REPORTS
IN
INFORMATICS

ISSN 0333-3590

The size of spheres of multipermutations under the maximum distance

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REPORT NO 384 **March 2009**

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This report has URL http://www.ii.uib.no/publikasjoner/texrap/ps/2009-384.ps

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Abstract

Recursions and generating functions for the size of spheres of multipermutations under the maximum distance are given.

1 Introduction

Let $S_{l,n}$ denote the set of sequences $(x_1, x_2, \ldots, x_{ln})$ where all $x_i \in [n] = \{1, 2, \ldots n\}$ and each element in $[n]$ occurs exactly l times in the sequence. Clearly, $S_{1,n} = S_n$, the set of permutations of $[n]$. We call the elements of $S_{l,n}$ multipermutations. Subsets of $S_{1,n}$, also known as permutation arrays, have been used as codes, for example for communication over power lines, see e.g. [10] and for flash memories, see [5]. Subsets of $S_{l,n}$ can be used as codes for multi level flash memories, see Shieh and Tsai [7]. They call such codes frequency permutation arrays (FPA). Another name for such codes is $[l, l, \ldots, l]$ constant composition codes over $\{1, 2, \ldots, n\}$. The distance between sequences used depends on the application. For the power line communication, the distance considered is the Hamming distance, and most of the work done on permutation arrays considers this distance. For the flash memory application, the relevant distance is the maximum distance (obtained from the l_∞ norm), that is

$$d_{\max}((x_1, x_2, \ldots, x_{ln}), (y_1, y_2, \ldots, y_{ln})) = \max\{|x_i - y_i| \mid 1 \le i \le ln\}.$$

A sphere of radius d and center $\rho \in S_{l,n}$ is the set of multipermutations $\tau \in S_{l,n}$ satisfying $d_{\max}(\pi, \rho) \le d$. Since $d_{\max}(p\pi, p\rho) = d_{\max}(\pi, \rho)$ for any permutation $p \in S_{ln}$, the size of a sphere of radius d is independent its center. We denote the size by $V_l(d, n)$.

Several bounds for binary codes under the Hamming distance, like the Hamming bound, the Varshamov-Gilbert bound, and the Plotkin bound, can directly be generalized to frequency permutation arrays under the maximum distance, and the size of spheres is an important part of these bounds. For binary codes, there are simple expressions for the size of spheres. However, it does not seem to be simple expressions for $V_l(d, n)$. The simplest case, $V_1(d, n)$ has been studied by a few authors since the 1950s, the main early paper is [4]. Kløve [2] gave a survey of known result as well as many new results. It is well known that $V_1(d, n)$ can be given as the value of the permanent of some matrix, and Shieh and Tsai [7] showed that the same is true for $V_l(d, n)$. The permanent of an $N \times N$ matrix $M = (m_{i,j})$ is defined by

$$\operatorname{per} M = \sum_{\pi \in S_N} m_{1,\pi_1} \cdots m_{N,\pi_N}. \tag{1}$$

Shieh and Tsai [7] showed that if $M_{l,n}$ is the $ln \times ln$ matrix defined by

$$m_{i,j} = 1 \text{ if } \left| \left\lceil \frac{i}{l} \right\rceil - \left\lceil \frac{j}{l} \right\rceil \right| \le d \text{ and } m_{i,j} = 0 \text{ otherwise,}$$

then $V_l(d, n) = \operatorname{per} M/(l!)^n$.

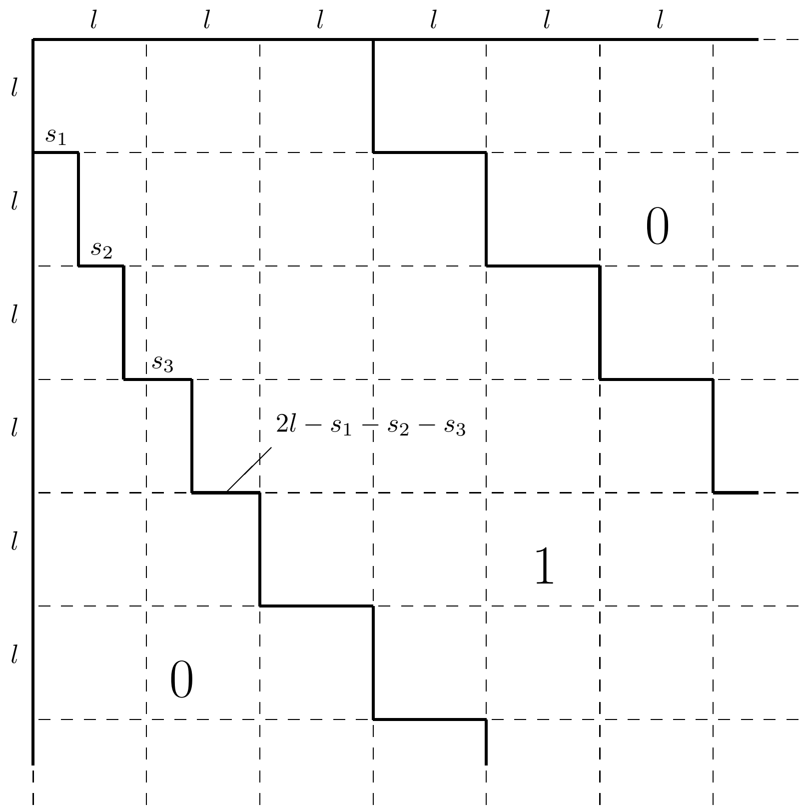
It is known that $V_1(d, n)$ satisfies a linear recurrence in n and that the generating function $\sum_{n=0}^{\infty} V_1(d, n)x^n$ is a rational function. In this paper we show that $V_l(d, n)$ also satisfies some linear recurrence. In [2] we studied recursions for $l = 1$ (and all d), and in [3] we studied recursions for $d = 1$ (and all l). In this paper we will give a recursion for general d and l. In Section 2 we consider $d = 2$ and in Section 3 general d. In Section 6 we look at the length of the recursion.

The existence of a linear recursion for $V_l(d, n)$ implies that

$$\mu_{d,l} = \lim_{n \to \infty} V_l(d, n)^{\frac{1}{nl}} \tag{2}$$

exists. We will briefly discuss properties and the computation of $\mu_{d,l}$ in Section 5.

Figure 1: The form of the matrix $M_l(n, s_1, s_2, s_3)$. The upper left corner is shown.



2 Recursion for $d = 2$

Before we give the results for general d, we give the result for $d = 2$. Our main reason for this is that this special case is easier to explain, and once this special case is described, it is easy to see how the construction generalizes to any d. And so, let $d = 2$ and suppose that l is given. For each $n \geq 1$ we will define a class of $ln \times ln$ matrices $M_l(n, s_1, s_2, s_3)$, given by n and three parameters, $s_1, s_2, s_3 \in \{0, 1, \ldots, l\}$, where

$$l \leq s_1 + s_2 + s_3 \leq 2l.$$

Explicitly, $M_l(n, s_1, s_2, s_3) = (m_{i,j})$ where $1 \leq i, j \leq ln$, and where $m_{i,j} = 1$ if (and only if)

$0 < i \leq l$ and $0 < j \leq 3l$,
$l < i \leq 2l$ and $s_1 < j \leq 4l$,
$2l < i \leq 3l$ and $s_1 + s_2 < j \leq 5l$,
$3l < i \leq 4l$ and $s_1 + s_2 + s_3 < j \leq 6l$,
$rl < i \leq (r+1)l$ and $(r-2)l < j \leq (r+3)l$ for $4 \leq r \leq n-1$.

