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Frequency permutation arrays within distance one

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Abstract

The following problem is considered: how many permutations p of the sequence $\iota = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, n, n, \ldots, n)$, where each element occurs l times, satisfy $|p_i - \iota_i| \le 1$ for all $i = 1, 2, \ldots, ln$? It is shown that this number, denoted by $V_l(n)$, satisfies a linear recurrence of length $l+1$. A table of values of $V_l(n)$ and generating functions for $l \le 10$ are given.

1 Introduction

Let $S_{l,n}$ denote the set of permutations of the sequence

$$\iota = (1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, n, n, \ldots, n),$$

where each element occurs l times, or equivalently, $S_{l,n}$ is the $([l, l, \ldots, l], 1)$ constant composition code over $\{1, 2, \ldots, n\}$. A *frequency permutation array* with minimum distance d is a subset of $S_{l,n}$ where the distance between the permutations are at least d, or equivalently, a $([l, l, \ldots, l], d)$ constant composition code. The distance we consider here is the distance obtained from the l_∞ norm, that is, we require that

$$|p_i - \iota_i| \le d \text{ for all } i, \ 1 \le I \le ln. \tag{1}$$

The number of permutations in $S_{l,n}$ satisfying (1) we denote by $V_l(d, n)$. Such frequency permutation arrays were studied by Shieh and Tsai [3]. For $l = 1$ we get a *permutation array*, and $V_1(d, n)$ has been studied by a number of authors since the 1950s. Kløve [1] gave a survey of known result as well as many new results. It is well known that $V_1(d, n)$ can be given as the value of the permanent of some matrix, and Shieh and Tsai [3] showed that the same is true for $V_l(d, n)$. The permanent of an $n \times n$ matrix $M = (m_{i,j})$ is defined by

$$\operatorname{per} M = \sum_{p \in S_{1,n}} m_{1,p_1} \cdots a_{n,p_n}. \tag{2}$$

It is known that $V_1(d, n)$ satisfies a linear recurrence in n and that the generating function $\sum_{n=0}^{\infty} V_1(d, n)x^n$ is a rational function.

Shieh and Tsai [3] showed that if M is the $ln \times ln$ matrix defined by

$$m_{i,j} = 1 \text{ if } \left| \left\lceil \frac{i}{l} \right\rceil - \left\lceil \frac{j}{l} \right\rceil \right| \le d \text{ and } m_{i,j} = 0 \text{ otherwise,}$$

then $V_l(d, n) = \operatorname{per} M/(l!)^n$. The known methods can be used to show that $V_l(d, n)$ satisfies some linear recurrence. However, the actual determinantion of the recurrence will probably be quite complicated in general. In this paper we will study the recursion for $d = 1$ (and all l). For convenience, we drop d from the notation and write $V_l(n) = V_l(1, n)$.

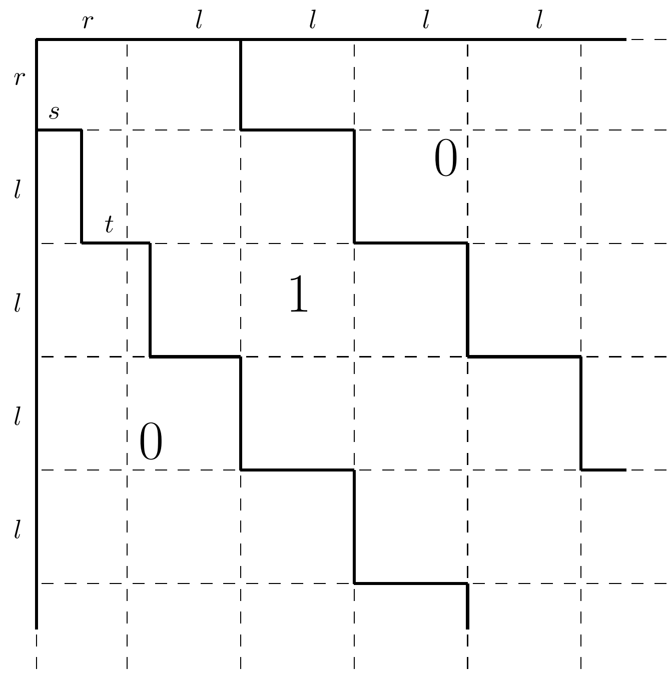
2 The main result

Suppose that l is given. We will define a class of matrices $M_l(n, r, s, t)$, given by four parameters, n, r, s, t, where

$$n \ge 1, \quad 1 \le r \le l, \quad 0 \le s \le r, \quad 0 \le t \le l, \text{ and } r \le s + t \le r + l.$$

