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Polar permutation graphs*

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Abstract

Polar graphs generalise bipartite, cobipartite, and split graphs, and they constitute a special type of matrix partitions. A graph is polar if its vertex set can be partitioned into two, such that one part induces a complete multipartite graph and the other part induces a disjoint union of complete graphs. Deciding whether a given arbitrary graph is polar, is an NP-complete problem. Here we show that for permutation graphs this problem can be solved in polynomial time. The result is surprising, as related problems like achromatic number and cochromatic number are NP-complete on permutation graphs. We give a polynomial-time algorithm for recognising graphs that are both permutation and polar. Prior to our result, polarity has been resolved only for chordal graphs and cographs.

1 Introduction

Many graph problems can be formulated as finding a partition of the vertices such that various parts satisfy certain properties internally, and at the same time certain other properties are satisfied regarding the interaction between these parts. Examples of such problems are the broad variety of colouring and homomorphism problems, and matrix partition. The latter was posed by Feder et al. [13], and it asks for a partition of the vertex set of a graph into subsets $A_1, \ldots, A_k$, such that each subset is either a clique or an independent set, and pairs of subsets are completely adjacent or completely non-adjacent, depending on a given pattern. If the pattern says that A_i, A_j should be completely adjacent for A_i, A_j independent sets, and completely non-adjacent for A_i, A_j cliques, we get exactly the polar graphs.

Polar graphs were defined already in 1985 by Tyshkevich and Chernyak [25]. A graph is *polar* if its vertex set can be partitioned into A and B such that A induces a complete multipartite graph and B induces a cluster graph, i.e., a disjoint union of complete graphs. Such a partition is called *polar*. As a polar partition into A and B implies that A induces a $\overline{P_3}$-free graph and B induces a P_3-free graph, polar graphs are self-complementary, and they contain the well-known classes of split graphs, bipartite graphs, and cobipartite graphs. If A is simply an independent set, then the graph (and the partition) is called *monopolar*. In addition to fitting into the matrix partition problem [13] described above, polar partitions can be seen as generalised colourings [5].

Recognising polar graphs is an NP-complete problem [6]. Notice, however, that "admitting a polar partition" can be expressed in monadic second order logic without using edge set

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quantification, and hence polar graphs of bounded treewidth or bounded clique-width can be recognised in polynomial time, by the results of [1, 7] and [8]. Consequently, it is of interest to find out where the boundary goes between subclasses of polar graphs that are recognisable in polynomial time and those whose recognition is intractable. When it comes to graph classes of unbounded treewidth and clique-width whose intersection with polar graphs can be recognised in polynomial time, so far we know only of chordal graphs [15, 10].

In this paper we prove that it can be decided in polynomial time whether a given permutation graph is polar, and we give an $O(n + m^4)$-time algorithm for this. Permutation graphs are a well-studied graph class with a large number of theoretical applications [16, 4], and they do not have bounded treewidth or clique-width [17]. Although many NP-complete problems become tractable on permutation graphs, well-known colouring problems, like cochromatic number [27, 14] and achromatic number [2], are NP-complete on this graph class. Our result is obtained in two steps. First we give an algorithm for recognising monopolar permutation graphs. This algorithm is then used as a subroutine for the recognition of polar permutation graphs.

Other results on polynomial-time recognisable subclasses of polar graphs include [24] which studies polar partitions where the size of each independent set and clique is bounded, [15, 10] which give forbidden subgraph characterisations and a recognition algorithm for polar chordal graphs, and [12] which gives similar results for polar cographs. In addition, [20] and [11] give respectively a forbidden subgraph characterisation and a polynomial-time recognition algorithm for bipartite graphs whose line graphs are polar. Finally, [9] gives a polynomial-time recognition algorithm for monopolar claw-free graphs. Another research direction is to study which NP-complete problems become tractable on polar graphs. For example, [21] gives polynomial-time algorithms for finding a minimum maximal independent set in some subclasses of polar graphs. This problem remains NP-hard in polar graphs admitting a polar partition where the size of every independent set is at most one and the size of every clique is at most two.

2 Definitions and notation

Our input graphs are simple, finite, and undirected. Only in Section 3, we will use directed graphs (*digraphs*) as auxiliary tools.

Let G be a graph. We denote its vertex set by $V(G)$ and its edge set by $E(G)$. An edge between vertices u and v is denoted by uv. If uv is an edge of G then u and v are *adjacent* in G. For a vertex x of G, the *neighbourhood* of x, denoted as $N_G(x)$, is the set of vertices that are adjacent to x. Let X be a set of vertices of G. Then we define $N_G[X] = X \cup \bigcup_{x \in X} N_G(x)$. The subgraph of G *induced* by X is denoted as $G[X]$ and defined as the graph on vertex set X and edge set the set of edges of G that join only vertices in X. By $G \setminus X$, we denote the graph $G[V(G) \setminus X]$. A graph is called *complete* if every pair of vertices is adjacent. A set $X \subseteq V(G)$ is called a *clique* if $G[X]$ is complete, and it is called an *independent set* if $G[X]$ has no edges.

The *disjoint union* of two graphs G and H is the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$; the disjoint union of more than two graphs is defined analogously. The *complement* of G, denoted as $\overline{G}$, is the graph on vertex set $V(G)$ and edge set $\{uv \mid u, v \in V(G) \text{ and } u \neq v\} \setminus E(G)$. A *multipartite graph* is the complement of the disjoint union of complete graphs. Equivalently, the vertex set of a multipartite graph admits a unique partition into maximal independent sets.

For a given graph G, a partition (A, B) of $V(G)$, where A or B can also be empty, is called *polar* if $G[A]$ is a complete multipartite graph and $G[B]$ is a disjoint union of complete graphs. Equivalently, (A, B) is a polar partition for G if and only if neither $\overline{G}[A]$ nor $G[B]$ contains an induced path on 3 vertices as an induced subgraph. Note that (A, B) is a polar partition for G if and only if (B, A) is a polar partition for $\overline{G}$. A polar partition (A, B) for G is called *monopolar* if A is an independent set in G. We say that a polar partition (A, B) for G is B*-maximal* if there is no polar partition (A', B') for G with $B \subset B'$. Note that, for a polar partition (A, B), if there is a vertex u in A without a neighbour in B then $(A \backslash \{u\}, B \cup \{u\})$ is also a polar partition for G. Hence the following result is immediate.