The form of $M(n, s_1, s_2, s_3)$ is illustrated in Fig. 1. The 1s are located symmetrically around the main diagonal, except for the first $2l$ rows and columns. The matrix $M_{l,n}$ given by Shieh and Tsai [7] mentioned above is $M_l(n, 0, 0, l)$ in this notation.

Let $B_l(n, s_1, s_2, s_3) = \operatorname{per} M_l(n, s_1, s_2, s_3)$. Our main goal for $d = 2$ is to determine

$$V_l(2, n) = B_l(n, 0, 0, l)/(l!)^n.$$

We will show that $B_l(n, s_1, s_2, s_3)$ satisfies a linear recursion in n for all choices of s_1, s_2, s_3. In particular, $V_l(2, n)$ satisfies a linear recursion in n.

One way to evaluate a permanent is to start by expanding by the first row, next evaluate the permanent of each of the resulting matrices by the same method, etc. To make this work, we have to keep track of the possible matrices that will occur. For $l = 1$, this is how we obtained the first recursion in [2], and for $d = 1$ this was done in [3]. Here, we will give a variant of the method in [3] which is more efficient, we will only need to keep track of the matrices after having expanded l rows, $2l$ rows, etc. We will show that we then only get matrices of the form $M(n', s_1, s_2, s_3)$, and we obtain a recursion for $B_l(n, s_1, s_2, s_3)$.

The evaluation by expanding is done as follows. The first row of $M_l(n, s_1, s_2, s_3)$ has the value 1 in exactly the first $3l$ positions. For each j, $1 \leq j \leq 3l$, let M_j be the $(ln-1) \times (ln-1)$ matrix obtained by removing the first row and the jth column of $M_l(n, s_1, s_2, s_3)$. Then

$$\text{per}\, M_l(n, s_1, s_2, s_3) = \sum_{j=1}^{3l} \text{per}\, M_j.$$

We call this expansion at the first level.

At the second level we expand each M_j. We see that the $3l-1$ matrices we obtain when expanding M_j are the matrices we obtain by removing from $M_l(n, s_1, s_2, s_3)$ its first two rows, its column j and another of its first $3l$ columns. Hence, the possible matrices we get when expanding at the second level are obtained by removing from $M_l(n, s_1, s_2, s_3)$ its first two rows and two of its first $3l$ columns in all possible ways.

Similarly for the next level of the expansion, etc. When expanding at the lth level, we get the matrices obtained by removing from $M_l(n, s_1, s_2, s_3)$ its first l rows and l of its first $3l$ columns, and the permanent of $M_l(n, s_1, s_2, s_3)$ is the sum of the permanents of these matrices. Moreover, the resulting matrices are of the form $M_l(n-1, t_1, t_2, t_3)$, and each such matrix may be obtained in many different ways when removing l columns from $M_l(n, s_1, s_2, s_3)$. Moreover, for each set of l columns, they can be picked in $l!$ ways since the order matters. If not all the s_1 first columns have been removed, the resulting matrix will have an all zero column and so have permanent equal to zero. Therefore, we can assume that the first s_1 columns have been removed. Of the remaining $l - s_1$ columns, some of those removed may be from the next s_2 columns, denote the number of these by $s_2 - t_1$ (so that t_1 of these columns were not removed). Similarly, some of the next s_3 columns may have been removed, with t_2 remaining, and some of the next $2l - s_1 - s_2 - s_3$ with t_3 remaining. Finally, since we have removed l columns in all, we have removed

$$l - \big(s_1 + (s_2 - t_1) + (s_3 - t_2) + (2l - s_1 - s_2 - s_3 - t_3)\big) = t_1 + t_2 + t_3 - l$$

columns from the l columns numbered $2l+1, 2l+2, \ldots 3l$. The resulting matrix is $M_l(n-1, t_1, t_2, t_3)$. This matrix will therefore appear

$$l! \binom{s_2}{t_1} \binom{s_3}{t_2} \binom{2l - s_1 - s_2 - s_3}{t_3} \binom{l}{t_1 + t_2 + t_3 - l}$$

times in the total sum. Hence we get the following recursion.

$$B_l(n, s_1, s_2, s_3) = \sum_{t_1=0}^{s_2} \sum_{t_2=0}^{s_3} \sum_{t_3=\max\{0, l-t_1-t_2\}}^{2l-s_1-s_2-s_3} l! \binom{s_2}{t_1} \binom{s_3}{t_2} \binom{2l - s_1 - s_2 - s_3}{t_3} \cdot \binom{l}{t_1 + t_2 + t_3 - l} B_l(n-1, t_1, t_2, t_3). \tag{3}$$

We have $B_l(1, s_1, s_2, s_3) = l!$ for all s_1, s_2, s_3. Hence, (3) and induction show that $(l!)^n$ divides $B_l(n, s_1, s_2, s_3)$. Let

$$b_l(n, s_1, s_2, s_3) = B_l(n, s_1, s_2, s_3)/(l!)^n.$$

Then $V_l(2, n) = b_l(n, 0, 0, l)$, and

$$b_l(n, s_1, s_2, s_3) = \sum_{t_1=0}^{s_2} \sum_{t_2=0}^{s_3} \sum_{t_3=\max\{0,l-t_1-t_2\}}^{2l-s_1-s_2-s_3} \binom{s_2}{t_1}\binom{s_3}{t_2}\binom{2l-s_1-s_2-s_3}{t_3} \cdot \binom{l}{t_1+t_2+t_3-l} b_l(n-1, t_1, t_2, t_3).$$

Starting from the initial values $b_l(1, s_1, s_2, s_3) = 1$, this gives an efficient recursion for evaluating $V_l(2, n)$. In Appendix 1 we give the values of $V_l(2, n)$ for $l \le 5$ and $n \le 30$ as an illustration.

3 Recursion for general d

We just list the definitions and results for general d. The proof is similar to the case $d = 2$ and is omitted. Let $X_{d,l}$ be the set of sequences $(s_1, s_2, \ldots, s_{2d-1})$ where $s_1, s_2, \ldots, s_{2d-1} \in \{0, 1, \ldots, l\}$ and

$$(d-1)l \le s_1 + s_2 + \cdots + s_{2d-1} \le dl.$$

We denote the number of such sequences by $\nu(d, l)$. We discuss the evaluation of $\nu(d, l)$ in Section 6.

For each sequence in $X_{d,l}$ we define a matrix $M_l(n, s_1, s_2, \ldots, s_{2d-1}) = (m_{i,j})$ where $1 \le i, j \le ln$, and where $m_{i,j} = 1$ if (and only if)

$0 < i \le l$ and $0 < j \le (d+1)l$,

$rl < i \le (r+1)l$ and $s_1 + s_2 + \cdots + s_r < i \le (r+d+1)l$, for $1 \le r \le 2d-1$,

$rl < i \le (r+1)l$ and $(r-d+1)l < j \le (r+d+1)l$, for $2d \le r \le n-1$.