The matrix M given by Shieh and Tsai [3] mentioned above is $M_l(n-1, l, 0, l)$ in this notation. The matrix $M_l(n, r, s, t)$ is an $(ln + r) \times (ln + r)$ matrix obtained by deleting the first $l - r$ rows and columns of $M_l(n, l, 0, l)$ and in addition some of

Figure 1: The form of the matrix $M_l(n, r, s, t)$. The upper left corner is shown.



the remaining elements are changed from 1 to 0. Explicitly, $M_l(n, r, s, t) = (m_{i,j})$ where $1 \le i, j \le l(n-1) + r$, and where $m_{i,j} = 1$ if (and only if)

$$1 \le i \le r \text{ and } 1 \le j \le l + r,$$
$$r + 1 \le i \le l + r \text{ and } s + 1 \le j \le 2l + r,$$
$$l + r + 1 \le i \le 2l + r \text{ and } s + t + 1 \le j \le 3l + r,$$
$$(a + 1)l + r + 1 \le i \le (a + 2)l + r \text{ and } al + 1 \le j \le (a + 2)l + r$$

for $2 \le a \le n - 2$,

$$(n - 1)l + r + 1 \le i \le nl + r \text{ and } (n - 1)l + 1 \le j \le nl + r.$$

The form of $M(n, r, s, t)$ is illustrated in Fig. 1. The 1s are located symmetrically around the main diagonal, except for the first $r + l$ rows and columns.

Let $A_l(n, r, s, t) = \text{per } M_l(n, r, s, t)$. Our main goal is then to determine

$$V_l(n) = A_l(n - 1, l, 0, l)/(l!)^n.$$

We will show that $V_l(n)$ satisfies a linear recursion in n of degree l. We also find its generating function.

We see that the first row of $M_l(n, r, s, t)$ starts with $r+l$ ones (and the remaining elements in the first row are zero). Expanding the permanent by the first row, we therefore get an expression for $A_l(n, r, s, t)$ in terms of some $A_l(n, r - 1, s', t')$ if $r > 1$ and in terms of some $A_l(n - 1, l, t', l - t')$ if $r = 1$. To simplify the notation a little bit, we let

$$B_l(n, u) = A_l(n - 1, l, u, l - u),$$

where we for now assume that n is fixed and we will show that all $A_l(n,r,s,t)$ can be expressed in terms of the $B_l(n,u)$, $u = 0,1,\ldots,l$. In this notation, $V_l(n) = B_l(n,0)/(l!)^n$.

First consider $r = 1$. Then $0 \le s \le 1$. If $s = 1$, the first column contains only one 1 and we get

$$A_l(n,1,1,t) = B_l(n,t). \tag{3}$$

If $s = 0$, the first t columns are identical and the next $l+1-t$ are also identical. Hence we get

$$A_l(n,1,0,t) = t\,B_l(n,t-1) + (l+1-t)B_l(n,t). \tag{4}$$

Now consider $r > 1$. Of the $r+l$ first columns, the first s are identical, the next t are identical, and the final $r+l-s-t$ are identical. Hence, expanding by the first row, we get

$$\begin{aligned} A_l(n,r,s,t) = s\,A_l(n,r-1,s-1,t) + t\,A_l(n,r-1,s,t-1) \\ + (r+l-s-t)A_l(n,r-1,s,t). \end{aligned} \tag{5}$$

Using (3), (4), and (5) repeatedly, we can determine all $A_l(n,r,s,t)$. We can in fact find an explicit expression for $A_l(n,r,s,t)$ in general.

Lemma 1. *For all r, s, t we have*

$$A_l(n,r,s,t) = r!\sum_{u=0}^{t}\binom{t}{u}\binom{r+l-s-t}{l-u}B_l(n,u). \tag{6}$$

Proof. To show the lemma, we first show that (3) and (4) are satisfied. For $r > 1$, we use (5) and induction.

First, (3). For $r = s = 1$, the sum in (6) is

$$1!\sum_{u=0}^{t}\binom{t}{u}\binom{l-t}{l-u}B_l(n,u) = B_l(n,t)$$

since all the other terms are zero. By (3), this is $A_l(n,1,1,t)$.

