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Computing minimum distortion embeddings into a path for bipartite permutation graphs and threshold graphs*

Pinar Heggernes† Daniel Meister† Andrzej Proskurowski‡

Abstract

The problem of computing minimum distortion embeddings of a given graph into a line (path) was introduced in 2004 and has quickly attracted significant attention with subsequent results appearing in recent STOC and SODA conferences. So far all such results concern approximation algorithms or exponential-time exact algorithms. We give the first polynomial-time algorithms for computing minimum distortion embeddings of graphs into a path when the input graphs belong to specific graph classes. In particular, we solve this problem in polynomial time for bipartite permutation graphs and threshold graphs.

1 Introduction

A metric space is defined by a set of points and a distance function between pairs of points. Given two metric spaces (U, d) and (U', d'), an *embedding* of the first into the second is a mapping $f : U \to U'$. The embedding has *distortion* c if for all $x, y \in U$, $d(x, y) \le d'(f(x), f(y)) \le c \cdot d(x, y)$. Low distortion embeddings between metric spaces are well-studied and have a long history. Embeddings of finite metric spaces into low dimensional geometric spaces have applications in various areas of computer science, like computer vision [21] and computational chemistry (see [10, 11] for an introduction and a list of applications).

Minimum distortion embeddings are difficult to compute. It is NP-hard even to approximate by a ratio better than 3 a minimum distortion embedding between two given finite 3-dimensional metric spaces [17].

Every finite metric space can be represented by a matrix whose entries are the distances between pairs of points, and hence corresponds to a graph. Kenyon et al. [12] initiated the study of computing a minimum distortion embedding of a given graph onto[1] another given graph, and they gave a parameterized algorithm for computing a minimum distortion embedding between an arbitrary unweighted graph and a bounded-degree tree. Subsequently, Badoiu et al. [3] gave a constant-factor approximation algorithm for computing minimum distortion embeddings of arbitrary unweighted graphs into trees.

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1 They study a more restricted version of the problem where both graphs have the same number of vertices.

Since then, computing a minimum distortion embedding for a given graph on n vertices into a path was identified as a fundamental problem. This is exactly the problem that we study in this paper. Badoiu et al. [2] gave an exponential-time exact algorithm and a polynomial-time $\mathcal{O}(n^{1/2})$-approximation algorithm for arbitrary unweighted input graphs, along with a polynomial-time $\mathcal{O}(n^{1/3})$-approximation algorithm for unweighted trees. They also showed that the problem is hard to approximate within a constant factor. In another paper Badoiu et al. [1] showed that the problem is hard to approximate by a factor polynomial in n, even for weighted trees. They also gave a better polynomial-time approximation algorithm for general weighted graphs, along with a polynomial-time algorithm that approximates the minimum distortion c embedding of a weighted tree into a path by a factor that is polynomial in c.

We initiate the study of designing polynomial-time algorithms for exact computation of minimum distortion embeddings into a path for input graphs of specific graph classes. In particular we give polynomial-time algorithms for the solution of this problem on bipartite permutation graphs and on threshold graphs. Minimum distortion into a path is very closely related to the widely known and extensively studied graph parameter bandwidth. The only difference between the two parameters is that a minimum distortion embedding has to be non-contractive, meaning that the distance in the embedding between two vertices of the input graph has to be at least their original distance, whereas there is no such restriction for bandwidth. Bandwidth is known to be one of the hardest graph problems; it is NP-hard even for very simple graphs like caterpillars of hair-length at most 3 [16], and it is hard to approximate by a constant factor even for trees [4]. Polynomial-time algorithms for the exact computation of bandwidth are known for very few graph classes, including bipartite permutation graphs [9] and threshold graphs (that are interval graphs) [13, 20]. However, simple examples exist to show that these bandwidth algorithms cannot be used to generate minimum distortion embeddings into a path for these graph classes. In fact, there exist very simple bipartite permutation graphs, like $K_{3,4}$, for which no optimal bandwidth layout corresponds to a minimum distortion embedding into a path. It should be noted that the bandwidth and the minimum distortion into a path of a graph can be very different. For example, it is common knowledge that a cycle of length n has bandwidth 2, whereas its minimum distortion into a path is $\Omega(n)$. In this paper, we also prove that the latter is exactly $n-1$.

The running times of the algorithms we present in this paper are $\mathcal{O}(n^2)$ for bipartite permutation graphs, and $\mathcal{O}(n)$ for threshold graphs. We would like to mention that our algorithms operate significantly different than known (non-trivial) bandwidth algorithms. Most algorithms for bandwidth take as input a graph and an integer k, and decide whether the bandwidth of the input graph is at most k. The bandwidth of the graph can afterwards be computed by binary search on possible values of k. As opposed to this approach, both of the algorithms that we present in this paper compute the minimum distortion into a path of a graph directly.

This paper is organized as follows. In the next section we give the necessary definitions and notation. In Section 3 we give the first preliminary results on simple graphs, like cycles. Sections 4 and 5 present the polynomial-time algorithms for threshold graphs and bipartite permutation graphs, respectively.

2 Definitions and notation

A graph is denoted by $G = (V, E)$, where V is the vertex set and E is the edge set of G. The set of neighbors of a vertex v is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. Similarly, for $S \subseteq V$, $N_G[S] = \bigcup_{v \in S} N_G[v]$. The *degree* of a vertex v is $d_G(v) = |N_G(v)|$. (We will omit the subscripts when the graph is clear from the context.) Two non-adjacent vertices u and v are called *false twins* if $N(u) = N(v)$. The subgraph of G induced by the vertices in S is denoted by $G[S]$. For $G' = G[S]$ and $v \in V \setminus S$, $G'+v$ denotes $G[S \cup \{v\}]$, and for any $v \in V$, $G-v$ denotes $G[V \setminus \{v\}]$. We study unweighted and connected input graphs. A u, v-path is a path between (and including) u and v. The *distance* $\mathrm{d}_G(u, v)$ between two vertices u and v in G is the number of edges in a shortest u, v-path in G. For any mapping f from V to (a subset of) $\mathbb{Z}$, the *distance* $\mathrm{d}_f(u, v)$ between u and v in f is $|f(u) - f(v)|$. We write $u \prec_f v$ when $f(u) < f(v)$. For a vertex v in G, every vertex u with $u \prec_f v$ is *to the left* of v, and every vertex w with $v \prec_f w$ is *to the right* of v in f. We will also informally write *leftmost* and *rightmost* vertices accordingly.