Then the permanent of $M_l(n, s_1, s_2, \ldots, s_{2d-1})$ is $l!\, b_l(n, s_1, s_2, \ldots, s_{2d-1})$, where $b_l(1, s_1, s_2, \ldots, s_{2d-1}) = 1$ for all $(s_1, s_2, \ldots, s_{2d-1}) \in X_{d,l}$ and

$$b_l(n, s_1, s_2, \ldots, s_{2d-1}) = \sum b_l(n-1, t_1, t_2, \ldots, t_{2d-1}) \binom{dl - \sum_{j=1}^{2d-1} s_j}{t_{2d-1}} \cdot \binom{l}{\sum_{j=1}^{2d-1} t_j - (d-1)l} \prod_{j=1}^{2d-2} \binom{s_{j+1}}{t_j}, \tag{4}$$

where the summation is over all sequences $(t_1, t_2, \ldots, t_{2d-1}) \in X_{d,l}$ such that

$$t_j \le s_{j+1} \text{ for } 1 \le j \le 2d-2 \text{ and } t_{2d-1} \le dl - \sum_{j=1}^{2d-1} s_j.$$

In particular,

$$V_l(d, n) = b_l(n, \overbrace{0, 0, \ldots, 0}^{d} \overbrace{1, 1, \ldots, 1}^{d-1}).$$

We see that if we list all the values $b_l(n, s_1, s_2, \ldots, s_{2d-1})$ for a fixed n in a vector $U(n)$ of length $\nu(d, l)$, then (4) essentially describes a matrix $T_{d,l}$ such that $U(n) = U(n-1)T_{d,l}$. From standard theory of linear recursion, if

$$G_{d,l}(x) = \sum_{i=0}^{\nu(d,l)} \gamma_i x^i$$

is the characteristic polynomial if $T_{d,l}$, then

$$\sum_{i=0}^{\nu(d,l)} \gamma_i b_l(n+i, s_1, s_2, \ldots, s_{2d-1}) = 0$$

for all n and all $(s_1, s_2, \ldots, s_{2d-1}) \in X_{d,l}$. In particular

$$\sum_{i=0}^{\nu(d,l)} \gamma_i V_l(d, n+i) = 0.$$

4 Generating functions

From the recursion we can also find the generating function $\sum_{n=0}^{\infty} V_l(d,n)x^n$. The method is standard and we have described it in more detail in [2] and [3]. The generating function is a rational function $f_{d,l}(x)/g_{d,l}(x)$ where $x^{\deg g_{d,l}(x)} g_{d,l}(1/x)$ divides $G_{d,l}$. In particular, $\deg g_{d,l}(x) \le \deg G_{d,l}(x) = \nu(d,l)$.

For $d = 2$, we have computed the generating functions for $l \le 5$. They are given in Appendix 2. It turns out that

$$\deg g_{2,l}(x) = \frac{(l+1)(l+2)(2l+3)}{6}$$

for these values, whereas

$$\nu(2,l) = \frac{(l+1)(l+2)(2l+3)}{6} + \frac{l(l+1)(2l+1)}{6}.$$

Probably $\deg g_{d,l}(x) < \nu(d,l)$ for all $d > 1$ and all l (for $d = 1$ we have $\deg g_{1,l}(x) = \nu(1,l) = l+1$). We note that in [2], where we treated $l = 1$ in detail, we gave two different constructions of recursions. The first construction in [2] is the special case $l = 1$ of the of the general construction above, and we showed that the $\deg g_{d,1}(x) < \nu(d,1)$ by giving a second construction that gives a shorter recursion (of approximately half the length of the recursion obtained from the first construction when d is large). The generating functions we have found for $d = 2$, $l \le 5$ give recursions of approximately half the length of the ones obtained by the construction above. This indicate that there may exist a "second construction" also for general l and $d > 1$. The second construction for $l = 1$ does not seem to generalize directly to larger l, however. The main point of the second construction in [2] is that expansion of the permanent is sometimes done by the first row, sometimes by the first column (depending on which have the least number of 1s). In a possible generalization to $l > 1$, we can not easily treat l levels at the time as we did for the construction above, and when we expand one level at a time (as we did for $d = 1$ in [3]), the number of matrices we have to consider grows by a factor of approximately l.

5 Computing $\mu_{d,l}$

From the theory of linear recurrences, we know that if

$$\sum_{n=0}^{\infty} V_l(d,n)x^n = \frac{f_{d,l}(x)}{g_{d,l}(x)}$$

(where $f_{d,l}(x)$ and $g_{d,l}(x)$ have no common factors), then $\mu_{d,l}$, defined by (2) above, is the largest positive root of the equation $g_{d,l}(1/x) = 0$.

Shieh and Tsai [7] gave the following bounds on $V_l(d,n)$.

$$\frac{(nl)!(2dl+l)^{nl}}{(nl)^{nl}(l!)^n 2^{2dl}} \le V_l(d,n) \le \frac{((2dl+l)!)^{nl/(2dl+l)}}{(l!)^n}.$$

This implies the following bounds on $\mu_{d,l}$.

$$\frac{2dl+l}{e(l!)^{1/l}} \le \mu_{d,l} \le \frac{((2dl+l)!)^{1/(2dl+l)}}{e(l!)^{1/l}} \tag{5}$$

(where $e = \exp(1) = 2.71\ldots$). We see that

$$\lim_{d\to\infty} \frac{\mu_{d,l}}{2d+1} = \frac{l}{e(l!)^{1/l}}.$$

For $l = 1$ this was known already by Lehmer [4]. We also see that

$$\lim_{l\to\infty} \mu_{d,l} = 2d+1$$

for all d.

From the polynomials $g_{1,l}(x)$ given in [3] and the polynomials $g_{2,l}(x)$ given in Appendix 2, we have computed $\mu_{1,l}$ for $l \le 8$ and $\mu_{2,l}$ for $l \le 5$. We give the result in a table where we have also included the lower and upper bounds in (5) .

d	l	lower bound	$\mu_{d,l}$	upper bound
1	2	1.56078	1.91548	2.11693
1	3	1.82207	2.09554	2.28227
1	4	1.99450	2.21786	2.38951
1	5	2.11817	2.30733	2.46564
1	6	2.21185	2.37609	2.52295
1	7	2.28563	2.43086	2.56788
1	8	2.34545	2.47569	2.60419
1	∞	3.00000	3.00000	3.00000
2	1	1.83940	2.33355	2.60517
2	2	2.60130	2.94454	3.20229
2	3	3.03678	3.30212	3.53495
2	4	3.32417	3.54123	3.75192
2	5	3.53028	3.71429	3.90652
2	∞	5.00000	5.00000	5.00000

6 On the length of the recursion

The length of the recursion is $\nu(d,l)$. We will now consider the computation of $\nu(d,l)$ and also give an approximation.

For small values of d, we can find $\nu(d,l)$ directly. Clearly, $\nu(1,l) = l+1$. We have

$$\begin{aligned}\nu(2,l) &= \sum_{s_1=0}^{l}\sum_{s_2=0}^{l}\sum_{s_3=\max\{0,l-s_1-s_2\}}^{\min\{2l-s_1-s_2,l\}} 1 \\ &= \sum_{s_1=0}^{l}\sum_{s_2=0}^{l-s_1-1}\sum_{s_3=l-s_1-s_2}^{l} 1 + \sum_{s_1=0}^{l}\sum_{s_2=l-s_1}^{l}\sum_{s_3=0}^{2l-s_1-s_2,l} 1 \\ &= \frac{2l^3+3l^2+1}{6} + \frac{9l^3+2l^2+13l+6}{6} = \frac{(l+1)(2l^2+4l+3)}{3}.\end{aligned}$$

Already for $l = 3$ this direct computation becomes cumbersome. Another method is more efficient, and we describe this next.