Lemma 2.1 *Let (A, B) be a B-maximal polar partition for a graph G. Every vertex in A has a neighbour in B.*

Let $n \geq 1$ and π be a permutation over $\{1, \ldots, n\}$, i.e., a bijection between $\{1, \ldots, n\}$ and $\{1, \ldots, n\}$. We will denote π equivalently as a *permutation sequence* $(\pi(1), \ldots, \pi(n))$. The *inversion graph* of π has vertex set $\{1, \ldots, n\}$ and two vertices u, v are adjacent if $(u-v)(\pi^{-1}(u) - \pi^{-1}(v)) < 0$. A graph is a *permutation graph* if it is isomorphic to the inversion graph of a permutation sequence [16, 4]. Permutation graphs can be recognised in linear time [22]. Permutation graphs also have a geometric intersection model: for two horizontal lines, mark n points on each line, assign to each point on the upper line a point on the lower line, and connect the two points by a line segment. The corresponding graph has a vertex for every line segment and two vertices are adjacent if the corresponding line segments cross. This representation is called a *permutation diagram*. A graph is a permutation graph if and only if it has a permutation diagram. It is important to note that every induced subgraph of a permutation graph is a permutation graph. For our purposes, we assume that a permutation graph is given as a permutation sequence and equal to the defined inversion graph. Every permutation graph with permutation sequence π has a permutation diagram $\mathcal{D}$ in which the endpoints of the line segments on the lower line appear in the same order as they appear in π. For such pairs $(\mathcal{D}, \pi)$, we say that $\mathcal{D}$ *corresponds to* π. For convenience reasons, sometimes we will not distinguish between vertices of the graph and line segments in the permutation diagram; however, the meaning will always be clear.

3 Monopolar permutation graphs

In this section, we give a polynomial-time algorithm for recognising monopolar permutation graphs, that certifies a positive answer by outputting a monopolar partition. This algorithm will be the main subroutine of our polar permutation graph recognition algorithm in the next section. In fact the subroutine presented in this section solves a more general problem: given a permutation graph G and a set $R \subseteq V(G)$, does G have a monopolar partition (A, B) such that $A \subseteq R$. By choosing $R = V(G)$ we get exactly monopolar permutation graph recognition.

Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. A *scanline* in $\mathcal{D}$ is a pair (a, e) where $a, e \in \{0.5, 1.5, \ldots, n + \frac{1}{2}\}$ [3]. Scanlines can be interpreted as additional line segments in the permutation diagram, that partition the vertex set into three sets: vertices to the left of the scanline, vertices to the right of the scanline and vertices that intersect with the scanline. Formally, for $s = (a, e)$ a scanline of G:

- $\mathrm{L}(s) =_{\mathrm{def}} \{x \in V(G) \mid \ x < a \text{ and } \pi^{-1}(x) < e\}$
- $\mathrm{R}(s) =_{\mathrm{def}} \{x \in V(G) \mid \ a < x \text{ and } e < \pi^{-1}(x)\}$
- $\mathrm{int}(s) =_{\mathrm{def}} V(G) \setminus (\mathrm{L}(x) \cup \mathrm{R}(x))$.

The scanlines $(\frac{1}{2}, \frac{1}{2})$ and $(n + \frac{1}{2}, n + \frac{1}{2})$ play a particular role; they are denoted by the special symbols $\mathbf{0}$ and $\mathbf{1}$. Note that $\mathrm{L}(\mathbf{0}) = \mathrm{int}(\mathbf{0}) = \emptyset$ and $\mathrm{R}(\mathbf{1}) = \mathrm{int}(\mathbf{1}) = \emptyset$ and $\mathrm{R}(\mathbf{0}) = \mathrm{L}(\mathbf{1}) = V(G)$. A scanline $s = (a, e)$ is called *minimal separation line*[1] if $1 < e < n$ and $a - \frac{1}{2}, \pi(e - \frac{1}{2}), a + \frac{1}{2}, \pi(e + \frac{1}{2}) \notin \mathrm{int}(s)$ or if $s = \mathbf{0}$ or if $s = \mathbf{1}$. We will see later that the number of minimal separation lines of a permutation graph can be much smaller than the number of its scanlines. For two scanlines $s = (a, e)$ and $s' = (a', e')$, we write $s < s'$ if $a \leq a'$ and $e \leq e'$ and $s \neq s'$.

We solve the monopolar permutation graph recognition problem by deciding whether an auxiliary digraph has a directed path between two specified vertices. This auxiliary digraph is defined as follows. Let G be a permutation graph with permutation sequence π and permutation diagram $\mathcal{D}$, and let $R \subseteq V(G)$. By $\mathrm{aux}(\mathcal{D}, R)$, we denote the digraph with the following vertices and arcs:

- $\mathrm{aux}(\mathcal{D}, R)$ has a vertex for every minimal separation line of G
- for two minimal separation lines s and s' with $s < s'$, there is an arc from the corresponding vertex of s to the corresponding vertex of s' in $\mathrm{aux}(\mathcal{D}, R)$ if $G[\mathrm{int}(s) \cup \mathrm{int}(s') \cup (\mathrm{R}(s) \cap \mathrm{L}(s'))]$ has a monopolar partition (I, C) such that $\mathrm{int}(s) \cup \mathrm{int}(s') \subseteq I \subseteq R$ and C is a clique in G.

Observe that the properties of the partition (I, C) imply that $G[\mathrm{int}(s) \cup \mathrm{int}(s') \cup (\mathrm{R}(s) \cap \mathrm{L}(s'))]$ is a split graph. Observe also that $\mathrm{aux}(\mathcal{D}, R)$ is acyclic, which follows from the partial order $<$ on the minimal separation lines. We denote the vertices of $\mathrm{aux}(\mathcal{D}, R)$ that correspond to $\mathbf{0}$ and $\mathbf{1}$ as respectively $\mathbf{0}$ and $\mathbf{1}$. Note that $\mathbf{0}$ has no incoming arc and $\mathbf{1}$ has no outgoing arc in $\mathrm{aux}(\mathcal{D}, R)$. We prove that there is a correspondence between the $\mathbf{0}, \mathbf{1}$-paths in $\mathrm{aux}(\mathcal{D}, R)$ and desired monopolar partitions for G.