An *embedding into a path (line)* for a graph $G = (V, E)$ is a mapping $\mathcal{E} : V \to \mathbb{Z}$. In the rest of this paper we use simply *embedding* to mean an embedding into a path. An embedding $\mathcal{E}$ is *non-contractive* if $\mathrm{d}_\mathcal{E}(u, v) \geq \mathrm{d}_G(u, v)$ for every pair of vertices $u, v \in V$. The *distortion* $\mathrm{D}(G, \mathcal{E})$ of a non-contractive embedding is defined to be the smallest c such that $\mathrm{d}_\mathcal{E}(u, v) \leq c \cdot \mathrm{d}_G(u, v)$ for every pair of vertices $u, v \in V$. Since we consider only unweighted graphs, it is easy to see that $\mathrm{D}(G, \mathcal{E})$ is the smallest c such that $\mathrm{d}_\mathcal{E}(u, v) \leq c$ for every edge uv of G (see also [12]). A *minimum distortion embedding* is a non-contractive embedding for G of smallest possible distortion. In this paper, the *distortion* of G, denoted by $\mathrm{D}(G)$, is the distortion of a minimum distortion embedding for G. Hence, our purpose is to compute $\mathrm{D}(G)$ when G is a bipartite permutation graph or a threshold graph.

Each integer (*position*) between the smallest and the largest integers that are mapped to in an embedding will be called a *slot* of that embedding. Exactly n slots of a non-contractive embedding are occupied by the vertices of G, and the rest are called *empty slots*. For a given vertex v, we refer to the rightmost vertex to the left of v of a certain property by *the close vertex to the left of v* and specifying the property (*close vertex to the right* is defined symmetrically). For two vertices u, v where $u \prec_\mathcal{E} v$, a vertex w is *between* u and v in $\mathcal{E}$ if $\mathcal{E}(u) \leq \mathcal{E}(w) \leq \mathcal{E}(v)$. In particular, w can be equal to u or v. The *vertex ordering underlying* $\mathcal{E}$, $\mathrm{ord}(\mathcal{E})$, is an ordered list of the n vertices occupying the non-empty slots of $\mathcal{E}$ in increasing order of positions.

In general, a *vertex ordering* for $G = (V, E)$ is a mapping $\sigma : V \to \{1, 2, \ldots, |V|\}$, thus a restricted embedding. Since every ordering can be considered as a permutation of V, we will also give an ordering as an ordered list of vertices $\sigma = \langle x_1, x_2, \ldots, x_n \rangle$. For an integer $c \geq 0$, we call σ a *c-ordering* for G if for every edge uv of G, $\mathrm{d}_\sigma(u, v) \leq c$. The *bandwidth* of G, $\mathrm{bw}(G)$, is the smallest c such that G has a c-ordering. Note that for a minimum distortion embedding $\mathcal{E}$ of G, $\mathrm{ord}(\mathcal{E})$ is not necessarily an optimal bandwidth ordering for G. Similarly, adding a minimum number of empty slots to an optimal bandwidth ordering to achieve a non-contractive embedding does not necessarily result in a minimum distortion embedding for G. A simple example is C_n, the cycle on n vertices, for which minimum distortion embeddings are without empty slots (as we will show in the next section), but no optimal bandwidth ordering is a minimum distortion embedding.

Each of the graph classes studied in this paper will be introduced in the section that presents results on it. All graph classes mentioned in this paper can be recognized in linear time [5, 8].

3 Preliminary results on distortion

3.1 Minimum distortion embeddings of arbitrary graphs

In this subsection we present results on minimum distortion embeddings that will be useful for our proofs later in the paper. We start by showing that in a minimum distortion embedding we can always assume consecutive vertices to have the same distance in the embedding as they have in the graph.

Lemma 3.1 *Let $G = (V, E)$ be a connected graph, and let $\mathcal{E}$ be an embedding for G with* $\mathrm{ord}(\mathcal{E}) = \langle x_1, \ldots, x_n \rangle$*. If* $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) \geq \mathrm{d}_G(x_i, x_{i+1})$ *for every* $1 \leq i < n$ *then* $\mathcal{E}$ *is non-contractive.*

Proof. Assume for a contradiction that $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) \geq \mathrm{d}_G(x_i, x_{i+1})$ for every $1 \leq i < n$, but that $\mathcal{E}$ is not non-contractive. Then, there is a pair u, v of vertices of G such that $\mathrm{d}_{\mathcal{E}}(u, v) < \mathrm{d}_G(u, v)$. Among all such pairs we choose u and v with smallest $\mathrm{d}_{\mathcal{E}}(u, v)$. Without loss of generality, we can assume that u appears to the left of v in $\mathcal{E}$. If $u = x_i$ and $v = x_{i+1}$ for some $1 \leq i < n$ then $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) < \mathrm{d}_G(x_i, x_{i+1})$. So, there is a vertex w between u and v in $\mathcal{E}$, and by the choice of u and v, $\mathrm{d}_{\mathcal{E}}(u, w) \geq \mathrm{d}_G(u, w)$ and $\mathrm{d}_{\mathcal{E}}(w, v) \geq \mathrm{d}_G(w, v)$. However, since $\mathrm{d}_{\mathcal{E}}(u, v) = \mathrm{d}_{\mathcal{E}}(u, w) + \mathrm{d}_{\mathcal{E}}(w, v)$ and $\mathrm{d}_G(u, v) \leq \mathrm{d}_G(u, w) + \mathrm{d}_G(w, v)$, we obtain the desired contradiction. ■

Corollary 3.2 *Let $G = (V, E)$ be a connected graph. Then, G has a minimum distortion embedding $\mathcal{E}$ with* $\mathrm{ord}(\mathcal{E}) = \langle x_1, \ldots, x_n \rangle$ *such that* $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) = \mathrm{d}_G(x_i, x_{i+1})$ *for every* $1 \leq i < n$.