A *composition* of M of length m and maximal size l is a sequence $(t_1, t_2, \ldots, t_m)$ of integers $t_i \in \{0, 1, \ldots, l\}$ such that $\sum t_i = M$. We denote the number of such compositions by $\sigma_l(m, M)$. Clearly, $\sigma_l(m, M) = 0$ for $M > lm$. In this notation,

$$\nu(d, l) = \sum_{M=(d-1)l}^{dl} \sigma_l(2d-1, M). \tag{6}$$

No closed expression for $\sigma_l(m, M)$ is known in the general case. We do have the following expression, essentially due to MacMahon, see [6].

$$\sigma_l(m, M) = \sum_{\substack{\sum_{i=0}^{l} m_i = m, \\ \sum_{i=0}^{l} i m_i = M}} \frac{m!}{m_0!\, m_1! \cdots m_l!}.$$

Another expression was given by Stanley [9], problem 29, page 117. Slightly reformulated, it is as follows.

$$\sigma_l(m, M) = \sum_{s=0}^{\lfloor M/(l+1) \rfloor} (-1)^s \binom{m-1+M-s(l+1)}{m-1} \binom{m}{s}.$$

Two recursive methods to compute $\sigma_l(m, M)$ were given in [6]. For numerical computing of the $\nu(d, l)$ is it probably simplest to combine (6) with the following recursion ([6], Lemma 2):

$$\sigma_l(0,0) = 1, \quad \sigma_l(0, M) = 0 \text{ for } M > 0,$$
$$\sigma_l(m, M) = \sum_{j=0}^{l} \sigma_l(m-1, M-j) \text{ for } m \geq 1.$$

To find the polynomial $\nu(d, l)$ for a fixed d, another recursion is more efficient. Define the integers $A_{m,b,i}$ for $0 \leq b \leq m-1$ and $0 \leq i \leq m-1$, recursively as follows:

$$A_{m,0,i} = 0 \text{ for } 0 \leq i \leq m-2, \quad A_{m,0,m-1} = 1,$$
$$A_{m,b,0} = \sum_{j=0}^{m-2} \left\{ A_{m-1,b-1,j} \binom{l-b+j+2}{j+1} - A_{m-1,b,j} \binom{j+1-b}{j+1} \right\} \text{ for } 1 \leq b \leq m-1,$$
$$A_{m,b,i} = A_{m-1,b,i-1} - A_{m-1,b-1,i-1} \text{ for } 1 \leq b \leq m-1, 1 \leq i \leq m-1.$$

Then, by Lemma 4 in [6],

$$\sigma_l(m, M) = \sum_{i=0}^{m-1} A_{m,b,i} \binom{M-b(l+1)+i}{i} \tag{7}$$

for $bl \leq M \leq (b+1)l$.

Remark. Lemma 4 in [6] only states this result for $bl < M \leq (b+1)l$, but it is easy to check that its proof is valid also for $M = bl$.

Combining (6) and (7) we get

$$\begin{aligned}
\nu(d,l) &= \sum_{M=(d-1)l}^{dl} \sigma_l(2d-1, M) \\
&= \sum_{M=(d-1)l}^{dl} \sum_{i=0}^{2d-2} A_{2d-1,d-1,i} \binom{M-(d-1)(l+1)+i}{i} \\
&= \sum_{i=0}^{2d-2} A_{2d-1,d-1,i} \sum_{M=(d-1)l}^{dl} \binom{M-(d-1)(l+1)+i}{i} \\
&= \sum_{i=0}^{2d-2} A_{2d-1,d-1,i} \left\{ \binom{l-d+2+i}{i+1} - \binom{i+1-d}{i} \right\}.
\end{aligned}$$

We state this as a theorem.

Theorem 1. *For $d \geq 1$ and $l \geq 1$ we have*

$$\nu(d,l) = \sum_{i=0}^{2d-2} A_{2d-1,d-1,i} \left\{ \binom{l-d+2+i}{i+1} - \binom{i+1-d}{i} \right\}.$$

The $A_{m,b,i}$ are polynomials in l. From the recursive definition, we see by induction that the degree of $A_{m,b,i}$, as a polynomial of l, is at most $m-1-i$. From Theorem 1 we get the following corollary.

Corollary 1. *$\nu(d,l)$ is a polynomial in l of degree (at most) $2d-1$.*

We note that for $x=-1$, we get $\binom{x-d+2+i}{i+1} - \binom{i+1-d}{i} = 0$. Hence $l+1$ divides the polynomial $\binom{l-d+2+i}{i+1} - \binom{i+1-d}{i}$. Therefore, Theorem 1 also gives the following corollary.

Corollary 2. *$l+1$ divides $\nu(d,l)$ (as polynomials).*

As an illustration, we have computed $\nu(d,l)$ for $d \leq 10$ using Theorem 1. The polynomials are listed in Appendix 3.

In general, for a fixed d we get $\nu(d,l) \sim \omega_d \, l^{2d-1}$ for some constant ω_d. This follows from a lemma by Laplace, see e.g. [9], problem 51, page 121 (solution page 166).

Lemma 1. *For positive integers n and k, the (euclidian) volume in R^n defined by all $(x_1, x_2, \ldots, x_n) \in R^n$ such that*

$$0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq n \text{ and } k-1 \leq x_1 + x_2 + \cdots + x_n \leq k$$

is given by $A(n,k)/n!$, where $A(n,k)$ is the Eulerian number.

An expression for $A(n,k)$ is

$$A(n,k) = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k+1-j)^n.$$

From this lemma we get the following corollary.

Corollary 3. *For positive integers n and k, the (euclidean) volume in R^n defined by all $(x_1, x_2, \ldots, x_n) \in R^n$ such that*

$$0 \leq x_i \leq l \text{ for } 1 \leq i \leq n \text{ and } (k-1)l \leq x_1 + x_2 + \cdots + x_n \leq kl$$

is given by $A(n,k)\, l^n/n!$.

Since $\nu(d,l)$ is the number of *integer* sequences $(s_1, s_2, \ldots, s_{2d-1})$ where $0 \leq s_i \leq l$ for $1 \leq i \leq 2d-1$ and

$$(d-1)l \leq s_1 + s_2 + \cdots + s_{2d-1} \leq dl,$$

we get

$$\nu(d,l) \sim \omega_d \, l^{2d-1},$$

where

$$\omega_d = \frac{A(2d-1, d-1)}{(2d-1)!} = \frac{1}{(2d-1)!} \sum_{j=0}^{d} (-1)^j \binom{2d}{j} (d-j)^{2d-1}.$$

Carlitz et al. [1] showed that

$$\frac{A(2d-1, d-1)}{(2d-1)!} \sim \sqrt{\frac{3}{d\pi}}.$$

Hence we also get

$$\omega_d \sim \sqrt{\frac{3}{d\pi}} \approx \frac{0.977205}{\sqrt{d}},$$

and

$$\nu(d,l) \approx \sqrt{\frac{3}{d\pi}} \, l^{2d-1}.$$

Some values of ω_d are given in the following table.

d	ω_d	$\omega_d \sqrt{d}$
5	$\frac{15619}{36288}$	0.962443
10	$\frac{37307713155613}{121645100408832}$	0.969849
15	$\frac{18497593486903125823791655511}{73681349947830849621196800000}$	0.972307
20	$\frac{10572354363336924802260977429426060187229}{48566385907612960377714956523578327040000}$	0.973534
25	$\frac{6753619678232365559234568647012257532483376611543319764453}{3465993527260783822633915460520201577706853740052480000000}$	0.974269
50		0.975738
100		0.976472

References

[1] L. Carlitz, D. C. Kurtz, R. Scoville, O. P. Stackelberg, "Asymptotic properties of Eulerian numbers," Z. Wahrsch. verw. Geb., 23, 1972, pp. 47–54.

[2] T. Kløve, "Spheres of Permutations under the Infinity Norm - Permutations with limited displacement," Department of Informatics, University of Bergen, Report No. 376, November 2008.
http://www.ii.uib.no/publikasjoner/texrap/pdf/2008-376.pdf

[3] T. Kløve, "Frequency permutation arrays within distance one," Department of Informatics, University of Bergen, Report No. 382, January 2009.
http://www.ii.uib.no/publikasjoner/texrap/pdf/2009-382.pdf

[4] D. H. Lehmer, "Permutations with strongly restricted displacements, " in Combinatorial Theory and its Applications II, P. Erdös, A. Rényi and V. T. Sós (eds.), Amsterdam: North Holland Publ., 1970, pp. 755–770.