For $r = 1$ and $s = 0$, there are two nonzero terms in the sum, namely for $u = t$ and $u = t-1$, and so

$$\begin{aligned} &1!\sum_{u=0}^{t}\binom{t}{u}\binom{l+1-t}{l-u}B_l(n,u) \\ &= t\cdot 1\cdot B_l(n,t-1) + 1\cdot(l+1-t)\cdot B_l(n,t) \\ &= A_l(n,1,0,t) \end{aligned}$$

by (4).

For $r > 1$, we assume that the expression (6) is true for $r-1$. Then we get

$$\begin{aligned}
A_l(n,r,s,t) &= s\,A_l(n,r-1,s-1,t) + t\,A_l(n,r-1,s,t-1) \\
&\quad + (r+l-r-t)A_l(n,r-1,s,t) \\
&= s(r-1)!\sum_{u=0}^{t}\binom{t}{u}\binom{r+l-s-t}{l-u}B_l(n,u) \\
&\quad + t(r-1)!\sum_{u=0}^{t-1}\binom{r+l-s-t}{l-u}B_l(n,u) \\
&\quad + (r+l-s-t)(r-1)!\sum_{u=0}^{t}\binom{t}{u}\binom{r-1+l-s-t}{l-u}B_l(n,u) \\
&= (r-1)!\sum_{u=0}^{t}\binom{t}{u}\binom{r+l-s-t}{l-u}B_l(n,u) \\
&\qquad \cdot \Big\{s+(t-u)+(r-t-s+u)\Big\} \\
&= r!\sum_{u=0}^{t}\binom{t}{u}\binom{r+l-s-t}{l-u}B_l(n,u).
\end{aligned}$$

This completes the induction and the proof of the lemma. □

Choosing $r = l$ and $t = l - s$ in (6) and reducing n by one, we get the following corollary.

Corollary 1.

$$B_l(n,s) = l!\sum_{u=0}^{l-s}\binom{l-s}{u}\binom{l}{u}B_l(n-1,u). \tag{7}$$

Since

$$B_l(0,0) = 1 \text{ and } B_l(0,s) = 0 \text{ for } s = 1,2,\ldots,l,$$

(7) and induction proves the following.

Corollary 2. *For all l, s and n, $(l!)^n$ divides $B_l(n,s)$.*

Let

$$b_l(n,s) = \frac{B_l(n,s)}{(l!)^n}.$$

By Corollary 2, the $b_l(n,s)$ are integers, and by Corollary 1 they satisfy the following recursion:

$$b_l(n,s) = \sum_{u=0}^{l-s}\binom{l-s}{u}\binom{l}{u}b_l(n-1,u). \tag{8}$$

We note that $V_l(n) = b_l(n,0)$.

Let $T_l = (t_{u,s})$ denote the $(l+1)\times(l+1)$ matrix defined by

$$t_{u,s} = \binom{l-s}{u}\binom{l}{u}.$$

Then

$$(b_l(n,0), b_l(n,1), \ldots, b_l(n,l)) = (b_l(n-1,0), b_l(n-1,1), \ldots, b_l(n-1,l))\,T_l.$$

In particular, if

$$f(x) = \sum_{i=0}^{l} c_i x^i \tag{9}$$

is the characteristic polynomial of T_l, then

$$\sum_{i=0}^{l} c_i b_l(n+i, s) = 0.$$

Hence we have proven the following theorem.

Theorem 1. *For each $l \geq 1$, the sequence $V_l(0), V_l(1), V_l(2), \ldots$ satisfies the linear recurrence of length $l+1$ determined by the characteristic polynomial of the matrix*

$$T_l = \left[\binom{l-s}{u}\binom{l}{l-u}\right]_{0\leq u\leq l, 0\leq s\leq l}.$$

The same is true for the sequences $b_l(0,s), b_l(1,s), b_l(2,s), \ldots$ for $0 \leq s \leq l$.

Computing $b_l(n,s)$ for some range of n, an efficient way is to use (8), with the initial values $b_l(0,0) = 1$ and $b_l(0,s) = 0$ for $1 \leq s \leq l$. The tables of $V_l(n) = b_l(n,0)$ in the first appendix are computed this way.

Keeping n fixed

In the discussion above we considered the sequence $b_l(n,s)$ for fixed l and s. Alternatively, we can keep n and s fixed. We can get explicit expressions (involving sums).