Lemma 3.1 *Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let $R \subseteq V(G)$. If $\mathrm{aux}(\mathcal{D}, R)$ has a $\mathbf{0}, \mathbf{1}$-path then G has a monopolar partition (A, B) with $A \subseteq R$.*

Proof. Let P be a $\mathbf{0}, \mathbf{1}$-path in $\mathrm{aux}(\mathcal{D}, R)$ and let $s_0 < \cdots < s_r$ be the minimal separation lines corresponding to the vertices on P. Note that $s_0 = \mathbf{0}$ and $s_r = \mathbf{1}$. By definition of $\mathrm{aux}(\mathcal{D}, R)$, there is a monopolar partition (I_i, C_i) for $G[\mathrm{int}(s_{i-1}) \cup \mathrm{int}(s_i) \cup (\mathrm{R}(s_{i-1}) \cap \mathrm{L}(s_i))]$ for every $1 \leq i \leq r$ such that $\mathrm{int}(s_{i-1}) \cup \mathrm{int}(s_i) \subseteq I_i \subseteq R$ and C_i is a clique of G. We show that $(I_1 \cup \cdots \cup I_r, C_1 \cup \cdots \cup C_r)$ is a monopolar partition for G. By definition, $C_1 \cup \cdots \cup C_i \subseteq \mathrm{L}(s_i)$ and $C_{i+1} \cup \cdots \cup C_r \subseteq \mathrm{R}(s_i)$ for every $1 \leq i < r$. With the properties of permutation diagrams, this implies that $G[C_1 \cup \cdots \cup C_r]$ is the disjoint union of $G[C_1], \ldots, G[C_r]$. Thus, $G[C_1 \cup \cdots \cup C_r]$ is the disjoint union of complete graphs. Now, consider $I_1 \cup \cdots \cup I_r$. Let $u, v \in I_1 \cup \cdots \cup I_r$. If there is $1 \leq i \leq r$ such that $u, v \in I_i$ then u and v are non-adjacent due to the definition of $\mathrm{aux}(\mathcal{D}, R)$. Otherwise, if there is no such i, there is $1 \leq i' \leq r$ that separates u and v, i.e., $u \in \mathrm{L}(s_{i'})$ and $v \in \mathrm{R}(s_{i'})$ or $v \in \mathrm{L}(s_{i'})$ and $u \in \mathrm{R}(s_{i'})$. By the properties of permutation diagrams, u and v are non-adjacent in G. Hence, $I_1 \cup \cdots \cup I_r$ is an independent set in G. Furthermore, since $I_i \subseteq R$ for every $1 \leq i \leq r$, $I_1 \cup \cdots \cup I_r \subseteq R$. Thus, the defined partition is a desired monopolar partition for G. It is an easy observation that $(I_1 \cup \cdots \cup I_r, C_1 \cup \cdots \cup C_r)$ is indeed a partition of $V(G)$. ■

For the converse of Lemma 3.1, we need the existence of minimal separation lines between chosen sets of vertices.

[1] Minimal separation lines correspond exactly to minimal separators [23].

Lemma 3.2 ([23]) *Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let X and Y be sets of vertices of G that induce connected subgraphs of G. If $N_G[X] \cap Y = \emptyset$ then there is a minimal separation line s such that $X \subseteq \mathrm{L}(s)$ and $Y \subseteq \mathrm{R}(s)$ or such that $Y \subseteq \mathrm{L}(s)$ and $X \subseteq \mathrm{R}(s)$.*

Lemma 3.3 *Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let $R \subseteq V(G)$. If G has a monopolar partition (A, B) with $A \subseteq R$ then there is a $\mathbf{0}, \mathbf{1}$-path in $\mathrm{aux}(\mathcal{D}, R)$.*

Proof. Let (A, B) be a monopolar partition for G with $A \subseteq R$. Let $C_1, \ldots, C_r$ be the sets of vertices that induce the connected components of $G[B]$. Without loss of generality, we can assume that $\min C_1 < \cdots < \min C_r$. With the properties of permutation diagrams, it follows that $\min C_1 \leq \max C_1 < \cdots < \min C_r \leq \max C_r$. Due to Lemma 3.2, for every $1 \leq i \leq r-1$, there is a minimal separation line s_i of G with $C_i \subseteq \mathrm{L}(s_i)$ and $C_{i+1} \subseteq \mathrm{R}(s_i)$. For applying Lemma 3.2, it is important to remember that there are no edges between vertices in C_i and C_{i+1}. Let $s_0 =_{\mathrm{def}} \mathbf{0}$ and $s_r =_{\mathrm{def}} \mathbf{1}$. With these definitions, there is a vertex of $\mathrm{aux}(\mathcal{D}, R)$ for every minimal separation line $s_0, \ldots, s_r$. We show that the vertices of $\mathrm{aux}(\mathcal{D}, A')$ that correspond to $s_0, \ldots, s_r$ form a path. Let $1 \leq i \leq r$. By definition of s_{i-1} and s_i and the assumptions about $C_1, \ldots, C_r$, we have that

- $C_1, \ldots, C_i \subseteq \mathrm{L}(s_i)$ and $(C_{i+1} \cup \cdots \cup C_r) \cap \mathrm{L}(s_i) = \emptyset$
- $C_i, \ldots, C_r \subseteq \mathrm{R}(s_{i-1})$ and $(C_1 \cup \cdots \cup C_{i-1}) \cap \mathrm{R}(s_{i-1}) = \emptyset$.

Hence, $(C_1 \cup \cdots \cup C_r) \cap \mathrm{R}(s_{i-1}) \cap \mathrm{L}(s_i) = C_i$. With this result, it also follows that $\mathrm{int}(s_1) \cup \cdots \cup \mathrm{int}(s_{r-1}) \subseteq A$. Hence, $(\mathrm{int}(s_{i-1}) \cup \mathrm{int}(s_i) \cup ((\mathrm{R}(s_{i-1}) \cap \mathrm{L}(s_i)) \setminus C_i), C_i)$ is a monopolar partition for $G[\mathrm{int}(s_{i-1}) \cup \mathrm{int}(s_i) \cup (\mathrm{R}(s_{i-1}) \cap \mathrm{L}(s_i))]$ of the required form for the existence of the arc from the corresponding vertex of s_{i-1} to the corresponding vertex of s_i in $\mathrm{aux}(\mathcal{D}, R)$. Hence, there is a $\mathbf{0}, \mathbf{1}$-path in $\mathrm{aux}(\mathcal{D})$. ■

Note that a $\mathbf{0}, \mathbf{1}$-path in $\mathrm{aux}(\mathcal{D}, R)$ does not correspond to a specific monopolar partition for G but can represent many partitions. The main reason is that the same vertex can belong to a clique or to the independent set in different monopolar partitions.

It remains to prove the running time. The following result from the literature gives an upper bound on the size of the auxiliary digraph that is constructed.