Proof. Let $\mathcal{F}$ be a minimum distortion embedding for G, and let $\mathrm{ord}(\mathcal{F}) = \langle x_1, \ldots, x_n \rangle$. Obtain $\mathcal{E}$ by placing x_1 in the slot at position 1 and x_{i+1} at distance $\mathrm{d}_G(x_i, x_{i+1})$ to the right of x_i for every $1 \leq i < n$. Informally spoken, $\mathcal{E}$ is obtained from $\mathrm{ord}(\mathcal{F})$ by adding the minimum number of necessary empty slots between consecutive vertices. Then, $\mathcal{E}$ satisfies the condition of Lemma 3.1, thus is non-contractive. It holds that $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) \leq \mathrm{d}_{\mathcal{F}}(x_i, x_{i+1})$, $1 \leq i < n$, so that $\mathrm{D}(G, \mathcal{E}) \leq \mathrm{D}(G, \mathcal{F})$. Thus, $\mathcal{E}$ is a minimum distortion embedding for G. ■

About the above result, note in particular that there are no empty slots between consecutive vertices in $\mathcal{E}$ that are adjacent in G. We say that an embedding *does not contain unnecessary empty slots* if it satisfies the distance condition of Corollary 3.2, i.e., consecutive vertices in the embedding are at distance exactly their distance in the graph.

A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets. We denote such a graph by $G = (A, B, E)$ where $A \cup B$ is the vertex set of G, and A and B are independent sets, also called *color classes*. If G is a connected bipartite graph, then the partition of the vertex set into the two color classes is unique.

Corollary 3.3 *The distortion of a connected bipartite graph is an odd integer.*

Proof. Let $G = (A, B, E)$ be a connected bipartite graph, and let $\mathcal{E}$ be a minimum distortion embedding for G. Let $\langle x_1, \ldots, x_n \rangle$ be the vertex ordering underlying $\mathcal{E}$. According to Corollary 3.2, we can choose $\mathcal{E}$ such that $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1}) = \mathrm{d}_G(x_i, x_{i+1})$. Then, x_i and x_{i+1} belong to the same color class if and only if $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+1})$ is even. By induction, it can be shown that the vertices at even distance from x_i in $\mathcal{E}$ are exactly the vertices from the color class of x_i. Hence, u and v belong to the same color class of G if and only if $\mathrm{d}_{\mathcal{E}}(u, v)$ is even. Since adjacent vertices of G belong to different color classes, every edge joins two vertices at odd distance in $\mathcal{E}$. Thus, $\mathrm{D}(G, \mathcal{E})$ is odd. ∎

Lemma 3.4 *For every connected graph G,* $\mathrm{D}(G) \geq \mathrm{bw}(G)$.

Proof. Let $\mathcal{E}$ be a minimum distortion embedding for G with $\mathrm{ord}(\mathcal{E}) = \langle x_1, \ldots, x_n \rangle$. For every pair x_i, x_{i+r} of adjacent vertices of G, $\mathrm{d}_{\mathcal{E}}(x_i, x_{i+r}) \geq r$. Thus, the bandwidth of ordering $\langle x_1, \ldots, x_n \rangle$ is at most $\mathrm{D}(G, \mathcal{E})$. ∎

In some of our proofs, we will identify a subgraph of a given graph and use the distortion of the subgraph as a lower bound for the distortion of the given graph. For this reason, we need the following lemmas. We say that a subgraph H of G is *distance-preserving* if $\mathrm{d}_H(u, v) \leq \mathrm{d}_G(u, v)$ for all $u, v \in V$. It follows directly that distances in H and G are then equal, since every path in H is a path in G. In particular, distance-preserving subgraphs are induced subgraphs.

Lemma 3.5 *Let $G = (V, E)$ be a graph and let H be a subgraph of G. If H is a distance-preserving subgraph of G, then* $\mathrm{D}(G) \geq \mathrm{D}(H)$.

Proof. Let $\mathcal{E}$ be a minimum distortion embedding for G, and let $\mathcal{F}$ be obtained from $\mathcal{E}$ by removing all vertices that are not in H. Let u and v be vertices of H. Clearly, $\mathrm{d}_{\mathcal{F}}(u, v) = \mathrm{d}_{\mathcal{E}}(u, v)$ and $\mathrm{D}(H, \mathcal{F}) \leq \mathrm{D}(G, \mathcal{E})$. Since H is distance-preserving and $\mathcal{E}$ is non-contractive, we obtain $\mathrm{d}_H(u, v) = \mathrm{d}_G(u, v) \geq \mathrm{d}_{\mathcal{E}}(u, v) = \mathrm{d}_{\mathcal{F}}(u, v)$. Hence, $\mathcal{F}$ is a non-contractive embedding for H, and thus $\mathrm{D}(H) \leq \mathrm{D}(G)$. ∎

For applying Lemma 3.5, the main task is to identify distance-preserving subgraphs. We give sufficient conditions for two easy situations.