We have, $b_l(0,0) = 1$ and $b_l(0,s) = 0$ for $0 \leq s \leq l$. From (8) we get, for $0 \leq s \leq l$,

$$\begin{aligned} b_l(1,s) &= 1, \\ b_l(2,s) &= \sum_{u=0}^{l-s}\binom{l-s}{u}\binom{l}{l-u} = \binom{2l-s}{l}, \\ b_l(3,s) &= \sum_{u=0}^{l-s}\binom{l-s}{u}\binom{l}{l-u}\binom{2l-u}{l}, \end{aligned}$$

$$\begin{aligned} b_l(4,s) &= \sum_{u=0}^{l-s}\binom{l-s}{u}\binom{l}{l-u}\sum_{j=0}^{l-u}\binom{l-u}{j}\binom{l}{l-j}\binom{2l-j}{l} \\ &= \sum_{j=0}^{l}\binom{l}{l-j}\binom{2l-j}{l}\sum_{u=0}^{l}\binom{l-s}{u}\binom{l}{l-u}\binom{l-u}{j} \\ &= \sum_{j=0}^{l}\binom{l}{l-j}\binom{2l-j}{l}\sum_{u=0}^{l}\binom{l-s}{u}\binom{l}{j}\binom{l-j}{u} \\ &= \sum_{j=0}^{l}\binom{l}{j}^2\binom{2l-j}{l}\sum_{u=0}^{l}\binom{l-s}{u}\binom{l-j}{l-j-u} \\ &= \sum_{j=0}^{l}\binom{l}{j}^2\binom{2l-j}{l}\binom{2l-s-j}{l-j}. \end{aligned}$$

We note that the sequence $\{b_l(3,0)\}$ is sequence A005258 in [4] and $\{b_l(4,0)\}$ is sequence A005258 in [4].

3 Generating functions

Let $h_{l,s}(x) = \sum_{n=0}^{\infty} b_l(n,s)x^n$ be the generating function of $b_l(n,s)$. Once we have the values of $b_l(n,s)$ for $n = 0, 1, \ldots, l$, then by the standard theory of linear recurrences, if $g_l(x) = \sum_{i=0}^{l} \gamma_{l,i} x^i$, where $\gamma_{l,i} = c_{l-i}$ (here the c_i are the coefficient of the characteristic polynomial (9) of T_l), then

$$f_{l,s}(x) = g_l(x) \sum_{n=0}^{\infty} b_l(n,s)x^n$$

is a polynomial of degree less than l, and so

$$h_{l,s}(x) = \frac{f_{l,s}(x)}{g_l(x)}.$$

Tables of $g_l(x)$ and $f_{l,s}(x)$ for $0 \le s < l \le 10$ are given in the second appendix. Note that by (8), $b_l(n,l) = b_l(n-1,0)$. In particular, this implies that

$$f_{l,l}(x) = x f_{l,0}(x).$$

Therefore, we have not included $f_{l,l}(x)$ in the tables.

We note that $\gamma_{l,l+1} = \det(M) = (-1)^{\lceil \frac{l+1}{2} \rceil} \prod_{i=0}^{l} \binom{l}{i}$. We use the notation $f_{l,s}(x) = \sum_{i=0}^{l} \phi_{l,s,i} x^i$. Some observed properties of these polynomials for the range we have computed ($l \le 10$) are given below. These properties may be true in general, but we have not studied this further.

- For $0 \le s < l$, we have $\deg f_{l,s}(x) = l - 1$, that is, $\phi_{l,s,l} = 0$.
- $\phi_{l,s,l-1} = (-1)^{s+1} \frac{\gamma_{l,l+1}}{\binom{l}{s}}$,
- $(-1)^{\lceil \frac{i}{2} \rceil} \gamma_{l,i} > 0$ for $0 \le i \le l+1$,
- $(-1)^{\lceil \frac{i}{2} \rceil} \phi_{l,0,i} > 0$ for $0 \le i \le l-1$,
- $(-1)^{\lceil \frac{i-1}{2} \rceil} \phi_{l,1,i} > 0$ for $0 \le i \le l-1$.