Lemma 3.4 ([23]) *A permutation graph on n vertices and m edges has at most $2 + n + m$ minimal separation lines, and they can be listed in linear time.*

Theorem 3.5 *There is an $\mathcal{O}(n + m^3)$-time algorithm that given a permutation graph G and a set $R \subseteq V(G)$, decides whether there is a monopolar partition (A, B) for G with $A \subseteq R$.*

Proof. The algorithm is as follows: on input G a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$ and $R \subseteq V(G)$, construct the auxiliary digraph $\mathrm{aux}(\mathcal{D}, R)$ and check whether $\mathrm{aux}(\mathcal{D}, R)$ has a $\mathbf{0}, \mathbf{1}$-path; accept if and only if a $\mathbf{0}, \mathbf{1}$-path exists. With Lemmata 3.1 and 3.3, the algorithm accepts if and only if a desired monopolar partition exists. It remains to consider the running time of the algorithm. Due to Lemma 3.4, there are at most $2 + |V(G)| + |E(G)|$ minimal separation lines of G, and they can be listed in linear time. For every pair of minimal separation lines, it needs to be checked whether it defines a monopolar partition of the special type for the induced subgraph. This can be done in linear time, for instance by applying the degree-sequence recognition algorithm for split graphs [18] and a preselection for the independent set and clique vertices (for respecting the restrictions on $\text{int}(s) \cup \text{int}(s')$ and $V(G) \setminus R$ in the definition of the auxiliary graph). Thus, the construction of $\text{aux}(\mathcal{D}, R)$ can be done in time $\mathcal{O}(n + m^3)$. A path from $\mathbf{0}$ to $\mathbf{1}$ in $\text{aux}(\mathcal{D}, R)$ can be found in time linear in the size of $\text{aux}(\mathcal{D}, R)$, which gives $\mathcal{O}(n + m^2)$ time. Hence, the problem can be solved in $\mathcal{O}(n + m^3)$ time. ∎

It is not difficult to modify the algorithm in the proof of Theorem 3.5 such that it also outputs a monopolar partition. This can be achieved for instance by assigning to every arc of the auxiliary digraph the monopolar partition that certifies the existence of this arc, following the proof of Lemma 3.1.

4 Polar permutation graphs

In the previous section, we have given an algorithm for recognising monopolar permutation graphs. This is a subclass of polar permutation graphs. In this section, we consider the complementary recognition problem, for polar permutation graphs that are not monopolar. We will devise an algorithm that, given a permutation graph, deletes a set of vertices with certain properties and checks whether the remaining graph is monopolar with a monopolar partition of specific properties. The main task is to characterise the sets of vertices that can be deleted and to define the properties of the desired monopolar partitions.

Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. A *trapezoid* in $\mathcal{D}$ is a pair (I_1, I_2) of intervals with $I_1 = \{i_1, \ldots, i_1'\}$ and $I_2 = \{i_2, \ldots, i_2'\}$ for $1 \leq i_1 \leq i_1' \leq |V(G)|$ and $1 \leq i_2 \leq i_2' \leq |V(G)|$. Trapezoids are the main substructures in permutation diagrams that we consider in this section. We define four sets for trapezoids, in analogy to scanlines. Let $\mathbf{T} = (I_1, I_2)$ be a trapezoid in $\mathcal{D}$ with $I_1 = \{i_1, \ldots, i_1'\}$ and $I_2 = \{i_2, \ldots, i_2'\}$. We define the *left side*, the *right side*, the *containment* and the *intersection* of $\mathbf{T}$:

- the left side: $\text{L}(\mathbf{T}) =_{\text{def}} \{x \in V(G) \mid \ x < i_1 \text{ and } \pi^{-1}(x) < i_2\}$
- the right side: $\text{R}(\mathbf{T}) =_{\text{def}} \{x \in V(G) \mid \ i_1' < x \text{ and } i_2' < \pi^{-1}(x)\}$
- the containment: $\text{con}(\mathbf{T}) =_{\text{def}} \{x \in V(G) \mid \ i_1 \leq x \leq i_1' \text{ and } i_2 \leq \pi^{-1}(x) \leq i_2'\}$
- the intersection: $\text{int}(\mathbf{T}) =_{\text{def}} V(G) \setminus (L(\mathbf{T}) \cap R(\mathbf{T}))$.

Note that $\text{con}(\mathbf{T}) \subseteq \text{int}(\mathbf{T})$. For $X \subseteq V(G)$, the *X-trapezoid* in $\mathcal{D}$ is the trapezoid $\mathbf{T} = (I_1, I_2)$ with $I_1 = \{\min X, \ldots, \max X\}$ and $I_2 = \{\min \pi^{-1}(X), \ldots, \max \pi^{-1}(X)\}$. Note that $X \subseteq \text{con}(\mathbf{T})$. Informally spoken, the X-trapezoid is the smallest trapezoid that contains X. Note that not every trapezoid is an X-trapezoid for some set X. Our main interest is in trapezoids with special properties. Let G have a polar partition (A, B) and let $Y \subseteq V(G)$. A trapezoid $\mathbf{T}$ in $\mathcal{D}$ is called *centre trapezoid for (A, B) in $\mathcal{D}$ (with Y in-cliqued)* if the following conditions are satisfied:

1) $G[\text{int}(\mathbf{T})]$ has a polar partition (A', C) with
   - C is a clique in G (and $Y \subseteq C$) and $C \subseteq \text{con}(\mathbf{T})$
   - $\text{int}(\mathbf{T}) \setminus \text{con}(\mathbf{T}) \subseteq A' \subseteq A$

2) one of the following two cases holds:

   – $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ is an independent set in G
   – $A \cap \mathrm{L}(\mathbf{T}) \neq \emptyset$ and $A \cap \mathrm{R}(\mathbf{T}) = \emptyset$ and there is a vertex $v \in \mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$ with $N_G(v) \cap \mathrm{L}(\mathbf{T}) \subseteq A$ such that $\{v\} \cup (A \cap \mathrm{L}(\mathbf{T})) \setminus N_G(v)$ is an independent set in G.

We show that such centre trapezoids indeed exist. A graph that is polar but not monopolar is called *multipolar*.

Lemma 4.1 *Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let G be multipolar and (A, B) a B-maximal polar partition for G. There are a trapezoid $\mathbf{T}$ in $\mathcal{D}$ and a clique X of G with $X \subseteq B$ and $1 \leq |X| \leq 2$ such that $\mathbf{T}$ is the X-trapezoid in $\mathcal{D}$ and $\mathbf{T}$ is a centre trapezoid for (A, B) in $\mathcal{D}$ with X in-cliqued.*

Proof. By Lemma 2.1, every vertex in A has a neighbour in B. Let the connected components of $G[B]$ be induced by the sets $C_1, \ldots, C_r$; without loss of generality we can assume that $\min C_1 < \cdots < \min C_r$. For every $x \in A$, denote by $\alpha(x)$ and $\omega(x)$ the respectively smallest and largest index i with C_i containing a neighbour of x.