Lemma 3.6 *Let u and v be two false twin vertices of a graph $G = (V, E)$. Let H be a connected subgraph of G that contains u and v. If $H - v$ is a distance-preserving subgraph of G then H is a distance-preserving subgraph of G.*

Proof. Let $H - v$ be distance-preserving. Let x and y be two vertices of H. If $x \neq v$ and $y \neq v$ then $\mathrm{d}_H(x, y) \leq \mathrm{d}_{H-v}(x, y)$ since adding vertices does not increase distances. Now, let $x = v$. If $y = u$ then u and v have a common neighbor in H (since H is connected) and G and thus $\mathrm{d}_H(v, u) = \mathrm{d}_G(v, u) = 2$. If $y \neq u$ then $\mathrm{d}_G(u, y) = \mathrm{d}_G(v, y)$. Let $(w_0, w_1, \ldots, w_s)$ be a shortest u, y-path in $H - v$. By $H - v$ being distance-preserving, $\mathrm{d}_G(u, y) = s$. Then, $(v, w_1, \ldots, w_s)$ is a v, y-path in H, so that v and y are at distance at most $s = \mathrm{d}_G(v, y)$ in H. Hence, H is a distance-preserving subgraph of G. ∎

Lemma 3.7 *Let u and v be two vertices of a graph $G = (V, E)$ such that $N_G(v) \subseteq N_G[u]$. Then, $G{-}v$ is a distance-preserving subgraph of G.*

Proof. Let a, b be vertices of $G{-}v$, and let P be a shortest a, b-path in G. If P does not contain v then $\mathrm{d}_{G-v}(a, b) = \mathrm{d}_G(a, b)$. Otherwise, if P contains v obtain P' by replacing v with u. If P' is a simple path, which means that no vertex appears more than once on P', P' is a path in $G{-}v$, and we conclude $\mathrm{d}_{G-v}(a, b) = \mathrm{d}_G(a, b)$. Suppose now that P' is not a simple path. Then, u occurs twice on P'. We obtain P'' from P' by cutting the piece from the first occurrence of u on P' until before the second occurrence of u. Then, P'' is an a, b-path in G of shorter length than P, which contradicts the choice of P. ■

3.2 Graph classes with easy minimum distortion embeddings

As a warm-up before we start with the more involved algorithms in the next sections, and as interesting independent results on their own, we present combinatorial results on the minimum distortion of proper interval graphs, cycles, complete bipartite graphs, and complete split graphs. The result for complete bipartite graphs is heavily needed for our results on bipartite permutation graphs.

A graph is an *interval graph* if sets of consecutive integers (intervals) can be assigned to its vertices such that two vertices are adjacent if and only if their intervals have a non-empty intersection. An interval graph is a *proper interval graph* if intervals can be assigned such that no interval is a subset of another. Proper interval graphs are equivalent to *unit interval graphs* meaning that there is an assignment with all intervals of the same length [18]. The vertex ordering by the smallest (or equivalently largest) element of the assigned intervals is called a *proper interval ordering.*

Theorem 3.8 *For every connected proper interval graph G,* $\mathrm{D}(G) = \mathrm{bw}(G)$ *.*

Proof. Let $G = (V, E)$ be a connected proper interval graph with proper interval ordering $\langle x_1, \ldots, x_n \rangle$. Let $\mathcal{E}$ be the non-contractive embedding without unnecessary empty slots with underlying vertex ordering $\langle x_1, \ldots, x_n \rangle$. Since G is connected, $x_i x_{i+1} \in E$ for every $1 \leq i < n$, so that there are no empty slots between the vertices in $\mathcal{E}$. For every pair x_i, x_j of adjacent vertices, where $i < j$, the set $\{x_i, x_{i+1}, \ldots, x_j\}$ is a clique in G [14, 8]. Hence, the maximum distance of two adjacent vertices is $\omega(G) - 1 = \mathrm{bw}(G)$. ■

The following three theorems show that the distortion of cycles, complete bipartite graphs and complete split graphs only depend on the number of vertices in the graphs.

Theorem 3.9 $\mathrm{D}(C_n) = n - 1$ *for* $n \geq 3$*.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a cycle in C_n. Vertex ordering $\sigma = \langle v_1, v_2, \ldots, v_n \rangle$ is a non-contractive embedding for C_n of distortion $\mathrm{d}_\sigma(v_1, v_n) = n - 1$. Hence, $\mathrm{D}(C_n) \leq n - 1$.

For the lower bound, let $\mathcal{E}$ be a minimum distortion embedding for C_n with the smallest number of non-adjacent consecutive vertices. Let $\mathrm{ord}(\mathcal{E}) = \langle x_1, \ldots, x_n \rangle$. For $1 \leq i < n$, we call position $\mathcal{E}(x_i)$ a *gap position* if $x_i x_{i+1} \notin E$. If $\mathcal{E}$ has no gap positions then $x_i x_{i+1} \in E$ for all $1 \leq i < n$, and $(x_1, \ldots, x_n)$ is a path in C_n. Then, $x_1 x_n \in E$ and $\mathrm{D}(C_n) = \mathrm{D}(C_n, \mathcal{E}) = \mathrm{d}_{\mathcal{E}}(x_1, x_n) = n - 1$. Now, assume that there is a gap position in $\mathcal{E}$. We construct a non-contractive embedding for C_n with a smaller number of gap positions and without increasing the distortion. Let $\mathcal{E}(x_j)$ be a gap position of $\mathcal{E}$. The number of empty slots between x_j and x_{j+1} in $\mathcal{E}$ is equal to $\mathrm{d}_{C_n}(x_j, x_{j+1}) - 1$. Let P be a shortest x_j, x_{j+1}-path in C_n. We obtain $\mathcal{F}$ from $\mathcal{E}$ by moving the vertices in P that are different from x_j and x_{j+1} into the empty slots between x_j and x_{j+1} respecting their order in P. Clearly, $\mathcal{F}$ is non-contractive. We determine the distortion of $\mathcal{F}$. Moved vertices are at distance 1 to their two neighbours, so that it holds for every pair u, v of adjacent vertices at distance more than 1 in $\mathcal{F}$ that $\mathcal{F}(u) = \mathcal{E}(u)$ and $\mathcal{F}(v) = \mathcal{E}(v)$, thus $\mathrm{d}_{\mathcal{F}}(u, v) = \mathrm{d}_{\mathcal{E}}(u, v)$. Hence, $\mathrm{D}(G, \mathcal{F}) \leq \mathrm{D}(G, \mathcal{E})$. We consider the number of non-adjacent consecutive vertices in $\mathcal{F}$. Let x_i and x_{i+1} be adjacent in $\mathcal{E}$. Note that x_i is moved if and only if x_{i+1} is moved. Then, x_i and x_{i+1} appear consecutively in $\mathcal{F}$. Hence, the number of non-adjacent consecutive vertices in $\mathcal{F}$ is at most the number in $\mathcal{E}$. However, since x_j and the close vertex to the right of x_j in $\mathcal{F}$ are adjacent, the number of consecutive non-adjacent vertices in $\mathcal{F}$ is smaller than the number in $\mathcal{E}$, which contradicts the choice of $\mathcal{E}$. Hence, $\mathcal{E}$ does not contain a gap position. ■

A bipartite graph $G = (A, B, E)$ is a *complete bipartite graph* if every vertex of A is adjacent to every vertex of B. Such a graph is denoted by $K_{n,m}$, where $n = |A|$ and $m = |B|$.