It is also possible to compute the $h_{l,s}(x)$ more directly. Let $\delta_{s,0}$ denote the Kronecker delta, that is $\delta_{0,0} = 1$ and $\delta_{s,0} = 0$ for $s \ne 0$. From (8) and the fact that $b_l(0,s) = \delta_{s,0}$, we get

$$\begin{aligned}
h_{l,s}(x) &= \delta_{s,0} + \sum_{n=1}^{\infty} b_l(n,s)x^n \\
&= \delta_{s,0} + \sum_{n=1}^{\infty} x^n \sum_{u=0}^{l} \binom{l-s}{u}\binom{l}{u} b_l(n-1,u) \\
&= \delta_{s,0} + \sum_{u=0}^{l} \binom{l-s}{u}\binom{l}{u} \sum_{n=1}^{\infty} b_l(n-1,u)x^n \\
&= \delta_{s,0} + \sum_{u=0}^{l} \binom{l-s}{u}\binom{l}{u} x h_{l,u}(x).
\end{aligned}$$

We can solve this set of $l+1$ equations and obtain the $h_{l,s}(x)$. In terms of matrices, we can write the equation set

$$(h_{l,0}(x), h_{l,1}(x), \ldots, h_{l,l}(x))\,(I - xT_l) = (1, 0, \ldots, 0),$$

where I the $(l+1) \times (l+1)$ identity matrix. Hence we get the following theorem.

Theorem 2. *For $l \geq 1$, $(h_{l,0}(x), h_{l,1}(x), \ldots, h_{l,l}(x))$ is the first row of*

$$(I - xT_l)^{-1}.$$

References

[1] T. Kløve, "Spheres of Permutations under the Infinity Norm - Permutations with limited displacement," Department of Informatics, University of Bergen, Report No. 376, November 2008.

[2] T. Kløve, T.-T. Lin, S.-C. Tsai, W.-G. Tzeng, "Efficient encoding and decoding with permutation arrays under infinity norm." Manuscript 2008.

[3] M.-Z. Shieh and S.-C. Tsai, "Decoding frequency permutation arrays under infinity norm." Manuscript submitted to ISIT 2009.

[4] N. J. A. Sloane, "The On-Line Encyclopedia of Integer Sequences," http://www.research.att.com/ njas/sequences/

Appendix, tables of $V_l(n)$

For each $l \leq 10$, we have included the values of $V_l(n)$ for $n \leq 20$ (provided the values can fit into one line, that is, have at most 70 digits). For $l = 1$ the values are the Fibonacci numbers.

$V_1(n)$:
1, 2,
3, 5,
8, 13,
21, 34,
55, 89,
144, 233,
377, 610,
987, 1597,
2584, 4181,
6765, 10946

$V_2(n)$:
1,
6,
19,
73,
264,
973,
3565,
13086,
48007,
176149,
646296,
2371321,
8700553,
31923030,
117128107,
429752305,
1576795176,
5785386229,
21227039605,
77883687150

$V_3(n)$:
1,
20,
147,
1445,
13040,
120685,
1108677,
10207204,
93913687,
864237977,
7952680800,
73181430953,
673420201433,
6196864537204,
57024000600459,
524739088121629,
4828687880190544,
44433943310167925,
408884434208138877,
3762584822284713668

$V_4(n)$:
1,
70,
1251,
33001,
778840,
18979501,
458283501,
11095029310,
268407261751,
6494628568501,
157140210849480,
3802140006795001,
91995514253325001,
2225900960991467350,
53857332463462696251,
1303118450607052410001,
31529924703022030340920,
762890097390817464052501,
18458696127973957622400501,
446621950121084699577258190

$V_5(n)$:
1,
252,
11253,
819005,
51955008,
3426862513,
223555999521,
14629457476284,
956506304490805,
62554067142502589,
4090653574960950144,
267509059547920101233,
17493706803334667625377,
1143999628072529268292860,
74811735155413338440277237,
4892306161272288597410401597,
319931875193202258763349275584,
20921913394524835983550484261681,
1368186457085199494502511814687265,
89472418108917375447330264236027196

$V_6(n)$:
1,
924,
104959,
21460825,
3725735664,
677246940469,
121533839931421,
21888321066877356,
3938124367151767675,
708743612574056867221,
127542377579526691799136,
22952473176498367236198409,
4130491868245063608463767529,
743318274070910835570229477500,
133766583990791214326410385255719,
24072462709511521506885699191291809,
4332049316570097565180842830546688336,
779590011562528975268061048428295217261,
140294013270250427479554925525141312195525,
25247129752917525999708412524109996498245516

$V_7(n)$:
1,
3432,
1004307,
584307365,
281152383072,
142648550730349,
71317516434003525,
35801871486260585544,
17952292795232366702935,
9004755701212513017959321,
4516329337951820803551129024,
2265217670079812211495904106153,
1136138585458716600268637214790361,
569840751629792139416688379526917800,
285808716115094273711695504392530084619,
143349934164043810258573400532731379769885,
71898446844750547021445142046885306393733664,
36061312115189434126727298190818148390989166709,
18086875091406590626472543912280627941050344054525,
9071634715222858104813326808457630531494707987562120,
4549959900223147403498497880417205059109504135655606127