[5] T.-T. Lin, S.-C. Tsai, W.-G. Tzeng, "Efficient encoding and decoding with permutation arrays," IEEE Intern. Symp. Information Theory, 2008, pp. 212–214.

[6] I. Naydenova and T. Kløve, "Generalized Bose-Lin codes, a class of codes detecting asymmetric errors, " IEEE Trans. Information Theory, 53, 2007, pp. 1188–1193.

[7] M.-Z. Shieh and S.-C. Tsai, "Decoding frequency permutation arrays under infinity norm." Manuscript submitted to ISIT 2009.

[8] N. J. A. Sloane, "The On-Line Encyclopedia of Integer Sequences,"
http://www.research.att.com/∼njas/sequences/

[9] R. Stanley, "What is Enumerative Combinatorics," second ed., Chapter 1 (Version 2.2.3, 24. February 2009),
http://www-math.mit.edu/∼rstan/ec/ch1.pdf

[10] A. J. H. Vinck, "Coded modulation for powerline communications, " AEÜ Int. J. Electron. and Commun., 54, 2000, pp. 45–49.

Appendix, tables of $V_l(2,n)$

For each $l \leq 5$, we have included the values of $V_l(2,n)$ for $n \leq 30$.

$V_1(2,n)$:
1,
2,
6,
14,
31,
73,
172,
400,
932,
2177,
5081,
11854,
27662,
64554,
150639,
351521,
820296,
1914208,
4466904,
10423761,
24324417,
56762346,
132458006,
309097942,
721296815,
1683185225,
3927803988,
9165743600,
21388759708

$V_2(2,n)$:
1,
6,
90,
786,
6139,
54073,
477228,
4113864,
35579076,
308945881,
2679325561,
23222971098,
201351085146,
1745886520422,
15137227297027,
131243141767393,
1137923361184848,
9866167034815440,
85542686564024352,
741681846818742097,
6430615722417399697,
55755456782298589758,
483417295385684950170,
4191379737416535376842,
36340578335963981789947,
315084221766193232580121,
2731879114043342105649060,
23686249544074420814317464,
205367219326632031251003468

$V_3(2, n)$:
1,
20,
1680,
61340,
1886431,
69496201,
2568223000,
91712960320,
3290467596440,
118724053748417,
4276273204804217,
153904262366842444,
5541519231941145440,
199545071017172522244,
7184755645113714298863,
258691998154725997048673,
9314545233907934721851472,
335381528796576643131475840,
12075785123501322139824319056,
434802491356562053648077727185,
15655568415749951175242827270209,
563696780587887749260530593413668,
20296551390368951276117746176119600,
730800730000286072797674937806710476,
26313322704376860235616283750860997359,
947441507678659152690546991358137409609,
34113723377013986574155024541718264474920,
1228303926845562546407588728737705181416896,
44226498504901457026109716494970551430847656

$V_4(2,n)$:
1,
70,
34650,
5562130,
708212251,
114774147001,
18679465660540,
2906167849870600,
454904037056013460,
71729455730285511001,
11285129375761977675001,
1773699532985462649188410,
278931562239767189408085850,
43869015908453746845566145990,
6898693708786029238293860809251,
1084865341390442288732669957148001,
170605963060816377946936433265175680,
26829411396875692269491197638918648400,
4219165662049303123773116859323196816720,
663502408038018748448058464247159216890001,
104341873350788394071924693694504070778210001,
16408717392067338871103828830379304070893724350,
2580421324598654367668863895234127817508960916250,
405794946455289176944249463868518316024763940672650,
63814981233441920799463600763450318436722590911956251,
10035491627130303719894763519343210818431070226780335001,
1578173188016415952171407375544957497245132745688595946420,
248182222455085215868443646494941162957177452913924510637400,
39028932930234202548832698372061496016964222910626480689250780

$V_5(2,n)$:
1,
252,
756756,
549676764,
298227062281,
218838390759073,
161446400503248672,
112632613848657302400,
79169699996993643966432,
56151546386557366024202177,
39717291593245217794329362081,
28058660061656964336359435570604,
19835819533825566529982592591911412,
14024417724324420598672399947721245804,
9914206081036463014882722168252570938889,
7008596284293402975749309111124669929079521,
4954676885097638926007640423100194180529855296,
3502659589845301193905028251874353899223998638208,
2476160267409321946445662150301548547825713614803904,
1750492069977099993617695861204414333904857504132837761,
1237490232072931405029781301450935949112692192184524220417,
874829200533708267279601442713056394858594226903974162717596,
618450208929741713644453490163133793621720303661713198326788756,
437206126713271660771172624406779729426859160661260215080381053692,
309077746685743196569847493784061900048383001921873806532692673661065,
218498887652047156870569796695108715135429114419066813146302911446876225,
154465226025904937372285035370321561654264346944052855436491085065008815488,
109197380141509750741925933786572475433390662020423361336205136335688514169600,
77195807294831479320075280615411171146694485973439617756494289638548283376478208

Appendix, table of the generating functions $f_{2,l}(x)/g_{2,l}(x)$ for $l \leq 5$

$$f_{2,1}(x) = 1 - x$$
$$g_{2,1}(x) = 1 - 2\,x - 2\,x^3 + x^5$$

$$\begin{aligned}f_{2,2}(x) = {} & 1 - 7\,x + 2\,x^2 - 52\,x^3 + 56\,x^4 + 132\,x^5 - 80\,x^6 - 80\,x^7 + 64\,x^8 \\ & - 80\,x^9 + 32\,x^{10}\end{aligned}$$
$$\begin{aligned}g_{2,2}(x) = {} & 1 - 8\,x + 4\,x^2 - 98\,x^3 + 64\,x^4 + 445\,x^5 - 194\,x^6 - 488\,x^7 + 652\,x^8 \\ & - 932\,x^9 + 544\,x^{10} - 320\,x^{11} + 16\,x^{12} + 112\,x^{13} - 32\,x^{14}\end{aligned}$$

$$\begin{aligned}f_{2,3}(x) = {} & 1 - 31\,x + 18\,x^2 - 5856\,x^3 + 11592\,x^4 + 329949\,x^5 - 165051\,x^6 \\ & - 5237784\,x^7 + 17799588\,x^8 - 104985396\,x^9 + 185382270\,x^{10} \\ & - 648773550\,x^{11} + 254062332\,x^{12} - 403944732\,x^{13} - 259841844\,x^{14} \\ & + 3689665830\,x^{15} - 2040982758\,x^{16} - 5836508136\,x^{17} + 5014992204\,x^{18} \\ & + 3849286212\,x^{19} - 917798607\,x^{20} - 704513619\,x^{21} - 1725057486\,x^{22} \\ & - 146677716\,x^{23} - 172186884\,x^{24} + 100442349\,x^{25} + 43046721\,x^{26}\end{aligned}$$
$$\begin{aligned}g_{2,3}(x) = {} & 1 - 32\,x + 30\,x^2 - 6926\,x^3 + 10338\,x^4 + 484180\,x^5 - 190920\,x^6 \\ & - 8194962\,x^7 + 32968386\,x^8 - 223701894\,x^9 + 437423949\,x^{10} \\ & - 1672218432\,x^{11} + 512697276\,x^{12} - 418712652\,x^{13} - 1551410172\,x^{14} \\ & + 14046926040\,x^{15} - 7944362064\,x^{16} - 34200623844\,x^{17} + 35768883636\,x^{18} \\ & + 27472210452\,x^{19} - 6756130701\,x^{20} - 4240400544\,x^{21} - 15479301690\,x^{22} \\ & - 3753981126\,x^{23} - 8670637062\,x^{24} - 170769708\,x^{25} - 2232052200\,x^{26} \\ & + 1151101206\,x^{27} + 889632234\,x^{28} - 143489070\,x^{29} - 43046721\,x^{30}\end{aligned}$$