Theorem 3.10 *Let m and n be integers satisfying $1 \leq n \leq m$. If $n+m$ is odd then $\mathrm{D}(K_{n,m}) = n + m - 2$, and if $n + m$ is even then $\mathrm{D}(K_{n,m}) = n + m - 1$.*

Proof. Let A and B be the two colour classes of $K_{n,m}$ with $|A| = n$ and $|B| = m$.

First we prove a lower bound on the distortion of $K_{n,m}$. Clearly, $\mathrm{D}(K_{1,1}) = 1$. Assume in the following that $m \geq 2$. Let $\mathcal{E}$ be a non-contractive embedding for $K_{n,m}$. The distance between consecutive vertices from the same color class is at least 2. Denote by a and a' the respectively leftmost and rightmost vertex from A, and denote by b and b' the respectively leftmost and rightmost vertex from B. It holds that $\mathrm{d}_{\mathcal{E}}(a, a') \geq 2n - 2$ and $\mathrm{d}_{\mathcal{E}}(b, b') \geq 2m - 2$, and $\mathrm{D}(\mathcal{E}) = \max\{\mathrm{d}_{\mathcal{E}}(a, b'), \mathrm{d}_{\mathcal{E}}(b, a')\}$. We distinguish two cases. If there is a vertex from A to the left of b or to the right of b' then the distortion of $\mathcal{E}$ is at least $2m - 1 \geq m + n - 1$. Now, let all vertices from A be between b and b'. Note that $\mathrm{d}_{\mathcal{E}}(b, a') = \mathrm{d}_{\mathcal{E}}(b, a) + \mathrm{d}_{\mathcal{E}}(a, a')$. So, $\mathrm{d}_{\mathcal{E}}(b, a') + \mathrm{d}_{\mathcal{E}}(a, b') = \mathrm{d}_{\mathcal{E}}(b, a') + \mathrm{d}_{\mathcal{E}}(a, a') + \mathrm{d}_{\mathcal{E}}(a', b') = \mathrm{d}_{\mathcal{E}}(a, a') + \mathrm{d}_{\mathcal{E}}(b, b')$. A lower bound on this sum is $2n - 2 + 2m - 2 = 2(n + m - 2)$. Hence, $\mathrm{D}(\mathcal{E}) \geq n + m - 2$, which already gives the lower bound in the case $n + m$ odd. Let $n + m$ be even. If $\mathrm{d}_{\mathcal{E}}(b, b') = 2m - 2$ then the slot at distance $n + m - 2$ is occupied by a vertex from B, and $\mathrm{d}_{\mathcal{E}}(b, a') \geq n + m - 1$. If $\mathrm{d}_{\mathcal{E}}(b, a') = n + m - 2$ then $\mathrm{d}_{\mathcal{E}}(b, b') \geq 2m - 1$ and $\mathrm{d}_{\mathcal{E}}(b, a') + \mathrm{d}_{\mathcal{E}}(a, b') \geq 2n - 2 + 2m - 1$, which gives $\mathrm{d}_{\mathcal{E}}(b, a') \geq n+m-1$ or $\mathrm{d}_{\mathcal{E}}(a, b') \geq n+m-1$. This completes the proof of the lower bound.

We prove an upper bound on the distortion by defining an embedding $\mathcal{E}$. Lay out the vertices from B in any order with exactly one empty slot between consecutive vertices. Denote by b and b' the respectively leftmost and rightmost vertex in $\mathcal{E}$. Let $p =_{\mathrm{def}} \mathcal{E}(b') - (n+m-2)$ or $p =_{\mathrm{def}} \mathcal{E}(b') - (n + m - 1)$, depending on whether $n + m$ is odd or even, respectively. Note that the slot at position p is empty in $\mathcal{E}$. Starting in the slot at position p and continuing towards the right, place the vertices from A in any order with one slot between consecutive vertices. This completes the definition of $\mathcal{E}$. Observe that $\mathcal{E}$ is a proper embedding. Furthermore, $\mathcal{E}$ is non-contractive, since vertices of the same colour class are at distance at least 2 from each other, and vertices from different colour classes are adjacent. Denote by a and a' the respectively leftmost and rightmost vertex from A in $\mathcal{E}$. It holds that $\mathrm{d}_{\mathcal{E}}(a,a') = 2n-2$ and $\mathrm{d}_{\mathcal{E}}(b,b') = 2m-2$. Then, $\mathrm{d}_{\mathcal{E}}(b,a') = \mathrm{d}_{\mathcal{E}}(b,b') - \mathrm{d}_{\mathcal{E}}(a,b') + \mathrm{d}_{\mathcal{E}}(a,a') \leq 2m-2-(n+m-2)+2n-2 = n+m-2$. Hence, if $n+m$ is odd then $\mathrm{D}(K_{n,m},\mathcal{E}) = n+m-2$, if $n+m$ is even then $\mathrm{D}(K_{n,m},\mathcal{E}) = n+m-1$. ∎

For natural numbers n and m, we denote by $S_{n,m}$ the graph whose vertices can be partitioned into a clique X of size n and an independent set I of size m that has all edges between X and I. Thus, $S_{n,m}$ can be obtained from a complete bipartite graph by making one color class into a clique. Note that $S_{1,m}$ coincides with $K_{1,m}$ and that $S_{n,1}$ is a complete graph.

Theorem 3.11 *Let n and m be natural numbers where $n \geq 2$ and $m \geq 2$. Then,* $\mathrm{D}(S_{n,m}) = n+m-2$.