$V_8(n)$:
1,
12870,
9793891,
16367912425,
22035642730200,
31502711112595885,
44295290280756102637,
62569846044620145720126,
88271313599568437690474935,
124573943375568956577577001653,
175789302808103093628255125104200,
248067238178807267808839236968317689,
350060517174005010391130870435408896201,
493989534503292754868353691591517256079190,
697095240068764884987582473125418987303636923,
983708797327298788273493429586552873326822654353,
1388164629137817856436379753509747271807084096142776,
1958914128119468029901132307723927945190375274909584405,
2764329581571820330816482241775044315481460863833230460405,
3900894854445014677762379303942960864562913439627571254125518

$V_9(n)$:
1,
48620,
96918753,
468690849005,
1778348612268800,
7216179759611167513,
2874529994648716999 3521,
115091311345187072928558460,
4601581231421727212859482693 05,
18405186258615556817302233513 16589,
736083102627831423482804749512 8099520,
294392202608319881913600060010 82378535233,
117739509112349463723345100673 103757213845377,
470889616655296059827467951144 765028870328114700,
188328366049457984016876017963 6870546935247688814737,
753203686482421199086495483228 783635335358614814 1111597,
301237555709358479059784840253 12642337692092111082459475840,
120477458857172786944857749445 9011311624810440907093504 99496681,
481839590235214778858627919090 15651155328614118131392851 0263717265,
192707742302937173669904302176 05739144603131055787760853398083 19254300

$V_{10}(n)$:
1,
184756,
970336269,
13657436403073,
146898506147371264,
1701765313758012953473,
19316611911260692278259065,
220494323800641046774094312836,
2513010804923623461613325281197757,
28653350779401783399803031179454720649,
326667439282263161656690619517700109863296,
3724347552175232603674450917904230607245244321,
42461049822986464700029958979381560005857008301553,
484096904839865210883595839396855576288524521889773780,
5519167755806190714476276496104887012101313653960556653357,
62923803056370379198572727452389636863587280852479303520268305,
717391637051748688222824953419630961585872405673287610725811666176,
8178952050692287035831370410106122047491833022842126660028825762249729

Appendix, generating functions

g_1	$1-x-x^2$
$f_{1,0}$	1

g_2	$1-3\,x-3\,x^2+2\,x^3$
$f_{2,0}$	$1-2\,x$
$f_{2,1}$	x

g_3	$1-7\,x-22\,x^2+15\,x^3+9\,x^4$
$f_{3,0}$	$1-6\,x-9\,x^2$
$f_{3,1}$	$x+3\,x^2$
$f_{3,2}$	$x-3\,x^2$

g_4	$1-19\,x-139\,x^2+314\,x^3+184\,x^4-96\,x^5$
$f_{4,0}$	$1-18\,x-88\,x^2+96\,x^3$
$f_{4,1}$	$x+16\,x^2-24\,x^3$
$f_{4,2}$	$x-4\,x^2+16\,x^3$
$f_{4,3}$	$x-14\,x^2-24\,x^3$

g_5	$1-51\,x-1026\,x^2+5375\,x^3+10575\,x^4-5250\,x^5-2500\,x^6$
$f_{5,0}$	$1-50\,x-825\,x^2+2750\,x^3+2500\,x^4$
$f_{5,1}$	$x+75\,x^2-450\,x^3-500\,x^4$
$f_{5,2}$	$x+5\,x^2+150\,x^3+250\,x^4$
$f_{5,3}$	$x-30\,x^2-25\,x^3-250\,x^4$
$f_{5,4}$	$x-45\,x^2-450\,x^3+500\,x^4$

g_6	$1-141\,x-7644\,x^2+111179\,x^3+498171\,x^4-842310\,x^5$ $-369900\,x^6+162000\,x^7$
$f_{6,0}$	$1-140\,x-6861\,x^2+78210\,x^3+207900\,x^4-162000\,x^5$
$f_{6,1}$	$x+321\,x^2-7560\,x^3-30150\,x^4+27000\,x^5$
$f_{6,2}$	$x+69\,x^2+798\,x^3+9540\,x^4-10800\,x^5$
$f_{6,3}$	$x-57\,x^2+882\,x^3-4320\,x^4+8100\,x^5$
$f_{6,4}$	$x-113\,x^2-1974\,x^3-540\,x^4-10800\,x^5$
$f_{6,5}$	$x-134\,x^2-4935\,x^3+32850\,x^4+27000\,x^5$