As the first case, assume that there are adjacent vertices $u, v \in A$ with $\omega(u) < \alpha(v)$. Note that $uv \in E(G)$ implies $\alpha(v) = \omega(u) + 1$. Let $C =_{\mathrm{def}} C_{\alpha(v)}$ and let $\mathbf{T}$ be the C-trapezoid in $\mathcal{D}$. Note that $u \in \mathrm{L}(\mathbf{T})$ and therefore $v \in \mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$ by the properties of permutation diagrams. Furthermore, $\mathrm{L}(\mathbf{T}) \cap N_G(v) \subseteq A$ by the definition of $\alpha(v)$ and $C \subseteq \mathrm{con}(\mathbf{T})$. For the first subcase, assume that $A \subseteq \mathrm{L}(\mathbf{T}) \cup \mathrm{int}(\mathbf{T})$, i.e., $A \cap \mathrm{R}(\mathbf{T}) = \emptyset$. A vertex in $A \cap \mathrm{L}(\mathbf{T})$ is either adjacent or non-adjacent to v. By the definition of complete multipartite graphs, the vertices in $A \cap \mathrm{L}(\mathbf{T})$ that are non-adjacent to v are in the same maximal independent set as v in $G[A]$. Hence, they are pairwise non-adjacent, which means that $\{v\} \cup (A \cap \mathrm{L}(\mathbf{T})) \setminus N_G(v)$ is an independent set in G. For the second subcase, let $A \cap \mathrm{R}(\mathbf{T})$ be non-empty. This means that both $A \cap \mathrm{L}(\mathbf{T})$ and $A \cap \mathrm{R}(\mathbf{T})$ are non-empty. Since vertices from $\mathrm{L}(\mathbf{T})$ and $\mathrm{R}(\mathbf{T})$ are pairwise non-adjacent by the properties of permutation diagrams, the definition of complete multipartite graphs implies that $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ is an independent set in G. Hence, in both subcases, $\mathbf{T}$ satisfies the second condition of the definition of centre trapezoid.

As the second case, assume that for all pairs $u, v \in A$ of adjacent vertices, $\alpha(u) \leq \alpha(v) \leq \omega(u)$ or $\alpha(v) \leq \alpha(u) \leq \omega(v)$. The following construction and argumentation is more difficult than it would be expected, since the condition of the second case does not compare all pairs of vertices from A but only adjacent vertices. Let $u, v \in A$ be a pair of adjacent vertices such that the intersection $\{\alpha(u), \ldots, \omega(u)\} \cap \{\alpha(v), \ldots, \omega(v)\}$ is of smallest size. Without loss of generality, we can assume that $\alpha(u) \leq \alpha(v)$. Let $C =_{\mathrm{def}} C_{\alpha(v)}$ and let $\mathbf{T}$ be the C-trapezoid in $\mathcal{D}$. For the first subcase, let $\omega(v) \leq \omega(u)$. Then, $\alpha(u) \leq \alpha(v) \leq \omega(v) \leq \omega(u)$ with our assumptions, which implies $\alpha(x) \leq \alpha(v) \leq \omega(v) \leq \omega(x)$ for all vertices $x \in A \cap N_G(v)$. This follows from the choice of the pair u, v as of smallest intersection size. Hence, $A \cap N_G(v) \subseteq \mathrm{int}(\mathbf{T})$, and by the properties of complete multipartite graphs, $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ is an independent set in G. For the second subcase, let $\omega(u) < \omega(v)$. For every vertex $x \in A \cap \mathrm{L}(\mathbf{T})$, $\omega(x) < \alpha(v)$, which means by the assumptions of the case that $(A \cap \mathrm{L}(\mathbf{T})) \cap N_G(v) = \emptyset$. Hence, due to the properties of complete multipartite graphs, $\{v\} \cup (A \cap \mathrm{L}(\mathbf{T}))$ is an independent set in G. In the following, we distinguish between the cases $A \cap \mathrm{L}(\mathbf{T})$ non-empty and empty. First, let $A \cap \mathrm{L}(\mathbf{T})$ be non-empty. By the properties of permutation diagrams, the vertices in $A \cap \mathrm{L}(\mathbf{T})$ and $A \cap \mathrm{R}(\mathbf{T})$ are pairwise non-adjacent. Then, the properties of complete multipartite graphs imply that $\{v\} \cup (A \cap \mathrm{L}(\mathbf{T})) \cup (A \cap \mathrm{R}(\mathbf{T}))$ is an independent set in G. Second, let $A \cap \mathrm{L}(\mathbf{T})$ be empty. Let $A \cap \mathrm{R}(\mathbf{T})$ be non-empty and let $w \in A \cap \mathrm{R}(\mathbf{T})$. Since $u \in \mathrm{int}(\mathbf{T})$, $w \neq u$. Suppose that $uw \in E(G)$. Then, $\alpha(v) < \alpha(w)$ and $\alpha(v) \leq \omega(u) < \omega(v)$ yield a contradiction to the intersection size of u, v. Hence, u and w are non-adjacent, which implies that $\{u\} \cup (A \cap \mathrm{R}(\mathbf{T})) = \{u\} \cup (A \cap \mathrm{L}(\mathbf{T})) \cup (A \cap \mathrm{R}(\mathbf{T}))$ is an independent set in G. Hence, in both subcases, $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ is an independent set in G, which shows that $\mathbf{T}$ satisfies the second condition of the definition of centre trapezoids.

It remains to check whether the chosen trapezoids satisfy the first condition of the definition of centre trapezoids. We can consider the two cases above simultaneously. Let C and $\mathbf{T}$ be as defined above. First, we show that $(\mathrm{int}(\mathbf{T}) \setminus C, C)$ is a polar partition for $G[\mathrm{int}(\mathbf{T})]$. Since no edge of G joins vertices from different cliques among $C_1, \ldots, C_r$, $B \cap \mathrm{int}(\mathbf{T}) = C$. Hence, $\mathrm{int}(\mathbf{T}) \setminus B = \mathrm{int}(\mathbf{T}) \setminus C \subseteq A$. By the definition of C-trapezoid, $C \subseteq \mathrm{con}(\mathbf{T})$, which also implies that $\mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T}) \subseteq \mathrm{int}(\mathbf{T}) \setminus C \subseteq A$. Hence, $\mathbf{T}$ satisfies the first condition of the definition of centre trapezoids. To complete the proof of the lemma, let $x =_{\mathrm{def}} \min C$ and $y =_{\mathrm{def}} \max C$. Note that $x = y$ in case $|C| = 1$. Then, with the properties of permutation diagrams and the representation of cliques, it holds that $\mathbf{T}$ is the $\{x, y\}$-trapezoid and a centre trapezoid for (A, B) in $\mathcal{D}$ with $\{x, y\}$ in-cliqued. ■

Lemma 4.1 is the main tool of our algorithm. Informally, the algorithm removes a trapezoid from the graph and checks whether the remaining subgraph is monopolar. The main problem then is to combine the independent set of the monopolar partition with the complete multipartite graph of the polar partition of the removed trapezoid. Not every monopolar partition is suitable. The next lemma will be useful for choosing a suitable monopolar partition. Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let $\mathbf{T} = (I_1, I_2)$ be a trapezoid in $\mathcal{D}$ with $I_1 = \{i_1, \ldots, i_1'\}$ and $I_2 = \{i_2, \ldots, i_2'\}$ and let $S \subseteq \mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$. A vertex $x \in S$ is *left-endpoint close to* $\mathbf{T}$ *among the vertices in* S if $x < i_1$ and $x \geq y$ for all vertices $y \in S$ or if $\pi^{-1}(x) < i_2$ and $\pi^{-1}(x) \geq \pi^{-1}(y)$ for all vertices $y \in S$. *Right-endpoint close vertex* is defined symmetrically. Note that there are at most two left-endpoint close and at most two right-endpoint close vertices for every trapezoid and chosen set.