Proof. Let (X,I) be a partition of the vertex set of $S_{n,m}$ into a clique X of size n and an independent set I of size m.

First we prove a lower bound on the distortion of $S_{n,m}$. Let $\mathcal{E}$ be a minimum distortion embedding for $S_{n,m}$ with the smallest number of vertices from I between vertices from X. The leftmost and rightmost vertex in $\mathcal{E}$ are at distance at least $m+n-1$. If one of these two vertices is a vertex from X then the two vertices are adjacent and the distortion of $\mathcal{E}$ is at least $m+n-1$. Now, let the leftmost and rightmost vertex be vertices from I, denoted as b and b', respectively. Denote by a and a' the respectively leftmost and rightmost vertex from X in $\mathcal{E}$. It holds that $\mathrm{D}(S_{n,m},\mathcal{E}) = \max\{\mathrm{d}_{\mathcal{E}}(b,a'), \mathrm{d}_{\mathcal{E}}(a,b')\}$. Furthermore, $S =_{\mathrm{def}} \mathrm{d}_{\mathcal{E}}(b,a') + \mathrm{d}_{\mathcal{E}}(a,b') = \mathrm{d}_{\mathcal{E}}(b,b') + \mathrm{d}_{\mathcal{E}}(a,a')$, and $\mathrm{D}(S_{n,m},\mathcal{E}) \geq \frac{1}{2}S$. We distinguish three cases. First, let there be no vertex from I between vertices from X in $\mathcal{E}$. Then, $\mathrm{d}_{\mathcal{E}}(b,b') \geq 2m-2+n-1$ and $\mathrm{d}_{\mathcal{E}}(a,a') \geq n-1$. For the second case, let there be exactly one vertex from I between a and a' in $\mathcal{E}$. Then, $\mathrm{d}_{\mathcal{E}}(b,b') \geq 2m-2+n-2$ and $\mathrm{d}_{\mathcal{E}}(a,a') \geq n$. In both cases, we obtain $S \geq 2m-2+2n-2$, thus $\mathrm{D}(S_{n,m},\mathcal{E}) \geq n+m-2$. For the third case, let there be at least two vertices from I between a and a' in $\mathcal{E}$. Let c and c' be the close I-vertex to the right of a and to the left of a', respectively. We obtain $\mathcal{F}$ from $\mathcal{E}$ by removing c and c', moving all vertices to the left of c one position further to the right, all vertices to the right of c' one position further to the left, placing c at distance 2 to the left of b and c' at distance 2 to the right of b'. Since vertices from I are at distance at least 2 from each other in $\mathcal{F}$, $\mathcal{F}$ is a non-contractive embedding for $S_{n,m}$. Furthermore, $\mathrm{d}_{\mathcal{F}}(c,a') = \mathrm{d}_{\mathcal{E}}(b,a')$ and $\mathrm{d}_{\mathcal{F}}(a,c') = \mathrm{d}_{\mathcal{E}}(a,b')$, so that $\mathrm{D}(S_{n,m},\mathcal{F}) = \mathrm{D}(S_{n,m},\mathcal{E})$. Since the number of vertices from I between vertices from X in $\mathcal{F}$ is smaller than the number in $\mathcal{E}$, we obtain a contradiction to the choice of $\mathcal{E}$. This completes the proof of the lower bound.

We prove the upper bound on the distortion by defining an embedding. We distinguish two cases. Let m be even. Let $\mathcal{E}$ be a non-contractive embedding without unnecessary empty slots with underlying vertex ordering of the following form: first $\frac{m}{2}$ vertices from I, then the vertices from X, then $\frac{m}{2}$ vertices from I. It clearly holds that $\mathrm{D}(S_{n,2},\mathcal{E}) = n$ and $\mathrm{D}(S_{n,m},\mathcal{E}) = 2\cdot\frac{m-1}{2}+1+n-1$ for $m \geq 4$. In the case where m is odd, we define embedding $\mathcal{E}$ as: take the above defined embedding for $S_{n,m-1}$ and place the last vertex from I between two vertices from X. Then, $\mathcal{E}$ is a non-contractive embedding for $S_{n,m}$ of distortion $(n+(m-1)-2)+1$. ∎

4 Distortion of threshold graphs

A graph is a *split graph* if its vertices can be partitioned into a clique X and an independent set I. We call such a partition a *split partition* and denote it as (X, I). A split partition is not unique in general. A graph $G = (V, E)$ with split partition (X, I) is also denoted as (X, I, E). We refer to the vertices of X and I as X-vertices and I-vertices, respectively. Threshold graphs are split graphs, and they have various characterizations [5, 8]. For our purposes, the following characterization will serve as definition. A graph is a *threshold graph* if and only if it is split and the vertices of the independent set can be ordered by neighborhood inclusion, for any split partition of it [15]. Equivalently, the vertices of the clique can also be ordered by neighborhood inclusion [15]. Hence, for any split partition (X, I) of a threshold graph G, the I-vertices can be ordered as $a_1, a_2, \ldots, a_m$ such that $N(a_1) \subseteq N(a_2) \subseteq \cdots \subseteq N(a_m)$, and the X-vertices can be ordered as $b_1, b_2, \ldots, b_n$ such that $N(b_1) \supseteq N(b_2) \supseteq \cdots \supseteq N(b_n)$. In particular, this means that I-vertices of same degree have exactly the same neighborhood, and the same for X-vertices. Hence the given orderings correspond to a non-decreasing degree order for the I-vertices and a non-increasing degree order for the X-vertices. For simplicity, we will say *decreasing* instead of non-increasing, and *increasing* instead of non-decreasing. Every connected threshold graph has a universal vertex. Hence, every pair of vertices in a connected threshold graph is at distance at most 2. In $G = (X, I, E)$, if there is no X-vertex without a neighbor in I, there is an I-vertex a that is adjacent to all X-vertices. Then, $(X \cup \{a\}, I \setminus \{a\})$ is also a split partition for G. In the following, we assume for split partitions that an X-vertex of smallest degree has no neighbors outside X.