$$
\begin{aligned}
f_{2,4}(x) = 1 &- 139\,x + 510\,x^2 \\
&- 545820\,x^3 + 4015240\,x^4 \\
&+ 642752180\,x^5 - 1184846320\,x^6 \\
&- 202781973040\,x^7 + 3110346706880\,x^8 \\
&- 84504639458960\,x^9 + 699054187326560\,x^{10} \\
&- 10909047845271040\,x^{11} + 22520223513992960\,x^{12} \\
&- 164003603686667520\,x^{13} - 353020728591080960\,x^{14} \\
&+ 25564835847599697920\,x^{15} - 68038069156922388480\,x^{16} \\
&- 941636714266491453440\,x^{17} + 4392585849414829178880\,x^{18} \\
&+ 14587815200757090877440\,x^{19} - 54043775528833422131200\,x^{20} \\
&- 104664789099155870515200\,x^{21} + 331804852736101980569600\,x^{22} \\
&- 700286428181410487992320\,x^{23} + 2024447466880321758167040\,x^{24} \\
&- 1173769931180653031718912\,x^{25} + 2281798810904943606418636 8\,x^{26} \\
&- 5385113787749743995125760\,x^{27} + 23282429070311131900280832 0\,x^{28} \\
&- 294839908562393836541706240\,x^{29} + 813401228701402580668907520\,x^{30} \\
&- 1082599689657060736609812480\,x^{31} + 982496333776152326874071040\,x^{32} \\
&- 1082409150610915684390010880\,x^{33} - 2940062389555207822796390 40\,x^{34} \\
&+ 4544465897921525603264102 40\,x^{35} + 1295781791108252796022947840\,x^{36} \\
&+ 4194669599322479840880230 40\,x^{37} - 3401508229416992832466452480\,x^{38} \\
&- 1085178457958512807945175040\,x^{39} + 5345265380411489003639930880\,x^{40} \\
&- 2036934787493407929107742720\,x^{41} - 8437661080764112652835225 60\,x^{42} \\
&+ 848912730686675412446085120\,x^{43} - 650385677379266604894781440\,x^{44} \\
&+ 485193403425206231787110400\,x^{45} - 204414289999917682891161600\,x^{46} \\
&+ 36385625335178860733399040\,x^{47} - 7424850878753083679047680\,x^{48} \\
&- 2329573556514219664343040\,x^{49} + 201683628454381757241753 6\,x^{50} \\
&- 153177439332441840943104\,x^{51}
\end{aligned}
$$

$$
\begin{aligned}
g_{2,4}(x) = 1 &- 140\,x + 580\,x^2 \\
&- 571250\,x^3 + 3834760\,x^4 \\
&+ 729293869\,x^5 - 1239228150\,x^6 \\
&- 229949173200\,x^7 + 3782161379100\,x^8 \\
&- 111399105463700\,x^9 + 956891573949280\,x^{10} \\
&- 15356381337693600\,x^{11} + 28634849575781200\,x^{12} \\
&- 164976373723456400\,x^{13} - 869871603512962400\,x^{14} \\
&+ 4444341511074909696\,0\,x^{15} - 109712489101144256000\,x^{16} \\
&- 188859426711247734400\,0\,x^{17} + 899717522199411055360\,0\,x^{18} \\
&+ 2948078301398380421120\,0\,x^{19} - 1089316848254768346931\,20\,x^{20} \\
&- 1939912794289840914432\,00\,x^{21} - 1018715321289225691136\,00\,x^{22} \\
&- 2185546694825738085990\,400\,x^{23} + 4698983226566609797120\,000\,x^{24} \\
&- 6852785584230685403840\,512\,x^{25} + 6563534131933468796059\,6480\,x^{26} \\
&+ 4622526007419592641085\,440\,x^{27} + 8833021903141399656923\,13600\,x^{28} \\
&- 1140130356342916159368\,069120\,x^{29} + 3212966248855102748834\,660352\,x^{30} \\
&- 4399095976350934094158\,233600\,x^{31} + 3584891799576034483057\,459200\,x^{32} \\
&- 3307947198357476906473\,881600\,x^{33} - 2679241028908679605046\,476800\,x^{34} \\
&+ 3836734028441249565187\,768320\,x^{35} + 6843574712321546357676\,441600\,x^{36} \\
&+ 8489245267080407603478\,52800\,x^{37} - 2371447028005289516712\,5913600\,x^{38} \\
&- 2096303413447907391527\,9769600\,x^{39} + 6439584576124550740080\,3287040\,x^{40} \\
&- 2298475630168193508153\,7536000\,x^{41} - 1218614340887832763249\,4592000\,x^{42} \\
&+ 1122431210374504532032\,8806400\,x^{43} - 8165868435331537681199\,923200\,x^{44} \\
&+ 6824247692849247568810\,475520\,x^{45} - 4797688016847534612597\,964800\,x^{46} \\
&+ 2982952571852303430687\,129600\,x^{47} - 1719409796540001628559\,769600\,x^{48} \\
&+ 9010210090004421248286\,72000\,x^{49} - 3024949324038525925604\,72064\,x^{50} \\
&- 1098363334959519896268\,963840\,x^{51} + 8513048937343972590747\,6480\,x^{52} \\
&- 6222833472880449788313\,600\,x^{53} - 2170013723876259413360\,640\,x^{54} \\
&+ 1531774393324418409431\,04\,x^{55}
\end{aligned}
$$