g_7	$1-393\,x-59193\,x^2+2322404\,x^3+28870212\,x^4$ $-109325076\,x^5-164634169\,x^6+64790985\,x^7+26471025\,x^8$
$f_{7,0}$	$1-392\,x-56154\,x^2+1918742\,x^3+17656954\,x^4$ $-38319960\,x^5-26471025\,x^6$
$f_{7,1}$	$x+1323\,x^2-110005\,x^3-1683787\,x^4$ $+4934055\,x^5+3781575\,x^6$
$f_{7,2}$	$x+399\,x^2-3381\,x^3+268569\,x^4$ $-1404585\,x^5-1260525\,x^6$
$f_{7,3}$	$x-63\,x^2+12789\,x^3+12691\,x^4$ $+641067\,x^5+756315\,x^6$
$f_{7,4}$	$x-273\,x^2-5439\,x^3-156751\,x^4$ $-338541\,x^5-756315\,x^6$
$f_{7,5}$	$x-357\,x^2-29253\,x^3+307671\,x^4$ $-276115\,x^5+1260525\,x^6$
$f_{7,6}$	$x-385\,x^2-46893\,x^3+1148707\,x^4$ $+5870445\,x^5-3781575\,x^6$

g_8	$1 - 1107\,x - 466155\,x^2 + 50943066\,x^3 + 1698359832\,x^4$ $-17812454688\,x^5 - 58970736384\,x^6 + 80038748160\,x^7$ $+28677390336\,x^8 - 11014635520\,x^9$
$f_{8,0}$	$1 - 1106\,x - 454392\,x^2 + 46023712\,x^3 + 1275963136\,x^4$ $-9584418816\,x^5 - 17662754816\,x^6 + 11014635520\,x^7$
$f_{8,1}$	$x + 5328\,x^2 - 1512504\,x^3 - 79579136\,x^4$ $+871670016\,x^5 + 2035740672\,x^6 - 1376829440\,x^7$
$f_{8,2}$	$x + 1896\,x^2 - 189576\,x^3 + 4129216\,x^4$ $-150226944\,x^5 - 518418432\,x^6 + 393379840\,x^7$
$f_{8,3}$	$x + 180\,x^2 + 123456\,x^3 + 4014976\,x^4$ $+24159744\,x^5 + 217061376\,x^6 - 196689920\,x^7$
$f_{8,4}$	$x - 612\,x^2 + 32112\,x^3 - 4323200\,x^4$ $+19869696\,x^5 - 126443520\,x^6 + 157351936\,x^7$
$f_{8,5}$	$x - 942\,x^2 - 157176\,x^3 + 712768\,x^4$ $-62243328\,x^5 + 69543936\,x^6 - 196689920\,x^7$
$f_{8,6}$	$x - 1062\,x^2 - 316056\,x^3 + 16515520\,x^4$ $+118315008\,x^5 + 155947008\,x^6 + 393379840\,x^7$
$f_{8,7}$	$x - 1098\,x^2 - 411768\,x^3 + 33923680\,x^4$ $+639330048\,x^5 - 2611058688\,x^6 - 1376829440\,x^7$

g_9	$1 - 3139\,x - 3733258\,x^2 + 1141936935\,x^3 + 105766383267\,x^4$ $-2954887228278\,x^5 - 26984014044012\,x^6 + 79103022450480\,x^7$ $+98126014508928\,x^8 - 32338686530304\,x^9 - 11759522374656\,x^{10}$
$f_{9,0}$	$1 - 3138\,x - 3687777\,x^2 + 1082504250\,x^3 + 89860199580\,x^4$ $-2022819810480\,x^5 - 12869477292672\,x^6 + 20579164155648\,x^7$ $+11759522374656\,x^8$
$f_{9,1}$	$x + 21171\,x^2 - 19932318\,x^3 - 3531961260\,x^4$ $+128735136432\,x^5 + 1103288522112\,x^6 - 2141394506496\,x^7$ $-1306613597184\,x^8$
$f_{9,2}$	$x + 8301\,x^2 - 3847878\,x^3 - 109098090\,x^4$ $-11363442048\,x^5 - 188865785808\,x^6 + 489980098944\,x^7$ $+326653399296\,x^8$
$f_{9,3}$	$x + 1866\,x^2 + 846207\,x^3 + 196166610\,x^4$ $-1966952808\,x^5 + 41823698112\,x^6 - 184992486336\,x^7$ $-139994313984\,x^8$
$f_{9,4}$	$x - 1137\,x^2 + 798984\,x^3 - 67272606\,x^4$ $+3813224040\,x^5 - 1402059456\,x^6 + 101107004544\,x^7$ $+93329542656\,x^8$
$f_{9,5}$	$x - 2424\,x^2 - 648891\,x^3 - 81203310\,x^4$ $-3419960616\,x^5 - 23288295744\,x^6 - 69997156992\,x^7$ $-93329542656\,x^8$
$f_{9,6}$	$x - 2919\,x^2 - 2062908\,x^3 + 199842228\,x^4$ $-807352920\,x^5 + 60553096128\,x^6 + 34998578496\,x^7$ $+139994313984\,x^8$
$f_{9,7}$	$x - 3084\,x^2 - 3007863\,x^3 + 585198918\,x^4$ $+22357365660\,x^5 - 114671268432\,x^6 + 190547816256\,x^7$ $-326653399296\,x^8$
$f_{9,8}$	$x - 3129\,x^2 - 3497238\,x^3 + 903113388\,x^4$ $+58072548240\,x^5 - 864203582592\,x^6 - 2939880593664\,x^7$ $+1306613597184\,x^8$