In this section, we give an efficient algorithm for computing the distortion of threshold graphs. The algorithm is based on a structural result about minimum distortion embeddings for threshold graphs that we prove first. We split the proof into different partial results, the combination of which states that every threshold graph has a minimum distortion embedding such that the clique vertices and the independent-set vertices are ordered by degree. To give a brief outline, we start from an arbitrary minimum distortion embedding and modify it by moving vertices to obtain more structure. A similar idea was already applied in the proof of Theorem 3.9. When we say in the following that we "remove a vertex from the embedding" we mean that the position of the vertex becomes an empty slot. When we mean that even the empty slot is deleted, we mention this explicitly. Note that every embedding of a threshold graph can be partitioned into three sections: independent-set vertices to the left of all clique vertices, independent-set vertices to the right of all clique vertices, and all other vertices in between.

Lemma 4.1 *Let $G = (X, I, E)$ be a connected threshold graph. There is a minimum distortion embedding for G such that there are no empty slots between X-vertices.*

Proof. Let $\mathcal{E}$ be a minimum distortion embedding for G with the smallest number of empty slots between X-vertices. In particular, $\mathcal{E}$ does not contain unnecessary empty slots by Corollary 3.2, and hence there are no two consecutive empty slots. We show that $\mathcal{E}$ satisfies the lemma. Let b_l and b_r be the leftmost and rightmost X-vertex in $\mathcal{E}$, respectively. Assume for a contradiction that there is an empty slot at position p between b_l and b_r in $\mathcal{E}$. Let u and v be the vertices at positions $p-1$ and $p+1$, respectively. Since $2 \geq \mathrm{d}_G(u, v) = \mathrm{d}_{\mathcal{E}}(u, v)$, it follows that at least one of these two vertices is an I-vertex. Assume that v is an I-vertex, and if u is also an

I-vertex then assume that $d_G(u) \geq d_G(v)$. Otherwise, we repeat the arguments on the reverse of $\mathcal{E}$. Obtain $\mathcal{F}$ from $\mathcal{E}$ by removing v and moving all vertices to the left of v two positions to the right. Note that the slot to the immediate right of u in $\mathcal{F}$ (hence immediate right of v in $\mathcal{E}$) is either empty or occupied by an X-vertex that is adjacent to u. Hence, $\mathcal{F}$ is non-contractive. Let w be a universal vertex in G, and let a_l and a_r be the respectively leftmost and rightmost vertex in $\mathcal{F}$ (and thus $\mathcal{E}$). We obtain $\mathcal{F}'$ from $\mathcal{F}$ as follows:

– if $v \prec_{\mathcal{E}} w$ then place v at distance 2 to the left of a_l

– if $w \prec_{\mathcal{E}} v$ then place v at distance 2 to the right of a_r .

Then, $\mathcal{F}'$ is a non-contractive embedding for G. Furthermore, $\mathrm{D}(G, \mathcal{F}') \leq \mathrm{D}(G, \mathcal{E})$ since $\mathrm{d}_{\mathcal{F}'}(v, w), \mathrm{d}_{\mathcal{F}'}(a_l, w), \mathrm{d}_{\mathcal{F}'}(a_r, w) \leq \max\{\mathrm{d}_{\mathcal{E}}(a_l, w), \mathrm{d}_{\mathcal{E}}(a_r, w)\}$. Thus, $\mathcal{F}'$ is a minimum distortion embedding for G with fewer empty slots between b_l and b_r than $\mathcal{E}$, contradicting the choice of $\mathcal{E}$. ∎

In particular, non-contractive embeddings of threshold graphs without empty slots between X-vertices do not contain consecutive I-vertices between X-vertices.

Lemma 4.2 *Let $G = (X, I, E)$ be a connected threshold graph. There is a minimum distortion embedding for G without empty slots between X-vertices such that the X-vertices are ordered decreasingly by degree.*

Proof. Let $\mathcal{E}$ be a minimum distortion embedding for G without empty slots between X-vertices and without unnecessary empty slots. Let u be the leftmost universal vertex in $\mathcal{E}$. Without loss of generality we assume that there is an X-vertex of smallest degree to the right of u in $\mathcal{E}$; otherwise we use the reverse of $\mathcal{E}$ instead of $\mathcal{E}$. Let v be the rightmost X-vertex of smallest degree in $\mathcal{E}$. Denote by b_l and b_r the respectively leftmost and rightmost X-vertex in $\mathcal{E}$. Note that $\mathrm{D}(G, \mathcal{E}) \geq \mathrm{d}_{\mathcal{E}}(b_l, b_r)$. Denote by A_l and A_r the set of I-vertices to the left of b_l and to the right of b_r in $\mathcal{E}$, respectively. Without loss of generality we assume that all vertices of A_l are ordered increasingly by degree and all vertices of A_r are ordered decreasingly by degree, since ordering the I-vertices in this way cannot increase $\mathrm{D}(G, \mathcal{E})$. This assumption is of importance only for making later arguments easier. Let M be the set of vertices in A_r that are at distance more than $\mathrm{D}(G, \mathcal{E})$ to b_l. Hence no vertex in M is adjacent to b_l. Furthermore, the vertices in M appear consecutively in $\mathcal{E}$, and if M is not empty then the righmtost vertex in $\mathcal{E}$ is contained in M. Let w be the rightmost vertex in $\mathcal{E}$ that is not contained in M; hence either $w = b_r$, or $w \in I$ and to the right of b_r. In both cases, $\mathrm{d}_{\mathcal{E}}(b_l, w) \leq \mathrm{D}(G, \mathcal{E})$. Let M be non-empty and let w' be the leftmost vertex in M in $\mathcal{E}$; hence $\mathrm{d}_{\mathcal{E}}(b_l, w') \geq \mathrm{D}(G, \mathcal{E}) + 1$. If $\mathrm{d}_{\mathcal{E}}(b_l, w') \geq \mathrm{D}(G, \mathcal{E}) + 2$ then for any $a \in M$ there are at least $\mathrm{d}_{\mathcal{E}}(w', a) + 2$ vertices between (including) b_l and u in $\mathcal{E}$ that are not adjacent to a. If $\mathrm{d}_{\mathcal{E}}(b_l, w') = \mathrm{D}(G, \mathcal{E}) + 1$ then for any $a \in M$ there are at least $\mathrm{d}_{\mathcal{E}}(w', a) + 1$ vertices between (including) b_l and u in $\mathcal{E}$ that are not adjacent to a. These two cases are called "long" and "short" case, respectively. We will define a new embedding for G by mainly reordering the vertices between b_l and b_r.