$f_{2,5}(x) =$
$1 - 626\,x$
$+ 9575\,x^{2}$
$- 50951400\,x^{3}$
$+ 1453082750\,x^{4}$
$+ 1251162871950\,x^{5}$
$- 6037402310750\,x^{6}$
$- 7873777615852000\,x^{7}$
$+ 520928521729080000\,x^{8}$
$- 67794375889052412500\,x^{9}$
$+ 2436350772022410118750\,x^{10}$
$- 176963886657387618950000\,x^{11}$
$+ 1339801054049322870468750\,x^{12}$
$- 53911883604977992507000000\,x^{13}$
$- 705490370575098855857187500\,x^{14}$
$+ 18277102137056948654827312500 0\,x^{15}$
$- 16412713651304233870684921875 00\,x^{16}$
$- 14146080652253288698959721875 0000\,x^{17}$
$+ 25625718465151604582184039843 75000\,x^{18}$
$+ 46175148612849644246036266796 875000\,x^{19}$
$- 65874104348491400956575180698 2421875\,x^{20}$
$- 65510050070902211649670923768 55468750\,x^{21}$
$+ 73187213858657376257044084913 57421875\,x^{22}$
$- 11630150714407491816231426035 24609375000\,x^{23}$
$+ 13961545294086412638819138906 169433593750\,x^{24}$
$- 64943766074296670792691067112 302246093750\,x^{25}$
$+ 30896434365879883582237491238 22414550781250\,x^{26}$
$+ 59511885435212150318448334057 6650390625000\,x^{27}$
$+ 61566678825397908725797741327 4716552734375000\,x^{28}$
$- 20272631105781306895899222872 46357934570312500\,x^{29}$
$+ 49633627817353452123912171593 714936645507812500\,x^{30}$
$- 19825313915302926747032598951 0106778076171875000\,x^{31}$
$+ 18308174314194045675920802653 36653679199218750000\,x^{32}$
$- 56733323005053849374695155658 11006777343750000000\,x^{33}$
$+ 90216088593915202631627776424 480496215820312 5000\,x^{34}$
$- 69918309146337984291146840626 8768029785156250000 00\,x^{35}$
$- 21540469544072752166265638310 2287014465332031250000\,x^{36}$
$+ 37529200180031354655484856056 94186683349609375000000\,x^{37}$
$+ 52999381989326539633006344412 90667373657226562500000\,x^{38}$
$- 10329857348775752432660178316 388528637695312500000000 0\,x^{39}$
$- 17766987329649697763687895988 062961669921875000000000 0\,x^{40}$
$- 19100174353290601136732338159 11261120300292968750000000\,x^{41}$
$+ 21412342758432331892768996601 209208786010742187500000000\,x^{42}$
$+ 21903015979921835767963419953 556031265258789062500000000\,x^{43}$
$- 36065939297443591965928053884 0962663879394531250000000000\,x^{44}$
$+ 10199161749407687356567204052 3978522338867187500000000000\,x^{45}$
$+ 25177279693766882116537521712 27415083312988281250000000000\,x^{46}$
$- 23269326396324954539155382883 30364044189453125000000000000\,x^{47}$
$+ 58757004774229851465797785921 3881759643554687500000000000\,x^{48}$
$- 96562081170283622838702625428 92063522338867187500000000000\,x^{49}$
$- 11291897774311211258667042396 683834075927734375000000000000\,x^{50}$
$+ 17460667961764230686480096101 444599151611328125000000000000\,x^{51}$
$+ 63894221847672369477788359186 808776855468750000000000000000\,x^{52}$
$+ 38167083253129331339290862043 292884826660156250000000000000\,x^{53}$

$$
\begin{aligned}
&- 4299642425199092352932951987385889053344726562500000000000000\, x^{54} \\
&+ 392274630823932089877432107353078842163085937500000000000000000\, x^{55} \\
&- 718931469549777052276991555028626441955566406250000000000000000\, x^{56} \\
&+ 859100638473055417530530415685703277587890625000000000000000000\, x^{57} \\
&- 3187814041552818655649870419284391403198242187500000000000000000\, x^{58} \\
&- 228379222470391243581377156723213195800781250000000000000000000\, x^{59} \\
&- 6096423100272460572460973818925857543945312500000000000000000000\, x^{60} \\
&- 2610825791319620260515236830544471740722656250000000000000000000\, x^{61} \\
&- 3298111490505286678099020308923721313476562500000000000000000000\, x^{62} \\
&- 180855464472491856883503315687179565429687500000000000000000000\, x^{63} \\
&+ 7203447550910285526788482215404510498046875000000000000000000000\, x^{64} \\
&+ 2311596499153025958847589612007141113281250000000000000000000000\, x^{65} \\
&+ 986675479502306460906315088272094726562500000000000000000000000\, x^{66} \\
&+ 3705344339016012409812133312225341796875000000000000000000000000\, x^{67} \\
&- 2716669477334799164070415496826171875000000000000000000000000000\, x^{68} \\
&- 3143612422247305202184271812438964843750000000000000000000000000\, x^{69} \\
&+ 1461607834758423817209303379058837890625000000000000000000000000\, x^{70} \\
&+ 12728540357989110982638597488403320312500000000000000000000000000\, x^{71} \\
&+ 2974752737209722088724374771118164062500000000000000000000000000\, x^{72} \\
&- 12002848711693930145204067230224609375000000000000000000000000000\, x^{73} \\
&- 2970120332888169870898127555847167968750000000000000000000000000\, x^{74} \\
&+ 2614442382752940587699413299560546875000000000000000000000000000\, x^{75} \\
&+ 467526544335499238222837448120117187500000000000000000000000000\, x^{76} \\
&+ 132618803577366098761558532714843750000000000000000000000000000\, x^{77} \\
&- 73857244412208907306194305419921875000000000000000000000000000\, x^{78} \\
&- 50648531686393544077873229980468750000000000000000000000000000\, x^{79} \\
&+ 70881446071155369281768798828125000000000000000000000000000000\, x^{80} \\
&+ 15380830708891153335571289062500000000000000000000000000000000\, x^{81} \\
&- 2329896180890500545501708984375000000000000000000000000000000\, x^{82} \\
&- 192409474402666091918945312500000000000000000000000000000000\, x^{83} \\
&- 58861821889877319335937500000000000000000000000000000000000\, x^{84} \\
&- 12125819921493530273437500000000000000000000000000000000000\, x^{85} \\
&+ 1331791281700134277343750000000000000000000000000000000000\, x^{86} \\
&+ 46566128730773925781250000000000000000000000000000000000\, x^{87}
\end{aligned}
$$

$g_{2,5}(x) =$
$1 - 627\,x$
$+ 9950\,x^{2}$
$- 51560102\,x^{3}$
$+ 1426944700\,x^{4}$
$+ 1301619619501\,x^{5}$
$- 5999501956225\,x^{6}$
$- 8136333969539150\,x^{7}$
$+ 554664957509089550\,x^{8}$
$- 74895161747694808200\,x^{9}$
$+ 2731880797416811164000\,x^{10}$
$- 199894375202242903659250\,x^{11}$
$+ 1412661073678553470620000\,x^{12}$
$- 51224012290774368976000000\,x^{13}$
$- 1010244621753949267801075000\,x^{14}$
$+ 22844077722991856038151398125\,x^{15}$
$- 192662010771997825832236346875\,x^{16}$
$- 18727273390025613708066989237500\,x^{17}$
$+ 3420819660068357865817963945000000\,x^{18}$
$+ 6081995743447028383359873956281250\,x^{19}$
$- 86685936952454631962656193186523437\,x^{20}$
$- 807514780192455994518699509827929687\,x^{21}$
$- 7011133831148611495252609181152343750\,x^{22}$
$- 19074968816813255153564948926485351562\,x^{23}$
$+ 212282265297690723566300476358968750000\,x^{24}$
$- 134047745000219985037067059648944287109375\,x^{25}$
$+ 49222227286092299628746595176327446777343\,x^{26}$
$+ 280296793648672988890467254536651044921875\,x^{27}$
$+ 1080826758653208473346469957935660900878906250\,x^{28}$
$- 3586514403500594146512336766581865444335937500\,x^{29}$
$+ 8810146281648482647723717808761933706787109375\,x^{30}$
$- 35940277692694064996720740504686670082031250000\,x^{31}$
$+ 31314852041132910247078774384909593106933593750\,x^{32}$
$- 90130674800512426901206501077899096792968750000\,x^{33}$
$- 150390237931311858907371719578259375103759765625\,x^{34}$
$- 907567085067946087288114707633550162944335937500\,x^{35}$
$- 3766804448426809639357126632848217756555175781250\,x^{36}$
$+ 7764052547870215935509216148705726841463623046875\,x^{37}$
$+ 14940674753139764374978733590483527566558837890625\,x^{38}$
$- 33571207837178240082682547111353027977874755859375\,x^{39}$
$- 205974069354730601074743977590650443629455566406250\,x^{40}$
$- 540402230616912893652629065022188357839172363281250\,x^{41}$
$+ 588387183937184612686277418983105895543060302734375\,x^{42}$
$+ 6439675463698795858521961496601536743786621093750000\,x^{43}$
$- 10210329047749596690041018796762239575601196289062500\,x^{44}$
$+ 32676913921227473255710393087291932909820556640625000\,x^{45}$
$+ 69926432422673837046105451184576527811889648437500000\,x^{46}$
$- 727593969397483446192762582112931243415832519531250000\,x^{47}$
$+ 784666869012635655812817519892552706298828125000000000\,x^{48}$
$- 354982183271208447857882391283553264135742187500000000\,x^{49}$
$- 1311531763145504444601283196373309908905029296875000000\,x^{50}$
$- 3107907971947698240896535223909499537658691406250000000\,x^{51}$
$+ 18387035117920488949083140429048934661865234375000000000\,x^{52}$
$- 82156946688571982980604927721556750450134277343750000000\,x^{53}$