g_{10}	$1 - 8953\,x - 30255928\,x^2 + 26167186427\,x^3$ $+6764138451925\,x^4 - 523127064454800\,x^5 - 12617955113431500\,x^6$ $+101565785269440000\,x^7 + 267474303094050000\,x^8 - 306145784385000000\,x^9$ $-93630456360000000\,x^{10} + 32406091200000000\,x^{11}$
$f_{10,0}$	$1 - 8952\,x - 30080125\,x^2 + 25453146300\,x^3$ $+6170357191500\,x^4 - 416764490490000\,x^5 - 7880125909050000\,x^6$ $+43309237545000000\,x^7 + 61224365160000000\,x^8 - 32406091200000000\,x^9$
$f_{10,1}$	$x + 83425\,x^2 - 255730650\,x^3$ $-148914546750\,x^4 + 17884439536500\,x^5 + 497095510815000\,x^6$ $-3484893302100000\,x^7 - 5798375604000000\,x^8 + 3240609120000000\,x^9$
$f_{10,2}$	$x + 34805\,x^2 - 62361320\,x^3$ $-14123649300\,x^4 - 336837114000\,x^5 - 52101666450000\,x^6$ $+573973533000000\,x^7 + 1200511368000000\,x^8 - 720135360000000\,x^9$
$f_{10,3}$	$x + 10495\,x^2 + 1566720\,x^3$ $+6691267350\,x^4 - 569339505000\,x^5 + 1254439980000\,x^6$ $-135720663750000\,x^7 - 408934008000000\,x^8 + 270050760000000\,x^9$
$f_{10,4}$	$x - 945\,x^2 + 9963680\,x^3$ $+316601250\,x^4 + 344117605500\,x^5 + 6677031825000\,x^6$ $+30313934100000\,x^7 + 203364756000000\,x^8 - 154314720000000\,x^9$
$f_{10,5}$	$x - 5950\,x^2 - 144275\,x^3$ $-3090409500\,x^4 - 47843392500\,x^5 - 8059670100000\,x^6$ $+14074256250000\,x^7 - 135790830000000\,x^8 + 128595600000000\,x^9$
$f_{10,6}$	$x - 7952\,x^2 - 12506625\,x^3$ $+1230762300\,x^4 - 321315399000\,x^5 + 6648804945000\,x^6$ $-59447544300000\,x^7 + 106366932000000\,x^8 - 154314720000000\,x^9$
$f_{10,7}$	$x - 8667\,x^2 - 21619300\,x^3$ $+9550455000\,x^4 + 599180215500\,x^5 + 5974944615000\,x^6$ $+149547242250000\,x^7 - 37614213000000\,x^8 + 270050760000000\,x^9$
$f_{10,8}$	$x - 8887\,x^2 - 26845400\,x^3$ $+17532273900\,x^4 + 2556502038000\,x^5 - 74233369890000\,x^6$ $-282790683000000\,x^7 - 559819512000000\,x^8 - 720135360000000\,x^9$
$f_{10,9}$	$x - 8942\,x^2 - 29245875\,x^3$ $+22895839800\,x^4 + 4681211724000\,x^5 - 237920095125000\,x^6$ $-2909170800900000\,x^7 + 8460304524000000\,x^8 + 3240609120000000\,x^9$