Lemma 4.2 *Let G be a permutation graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. Let G be multipolar and (A, B) a B-maximal polar partition for G. Let $\mathbf{T}$ be a centre trapezoid for (A, B) in $\mathcal{D}$.*

1) Let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ be an independent set in G and let $A \cap \mathrm{con}(\mathbf{T})$ be non-empty.
Let x be an arbitrary vertex in $A \cap \mathrm{con}(\mathbf{T})$. Then, $\{x\} \cup (A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})))$ is an independent set in G.

2) Let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ be an independent set in G and let $A \cap \mathrm{con}(\mathbf{T})$ be empty.
If there is $x \in A \cap \mathrm{int}(\mathbf{T})$ such that $\{x\} \cup (A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})))$ is an independent set in G then x can be chosen as left-endpoint close or right-endpoint close to $\mathbf{T}$ among the vertices in $\mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$.

3) Let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ not be an independent set in G.
Let $A \cap \mathrm{L}(\mathbf{T}) \neq \emptyset$ and $A \cap \mathrm{R}(\mathbf{T}) = \emptyset$ and let there be a vertex $x \in \mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$ with $N_G(x) \cap \mathrm{L}(\mathbf{T}) \subseteq A$ such that $(A \cap \mathrm{L}(\mathbf{T})) \setminus N_G(x)$ is an independent set in G. Then, x can be chosen as left-endpoint close to $\mathbf{T}$ among the vertices in $\mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$.

Proof. We consider the three cases separately. For the first two cases, let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ be an independent set in G. If $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ is empty then the two cases trivially hold. So, let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ be non-empty. For the first case, let there be a vertex $x \in A \cap \mathrm{con}(\mathbf{T})$. Then, x is not adjacent to any vertex in $\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})$ and thus $\{x\} \cup (A \setminus \mathrm{int}(\mathbf{T}))$ is an independent set in G due to the properties of complete multipartite graphs.

For the second case, let $A \cap \mathrm{con}(\mathbf{T}) = \emptyset$. Assume that there is $x \in A \cap \mathrm{int}(\mathbf{T})$ such that $\{x\} \cup (A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})))$ is an independent set in G. By a symmetry argument for permutation diagrams, we can assume that $A \cap \mathrm{L}(\mathbf{T}) \neq \emptyset$ and x is smaller than the vertices in $\mathrm{con}(\mathbf{T})$. The three other cases are obtained from flipping the permutation diagram vertically or horizontally. Informally spoken, the upper endpoint of x in $\mathcal{D}$ is to the left of $\mathbf{T}$. By assumption, all vertices in $A \cap \mathrm{L}(\mathbf{T})$ are non-adjacent to x. Let y be the left-endpoint close vertex for $\mathbf{T}$ among the vertices in $\mathrm{int}(\mathbf{T})$ with the upper endpoint of y to the left of $\mathbf{T}$ in $\mathcal{D}$. Assume that $x \neq y$. Note that $x < y$. No vertex from $A \cap \mathrm{L}(\mathbf{T})$ is adjacent to y, since otherwise such a vertex would be adjacent also to x due to the properties of permutation diagrams. Thus, $\{y\} \cup (A \cap \mathrm{L}(\mathbf{T}))$ is an independent set in G. And with the properties of complete multipartite graphs, $\{y\} \cup (A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})))$ is an independent set in G.

For the third case, let $A \cap (\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T}))$ not be an independent set in G. By assumption about $\mathbf{T}$, $A \cap \mathrm{L}(\mathbf{T}) \neq \emptyset$ and $A \cap \mathrm{R}(\mathbf{T}) = \emptyset$ and there is a vertex x that satisfies the assumptions of the case. By a symmetry argument for permutation diagrams, we can assume that x is smaller than the vertices in $\mathrm{con}(\mathbf{T})$, i.e., the upper endpoint of x is to the left of $\mathbf{T}$ in $\mathcal{D}$. Let y be the left-endpoint close vertex for $\mathbf{T}$ among the vertices in $\mathrm{int}(\mathbf{T}) \setminus \mathrm{con}(\mathbf{T})$ with the upper endpoint of y to the left of $\mathbf{T}$ in $\mathcal{D}$. Assume that $y \neq x$. With the properties of permutation diagrams and the assumptions about x, $N_G(y) \cap \mathrm{L}(\mathbf{T}) \subseteq N_G(x) \cap \mathrm{L}(\mathbf{T}) \subseteq A$. We show that $(A \cap \mathrm{L}(\mathbf{T})) \setminus N_G(y)$ is an independent set in G. If there is a pair $u, v \in (A \cap \mathrm{L}(\mathbf{T})) \setminus N_G(y)$ of adjacent vertices then the properties of complete multipartite graphs imply that at least one of them is adjacent to y, which is a contradiction. ∎

We are ready to give the final algorithm. Our algorithm for recognising polar permutation graphs is called `Polar-Permutation-graphs`, and given in Figure 1. If the input graph is polar, the algorithm outputs a polar partition, thus provides a certificate.