$$
\begin{aligned}
&- 1243794458205953100580605244688870567321777343750000000000000\,x^{54}\\
&+ 2172817432594454387827931896201868017959594726562500000000000\,x^{55}\\
&- 3688284758047112957242527597187684745788574218750000000000000\,x^{56}\\
&+ 5200651344036302035894813892562040016174316406250000000000000\,x^{57}\\
&- 17970309439969373033274891122986148395538330078125000000000000\,x^{58}\\
&- 1283003565046541206411128362751114845275878906250000000000000\,x^{59}\\
&- 36124291872769963922254050986566834449768066406250000000000000\,x^{60}\\
&- 16058740434147762172763493746724321365356445312500000000000000\,x^{61}\\
&- 18258785083553543993074350272363901138305664062500000000000000\,x^{62}\\
&- 1218109905909306211171468485553550720214843750000000000000000\,x^{63}\\
&+ 49314138305344489472646685092953681945800781250000000000000000\,x^{64}\\
&+ 4850117624497198322742706210875511169433593750000000000000000\,x^{65}\\
&- 13344407021807015009221141069221496582031250000000000000000000\,x^{66}\\
&+ 25871855222601813163418379099607467651367187500000000000000000\,x^{67}\\
&- 21856330188690088304264102008342742919921875000000000000000000\,x^{68}\\
&+ 20939248138251240481049934267997741699218750000000000000000000\,x^{69}\\
&+ 35893721964375053693129880189895629882812500000000000000000000\,x^{70}\\
&+ 182809278802800455823655016422271728515625000000000000000000000\,x^{71}\\
&+ 44762046376416082445707070827484130859375000000000000000000000\,x^{72}\\
&- 223086894734007332391290247440338134765625000000000000000000000\,x^{73}\\
&- 53404500714882399512726962566375732421875000000000000000000000\,x^{74}\\
&+ 60973132141142341329756379127502441406250000000000000000000000\,x^{75}\\
&+ 12978340018833954565748572349548339843750000000000000000000000\,x^{76}\\
&- 1247205152485516204461455345153808593750000000000000000000000\,x^{77}\\
&- 2599130825874420295283198356628417968750000000000000000000000\,x^{78}\\
&+ 135969470882491592317819595336914062500000000000000000000000\,x^{79}\\
&+ 93067797388328593224287033081054687500000000000000000000000\,x^{80}\\
&+ 44629435294261388480663299560546875000000000000000000000000\,x^{81}\\
&+ 20739537825667299330234527587890625000000000000000000000000\,x^{82}\\
&- 10097087501855567097663879394531250000000000000000000000000\,x^{83}\\
&+ 249995683471485972404479980468750000000000000000000000000000\,x^{84}\\
&+ 38646789877675473690032958984375000000000000000000000000000\,x^{85}\\
&- 45548024517484009265899658203125000000000000000000000000000\,x^{86}\\
&- 6191411893814802169799804687500000000000000000000000000000\,x^{87}\\
&+ 2132751047611236572265625000000000000000000000000000000000\,x^{88}\\
&+ 224821269512176513671875000000000000000000000000000000000000\,x^{89}\\
&- 13783574104309082031250000000000000000000000000000000000000\,x^{90}\\
&- 4656612873077392578125000000000000000000000000000000000000\,x^{91}
\end{aligned}
$$

Appendix 3, $\nu(d,l)$ for $l \le 10$

$$
\begin{aligned}
\nu(1,l) &= l+1,\\
\nu(2,l) &= (l+1)(2l^2+4l+3)/3,\\
\nu(3,l) &= (l+1)(11\,l^4+44\,l^3+71\,l^2+54\,l+20)/20,\\
\nu(4,l) &= (l+1)(151\,l^6+906\,l^5+2335\,l^4\\
&\quad +3300\,l^3+2734\,l^2+1284\,l+315)/315,\\
\nu(5,l) &= (l+1)(15619\,l^8+124952\,l^7+444682\,l^6\\
&\quad +918764\,l^5+1208767\,l^4+1042412\,l^3\\
&\quad +582804\,l^2+198000\,l+36288)/36288,\\
\nu(6,l) &= (l+1)(655177\,l^{10}+6551770\,l^9+29794430\,l^8\\
&\quad +81112960\,l^7+146530181\,l^6+183878590\,l^5\\
&\quad +162895420\,l^4+101206440\,l^3+42736392\,l^2\\
&\quad +11373840\,l+1663200)/1663200,\\
\nu(7,l) &= (l+1)(27085381\,l^{12}+325024572\,l^{11}+1800605103\,l^{10}\\
&\quad +6088483390\,l^9+14000223963\,l^8+23082515016\,l^7\\
&\quad +28018754489\,l^6+25286101830\,l^5+16900442856\,l^4\\
&\quad +8210096312\,l^3+2786177808\,l^2+609497280\,l\\
&\quad +74131200)/74131200,\\
\nu(8,l) &= (l+1)(2330931341\,l^{14}+32633038774\,l^{13}\\
&\quad +213237013171\,l^{12}+861926141804\,l^{11}\\
&\quad +2408141739003\,l^{10}+4921524309702\,l^9\\
&\quad +7592416278733\,l^8+8991013223192\,l^7\\
&\quad +8225645724296\,l^6+5799204034624\,l^5\\
&\quad +3113095208496\,l^4+1241809855104\,l^3\\
&\quad +352311197760\,l^2+65424240000\,l\\
&\quad +6810804000)/6810804000,\\
\nu(9,l) &= (l+1)(2127599641825\,l^{16}+34041594269200\,l^{15}\\
&\quad +256340655118572\,l^{14}+1205857572816008\,l^{13}\\
&\quad +3966588745097342\,l^{12}+9676751782713288\,l^{11}\\
&\quad +18117372902624996\,l^{10}+26569292467501784\,l^9\\
&\quad +30868135523952441\,l^8+28537115693110024\,l^7\\
&\quad +20957006835229192\,l^6+12126026835494208\,l^5\\
&\quad +5439900247285392\,l^4+1840933284559488\,l^3\\
&\quad +448739759082240\,l^2+72392382720000\,l\\
&\quad +6586804224000)/6586804224000,
\end{aligned}
$$

$$
\begin{aligned}
\nu(10, l) = (l+1)(&186538565778065\, l^{18} + 3357694184005170\, l^{17} \\
&+ 28630897062842508\, l^{16} + 153663413655678048\, l^{15} \\
&+ 581733431430496302\, l^{14} + 1649862235026713988\, l^{13} \\
&+ 3633650243948073908\, l^{12} + 6356296462562545824\, l^{11} \\
&+ 8956713954322455321\, l^{10} + 10248755957827386714\, l^{9} \\
&+ 9553375268239035480\, l^{8} + 7245526459096165968\, l^{7} \\
&+ 4445106742777275152\, l^{6} + 2181304370946591648\, l^{5} \\
&+ 840556042054291584\, l^{4} + 247018063737415680\, l^{3} \\
&+ 52782375949678080\, l^{2} + 7527050946938880\, l \\
&+ 608225502044160)/608225502044160.
\end{aligned}
$$