences $\sigma = (y_i)_{i\in\{0,1,\dots,m\}}$ and $\sigma' = (z_i)_{i\in\{0,1,\dots,n\}}$ with $y_0 = z_0$ and $y_m = z_n$.*

Proof: We will use in this proof neutrality and associativity of descent data (cf. Proposition 16) often without explicit references.

(1) $\iff$ (2) follows from Lemma 38. (3) implies (4) because it is easy to see that from any alternating sequence from y_0 to y_m a proper sequence σ_0 can always be extracted. Given arbitrary σ and σ', the assumption yields $\xi_\sigma = \xi_{\sigma_0}$ and $\xi'_\sigma = \xi_{\sigma_0}$, hence $\xi_\sigma = \xi_{\sigma'}$.

(2) is a special case of (4) with $m = 2k + 2$ for $k \in \mathbb{N}$, $p = a$, and $n = 0$.

To close the circle, it remains to show that (2) implies (3). We prove this by induction over n. For the induction basis $n = 0$ we have $y_0 = z_0 = y_m$ and $\xi_{\sigma'} = id_{\gamma^{-1}(z_0)}$. For $m = 0$ and $m = 1$ we have $\xi_\sigma = id_{\gamma^{-1}(y_0)} = id_{\gamma^{-1}(y_m)} = id_{\gamma^{-1}(z_0)} = \xi_{\sigma'}$ thus we remain with four cases

1. $m = 2k + 2$, $k \in \mathbb{N}$, $p = a$: In this case, σ represents a domain cycle thus we have $\xi_\sigma = id_{\gamma^{-1}(y_0)} = \xi_{\sigma'}$ by assumption.

2. $m = 2k+2$, $k \in \mathbb{N}$, $p = r$: The reverse sequence $\sigma^- = (y_m, y_{m-1}, \ldots, y_0)$ falls into case 1 thus this case is ensured by the fact that $\xi_\sigma = \xi_{\sigma^-}^{-1}$.

3. $m = 2k+3$, $k \in \mathbb{N}$, $p = a$: $y_{m-1} \neq y_1$, since σ is proper, thus the alternating sequence $\sigma'' = (y_{m-1}, y_1, \ldots, y_{m-1})$ represents a domain cycle and we have $\xi_{\sigma''} = id_{\gamma^{-1}(y_{m-1})}$ by assumption. We have $\xi_\sigma = \xi^\tau_{y_{m-1},y_m} \circ \xi_{\sigma''} \circ \xi^\tau_{y_0,y_{m-1}}$ by construction and since $y_0 = y_m$, we get $\xi_\sigma = \xi^\tau_{y_{m-1},y_m} \circ id_{\gamma^{-1}(y_{m-1})} \circ \xi^\tau_{y_0,y_{m-1}} = \xi^\tau_{y_0,y_m} = id_{\gamma^{-1}(z_0)} = \xi_{\sigma'}$, as required.

4. $m = 2k+3$, $k \in \mathbb{N}$, $p = r$: In this case, $\sigma'' = (y_1, \ldots, y_{m-1}, y_1)$ represents a domain cycle and the proof is along the lines of case 3.

Now we show the induction step to $n \geq 1$ under the hypothesis that the assertion is true for all pairs (m, n') with $n' < n$.

1. $z_1 = y_0$: This means $z_1 = y_0 = z_0$ and thus $\xi_{\sigma'} = \xi_{\sigma'_1}$ for the sequence $\sigma'_1 = (z_1, \ldots, z_n)$. $\xi_\sigma = \xi_{\sigma'_1}$, however, holds by induction hypothesis.

2. $z_1 = y_k$ for some $1 \leq k \leq m$: By induction hypothesis we have $\xi_{\sigma_1} = \xi_{\sigma'_1}$ and $\xi_{\sigma_2} = \xi_{\sigma'_2}$ for the subsequences $\sigma_1 = (y_0, \ldots, y_k)$, $\sigma_2 = (y_k, \ldots, y_m)$, $\sigma'_1 = (z_0, z_1)$, $\sigma'_2 = (z_1, \ldots, z_n)$ thus we also obtain $\xi_\sigma = \xi_{\sigma_2} \circ \xi_{\sigma_1} = \xi_{\sigma'_2} \circ \xi_{\sigma'_1} = \xi_{\sigma'}$.