Theorem 4.3 *Algorithm* `Polar-Permutation-graphs` *recognises polar permutation graphs in* $\mathcal{O}(n + m^4)$ *time.*

Proof. For the correctness of the algorithm, let G be the input graph with permutation sequence π and corresponding permutation diagram $\mathcal{D}$. We first show that every 'yes' answer (in lines 2, 14, 20, 28, 34) is correct and the output partition is a proper polar partition for G. So, let the answer of the algorithm on input $G, \pi, \mathcal{D}$ be 'yes'. It is a simple check that the output vertex partition is indeed a partition of $V(G)$. If the answer is output in line 2 then G is monopolar, thus polar, and the output partition is a polar partition for G. We consider the four other cases. We consider the **for** loop during its last execution. Let $\mathbf{T}$ be the trapezoid defined in line 5 and let (A', C) be the polar partition for $G[\mathrm{int}(\mathbf{T})]$ chosen in line 8. Note that (A', C) has the properties of lines 6–7. In particular, no vertex from C has a neighbour in $G \setminus \mathrm{int}(\mathbf{T})$, which follows directly from $C \subseteq \mathrm{con}(\mathbf{T})$ and the properties of permutation diagrams. Hence, since C is a clique in G and $G[B]$ is an induced subgraph of $G \setminus \mathrm{int}(\mathbf{T})$ and the disjoint union of complete graphs, for each of the four cases for B, $G[B \cup C]$ is the disjoint union of complete graphs. It remains to show for each of the four cases that the first component of the output

Algorithm `Polar-Permutation-graphs`
Input permutation graph G with permutation sequence π and corresponding permutation diagram $\mathcal{D}$
Output answer 'yes' or 'no', and if 'yes' then a polar partition for G

begin
1 **if** G is monopolar **then**
2 **compute** a monopolar partition (A, B) for G; **return** 'yes' and (A, B)
3 **end if**;
4 **for** X a clique of G of size 1 or 2 **do**
5 **let** $\mathbf{T}$ be the X-trapezoid in $\mathcal{D}$;
6 **if** $G[\text{int}(\mathbf{T})]$ is polar and has polar partition (A', C) such that $\text{int}(\mathbf{T}) \setminus \text{con}(\mathbf{T}) \subseteq A'$ and
7 C is a clique of G and maximal with $X \subseteq C \subseteq \text{con}(\mathbf{T})$ **then**
8 **let** (A', C) be the computed polar partition for $G[\text{int}(\mathbf{T})]$;
9 **if** $A' \cap \text{con}(\mathbf{T}) \neq \emptyset$ **then**
10 **let** $x \in A' \cap \text{con}(\mathbf{T})$;
11 **if** $G \setminus \text{int}(\mathbf{T})$ has monopolar partition (A, B) with every vertex in A is adjacent to
12 every vertex in $A' \cap N_G(x)$ and non-adjacent to every vertex in $A' \setminus N_G(x)$ **then**
13 **let** (A, B) be the computed monopolar partition for $G \setminus \text{int}(\mathbf{T})$;
14 **return** 'yes' and $(A \cup A', B \cup C)$
15 **end if**
16 **else**
17 **if** $G \setminus \text{int}(\mathbf{T})$ has monopolar partition (A, B) with
18 every vertex in A is adjacent to every vertex in A' **then**
19 **let** (A, B) be the computed monopolar partition for $G \setminus \text{int}(\mathbf{T})$;
20 **return** 'yes' and $(A \cup A', B \cup C)$
21 **end if**;
22 **let** L and R be the sets of respectively left-endpoint close and right-endpoint close vertices
23 for $\mathbf{T}$ among the vertices in $\text{int}(\mathbf{T}) \setminus \text{con}(\mathbf{T})$;
24 **if** there are vertex $x \in L \cup R$ and monopolar partition (A, B) for $G \setminus \text{int}(\mathbf{T})$ with
25 every vertex in A is adjacent to every vertex in $A' \cap N_G(x)$ and non-adjacent to
26 every vertex in $A' \setminus N_G(x)$ **then**
27 **let** (A, B) be the computed monopolar partition for $G \setminus \text{int}(\mathbf{T})$;
28 **return** 'yes' and $(A \cup A', B \cup C)$
29 **end if**;
30 **if** there are vertex $x \in L$ and monopolar partition (A, B) for $G \setminus (\text{int}(\mathbf{T}) \cup (N_G(x) \cap \text{L}(\mathbf{T})))$
31 with $A \cap \text{R}(\mathbf{T}) = \emptyset$ and $\{x\} \cup A$ an independent set in G and
32 $G[A \cup A' \cup (N_G(x) \cap \text{L}(\mathbf{T}))]$ complete multipartite **then**
33 **let** (A, B) be the computed monopolar partition for $G \setminus \text{int}(\mathbf{T})$;
34 **return** 'yes' and $(A \cup A' \cup (N_G(x) \cap \text{L}(\mathbf{T})), B \cup C)$
35 **end if**
36 **end if**
37 **end if**
38 **end for**;
39 **return** 'no'
end.

Figure 1: The polar permutation graph recognition algorithm.

vertex partition induces a complete multipartite graph in G. The case is clear for the output in line 34 by the condition in line 32. Assume that the answer is output in line 14, let (A, B) be the monopolar partition for $G \setminus \text{int}(\mathbf{T})$ in line 13 and let x be the vertex chosen in line 10. Since A and $A' \setminus N_G(x)$ are independent sets in G, $A \cup (A' \setminus N_G(x))$ is an independent set in G due to the conditions in line 11–12. And by the properties of complete multipartite graphs and the adjacency condition in lines 11–12, every vertex in $A \cup (A' \setminus N_G(x))$ is adjacent to every vertex in $A' \cap N_G(x)$. Hence, $G[A \cup A']$ is complete multipartite. The cases for the output in lines 20 and 28 follow similarly. We conclude that the output partition is indeed a polar partition for G and G is polar.

For the converse, let G be polar. We show that `Polar-Permutation-graphs` answers 'yes'. If G is monopolar then `Polar-Permutation-graphs` returns answer 'yes' in line 2. Let G not be monopolar, and let (P, Q) be a Q-maximal polar partition for G. According to Lemma 4.1, there are a trapezoid $\mathbf{T}$ in $\mathcal{D}$ and a clique X of G with $X \subseteq Q$ and $1 \leq |X| \leq 2$ such that $\mathbf{T}$ is the X-trapezoid in $\mathcal{D}$ and $\mathbf{T}$ is a centre trapezoid for (P, Q) in $\mathcal{D}$ with X in-cliqued. Note that X and thus $\mathbf{T}$ can be chosen by `Polar-Permutation-graphs` in lines 4 and 5. Let (A', C) be a polar partition for $G[\text{int}(\mathbf{T})]$ as defined in condition 1 of the definition of centre trapezoids; such a partition exists by assumption. Without loss of generality, we can assume that every vertex in $A' \cap \text{con}(\mathbf{T})$ is non-adjacent to some vertex in C. This partition satisfies the conditions in lines 6–7, and `Polar-Permutation-graphs` continues execution in line 8. Observe for the following arguments that $P \cap \text{int}(\mathbf{T}) = A'$: if there is $u \in (P \cap \text{int}(\mathbf{T})) \setminus A'$ then $u \in C$. By the definition of (A', C) according to condition 1 of the definition of centre trapezoid, $u \in \text{con}(\mathbf{T})$, which means that u is not adjacent to any vertex in $Q \setminus \text{int}(\mathbf{T})$. Then, $(P \setminus \{u\}, Q \cup \{u\})$ is a polar partition for G, which contradicts the choice of (P, Q) as Q-maximal. Then, $P \cap (\text{L}(\mathbf{T}) \cup \text{R}(\mathbf{T})) = P \setminus A'$. We distinguish between the two cases in condition 2 of the definition of centre trapezoids. As the first main case, let $P \setminus A'$ be an independent set in G. Then, $G \setminus \text{int}(\mathbf{T})$ is monopolar with monopolar partition $(P \setminus A', Q \setminus C)$. As a first subcase, let $P \cap \text{con}(\mathbf{T}) = A' \cap \text{con}(\mathbf{T})$ be non-empty (line 9). For any vertex $x \in A' \cap \text{con}(\mathbf{T})$, $\{x\} \cup (P \setminus A')$ is an independent set in G and particularly in $G[P]$. With the properties of complete multipartite graphs, all vertices in $P \setminus A'$ are adjacent to all vertices in $A' \cap N_G(x)$ and non-adjacent to all vertices in $A' \setminus N_G(x)$. Then, the conditions in lines 11–12 are satisfied by partition $(P \setminus A', Q \setminus C)$, and the algorithm accepts. As a second subcase, let $A' \cap \text{con}(\mathbf{T})$ be empty. If all vertices in $P \setminus A'$ are adjacent to all vertices in A' then `Polar-Permutation-graphs` accepts in line 20. Let there be a vertex $y \in P \setminus A'$ that is non-adjacent to some vertex $x \in A'$. Then, x and y are in the same maximal independent set of $G[P]$, which implies that $\{x\} \cup (P \setminus A')$ is an independent set of G. Due to Lemma 4.2, there is a vertex z in $L \cup R$ of lines 22–23 such that $\{z\} \cup (P \setminus A')$ is an independent set of G. Analogous to the first subcase, `Polar-Permutation-graphs` accepts in line 28. As the second main case, let $P \setminus A'$ not be an independent set of G. According to condition 2 of the definition of centre trapezoids, $P \cap \text{L}(\mathbf{T}) \neq \emptyset$ and $P \cap \text{R}(\mathbf{T}) = \emptyset$ and there is a vertex $v \in A' \setminus \text{con}(\mathbf{T})$ with $N_G(v) \cap \text{L}(\mathbf{T}) \subseteq P$ and $\{v\} \cup (P \cap \text{L}(\mathbf{T})) \setminus N_G(v)$ is an independent set of G. Then, $((P \setminus A') \setminus N_G(v), Q \setminus C)$ is a monopolar partition for $G \setminus (\text{int}(\mathbf{T}) \cup (N_G(v) \cap \text{L}(\mathbf{T})))$. Due to Lemma 4.2, we can choose v from L. Then, `Polar-Permutation-graphs` accepts in line 34. This completes the correctness proof.

For the running time, observe that the **for** loop is executed at most $|V(G)| + |E(G)|$ times. First, we show that each **for** loop execution takes time $\mathcal{O}(n + m)$ plus the time for deciding whether a permutation graph has a monopolar partition (A, B) with $A \subseteq R$ for R a given set of vertices. Trapezoid $\mathbf{T}$ in line 5 can be computed in constant time from the given set X. The sets $\text{int}(\mathbf{T})$, $\text{con}(\mathbf{T})$, $\text{L}(\mathbf{T})$, $\text{R}(\mathbf{T})$ and L and R (in lines 22–23) can be computed in linear time straightforward by checking the endpoints of every vertex against the intervals of $\mathbf{T}$. We consider the conditional in lines 9–36. Assume that partition (A', C) for $\text{int}(\mathbf{T})$ is given. The test $A' \cap \text{con}(\mathbf{T}) \neq \emptyset$ (line 9) can be done in linear time. Vertex x in line 10 is chosen arbitrarily. For the conditional in lines 11-12, we need to compute an appropriate set R. We define R as the set of vertices in $\mathrm{L}(\mathbf{T}) \cup \mathrm{R}(\mathbf{T})$ that are adjacent to every vertex in $A' \cap N_G(x)$ and non-adjacent to every vertex in $A' \setminus N_G(x)$. This set can be computed in linear time. For the conditionals in lines 17–18 and 24–26, the corresponding sets R can be computed similarly. For the conditional in lines 30–32, R contains only vertices from $\mathrm{L}(\mathbf{T})$ due to the condition $A \cap \mathrm{R}(\mathbf{T}) = \emptyset$ and no vertices from $N_G(x)$. With the requirements on the computed polar partition, R can be computed in linear time analogous to the previous cases. The conditionals in lines 24–26 and 30–32 are executed several times with different choices for x. Since $|L \cup R| \leq 4$, the number of executions is constant. Hence, besides the running time for obtaining partitions (A', C) and (A, B), a **for** loop execution takes linear time. The existence of partition (A, B) with the restriction R can be decided in $\mathcal{O}(n + m^3)$ time due to Theorem 3.5. By the remark at the end of Section 3, a partition (A, B) can be computed in the same time if it exists.

It remains to give an algorithm for deciding the conditional in lines 6–7. Since the vertices in X are adjacent to all vertices in $\mathrm{con}(\mathbf{T})$, it suffices to compute a desired polar partition for $G[\mathrm{int}(\mathbf{T})] \setminus X$ and we only need to decide for vertices in $\mathrm{con}(\mathbf{T}) \setminus X$ whether they belong to the clique or to the complete multipartite graph. A polar partition for $G[\mathrm{int}(\mathbf{T})]$ of the type of lines 6–7 can be computed by applying the algorithm of Theorem 3.5 to the complement of $G[\mathrm{int}] \setminus X$ with set R chosen as $\mathrm{con}(\mathbf{T})$. Let the output polar partition be (A', C). If a vertex in A' is adjacent to all vertices in C then this vertex is moved to set C. Hence, in $\mathcal{O}(n + m^3)$ time, the conditional in lines 6–7 can be decided and a corresponding polar partition can be computed in the positive case. In total, we obtain $\mathcal{O}(n + m^4)$ running time. ∎

5 Concluding remarks and open problems

The main time consuming operation in the presented algorithm is the construction of the auxiliary digraph. One possibility to improve on the running time is to bound the number of arcs in the auxiliary digraph and to avoid testing all possible pairs of vertices.

Permutation graphs are both comparability and cocomparability graphs. An interesting question is whether polar comparability graphs, or equivalently polar cocomparability graphs, can be recognised in polynomial time. Another question is whether in permutation graphs that are not polar, a maximum induced polar subgraph can be computed in polynomial time.

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