3. $z_1 \neq y_k$ for all $0 \leq k \leq m$:

   (a) $(y_0, y_1) \in ker(p)$, $(z_0, z_1) \in ker(p)$: Then $\sigma_1 = (z_1, y_1, \ldots, y_m)$ is a proper alternating sequence. By induction hypothesis we have $\xi_{\sigma_1} = \xi_{\sigma'_1}$ for the alternating sequence $\sigma'_1 = (z_1, \ldots, z_n)$. Thus $\xi_\sigma = \xi_{\sigma_1} \circ \xi^{p!}_{y_0,z_1} = \xi_{\sigma'_1} \circ \xi^{p!}_{z_0,z_1} = \xi_{\sigma'}$.

   (b) $(y_0, y_1) \in ker(p)$, $(z_0, z_1) \in ker(-p)$: Then $\sigma_1 = (z_1, z_0 = y_0, y_1, \ldots, y_m)$ is a proper alternating sequence. By induction hypothesis we have $\xi_{\sigma_1} = \xi_{\sigma'_1}$ for the alternating sequence $\sigma'_1 = (z_1, \ldots, z_n)$, thus we obtain, finally, $\xi_\sigma = \xi_{\sigma_1} \circ \xi^{-p!}_{z_0,z_1} = \xi_{\sigma'_1} \circ \xi^{-p!}_{z_0,z_1} = \xi_{\sigma'}$. □

8.6 Twisting

In this section we show that the definition of a' in (33) yields a natural transformation $a' : I^L \to I^A$ and together with $\tau : I^A \to A$ and $\gamma : I^L \to L$ a pullback square over a.

Naturality: This is easy to see for the case of an operation symbol $op : Z' \to Z$ with $Z' \neq X$. Let therefore $op : X \to Z$.

Case 1: $Z \neq X$. Then by the definitions of a'_Z and op^{I^L}, we obtain for $(x, i) \in I^L_X = L^1_X + L^2_X + L^3_X$

$$a'_Z(op^{I^L}(x, i)) = op^A(a_X(x))$$

from naturality of a. Then the definition of op^{I^A} yields $op^{I^A}(a_X(x), j) = op^A(a_X(x))$ with $j = i$ if $x \neq x_1$ or $i = 3$ and $j = 3 - i$ if $x = x_1$ and $i \neq 3$. Then (33) yields

$$a'_Z(op^{I^L}(x, i)) = op^{I^A}(a'_X(x, i)).$$

Case 2: $Z = X$. In this case the definitions of a'_X and op^{I^A} yield

$$a'_X(op^{I^L}(x, i)) = (op^A(a_X(x)), 3)$$

by naturality of a for each $(x, i) \in I^L_X$. The right hand side equals $op^{I^A}(a_X(x), j)$ with the same choice of j as in Case 1. Thus

$$a'_X(op^{I^L}(x, i)) = op^{I^A}(a'_X(x, i))$$

by (33).

Commutativity of the square: On carriers I^L_Y for $Y \neq X$ this is again a consequence of the fact that τ_Y, γ_Y are identities and $a'_Y = a_Y$. And on I^L_X, one gets

$$\tau_X(a'_X(x,i)) = a_X(x)$$

because τ projects out i and a' respects the identifications of a. Since γ also deletes i, we also have

$$a_X(\gamma_X(x,i)) = a_X(x)$$

such that the square commutes.

Pullback property: By (21) the construction can be carried out pointwise and the consequence is clear for carriers of sort $Y \neq X$. For the carrier X the twist has to be considered: Given a pair $(x,(y,i)) \in L_X \times I^A_X$ with $a_X(x) = \tau_X(y,i)$ the element $(x,j) \in I^L_X$ with $j = i$, if $x \neq x_1$ or $i = 3$, and $j = 3 - i$ otherwise, can easily be shown to be unique with $a'_X(x,j) = (y,i)$ and $\gamma_X(x,j) = x$.