We begin by defining an area of $\mathcal{E}$. The *working interval* is the interval of slots between (including) positions $\mathcal{E}(b_l)$ and $\mathcal{E}(b_r)$ potentially extended by the positions:

– $\mathcal{E}(b_r) + 1$ if the slot at this position is non-empty and not occupied by w'

– $\mathcal{E}(b_l) - 1$ if the slot at this position is non-empty and we are in the short case and there are two X-vertices between b_l and u at distance 1 from each other.

As an auxiliary result we show for the case where the slot at position $\mathcal{E}(b_l) - 1$ in $\mathcal{E}$ is occupied that for every vertex $a \in M$ there are at least $\mathrm{d}_{\mathcal{E}}(w', a) + 2$ non-neighbours to the left of the leftmost neighbour of a in the working interval. By the discussion above we need only to consider the case $\mathrm{d}_{\mathcal{E}}(b_l, w') = \mathrm{D}(G, \mathcal{E}) + 1$. The claim is obviously true if position $\mathcal{E}(b_l) - 1$ belongs to the working interval. If position $\mathcal{E}(b_l) - 1$ does not belong to the working interval then all X-vertices between b_l and u are at distance at least 2 from each other. In particular, between every pair of consecutive X-vertices between b_l and u there is an I-vertex. Hence, the leftmost neighbour of x is at distance at most $\mathrm{D}(G, \mathcal{E}) - 1$, since the slot at distance $\mathrm{D}(G, \mathcal{E})$ to the left of x is occupied by an I-vertex.

Let β be an ordering of the vertices in the working interval of the following form: X-vertices are ordered decreasingly by degree and I-vertices are placed rightmost between two neighbors; for ease of argument we assume that X-vertices of the same degree keep their $\mathcal{E}$-order in β. We show that β exists by counting I-vertices. Assign I-vertices to X-vertices in the following way. Let y be an I-vertex in the working interval in $\mathcal{E}$:

– if $v \prec_{\mathcal{E}} y$ then assign y to the close vertex to the left

– if $u \prec_{\mathcal{E}} y \prec_{\mathcal{E}} v$ then assign y to the close vertex to the right

– if position $\mathcal{E}(b_l) - 1$ does not belong to the working interval: if $y \prec_{\mathcal{E}} u$ then assign y to the close vertex to the left

– if position $\mathcal{E}(b_l) - 1$ belongs to the working interval, let z be the leftmost X-vertex such that the close vertex to the right is an X-vertex: if $y \prec_{\mathcal{E}} z$ then assign y to the close vertex to the right, if $z \prec_{\mathcal{E}} y \prec_{\mathcal{E}} u$ then assign y to the close vertex to the left.

Note that every I-vertex in the working interval is assigned to an X-vertex, u and v have no assigned I-vertex (particularly since v has no I-vertex neighbors) and no X-vertex has two assigned I-vertices (particularly since the close vertex to the right of z is an X-vertex). For every X-vertex x, denote by $n(x)$ the number of I-vertices to the right of x in β. Clearly, $n(v) = 0$. Let x be an X-vertex satisfying $u \prec_{\beta} x \prec_{\beta} v$, and let x be assigned an I-vertex y. If the close vertex to the left of x is an X-vertex, then y is to the right of x; otherwise y could be placed between x and the close vertex to the left thus obtaining an ordering of the desired form with an I-vertex further to the right. Hence, $n(x)$ is at least the number of X-vertices to the right of x in β that are assigned an I-vertex, and if x is assigned an I-vertex and the close vertex to the left of x is an X-vertex then $n(x)$ is at least the number of X-vertices between x and v that are assigned an I-vertex (note that the difference between the two cases is 1). In particular, $n(u)$ is equal to the number of I-vertices in the working interval, which shows that β indeed exists.

We define an embedding $\mathcal{F}$ for G. We only give the underlying vertex ordering of the embedding; the actual embedding is obtained by adding the necessary (but no unnecessary) empty slots: place the vertices in $A_l \cup M$ ordered increasingly by degree, where vertices in A_l preserve their order in $\mathcal{E}$ and vertices in M appear in reverse order as in $\mathcal{E}$, then place the vertices from the working interval according to β, place the vertices in $A_r \setminus M$ in their order in $\mathcal{E}$. By definition, $\mathcal{F}$ is non-contractive, and there are empty slots only to the left of u and to the right of v in $\mathcal{F}$. In the following, we determine the distortion of $\mathcal{F}$. Since the working interval in $\mathcal{E}$ does not contain empty slots, $\mathrm{d}_{\mathcal{F}}(u, v)$ is exactly the size of the working interval. Furthermore, the slot at position $\mathcal{F}(u) - 1$ is non-empty if $A_l \cup M$ is non-empty, and the slot at position $\mathcal{F}(v) + 1$ is empty. We begin with vertices to the right of u in $\mathcal{F}$. According to the definition of w, either $b_r = w$ or $b_r \prec_{\mathcal{E}} w$. If $b_r \prec_{\mathcal{E}} w$ then $\mathrm{D}(G, \mathcal{E}) = \mathrm{d}_{\mathcal{E}}(b_l, w)$ (which also means long case for M non-empty) or $\mathrm{D}(G, \mathcal{E}) \geq \mathrm{d}_{\mathcal{E}}(b_l, w) + 1$ (which can mean short or long case for M non-empty). Hence, $\mathrm{d}_{\mathcal{F}}(u, w) \leq \mathrm{D}(G, \mathcal{E})$. Let $b_r = w$. Then, $A_r = M$. Let $\mathrm{D}(G, \mathcal{E}) = \mathrm{d}_{\mathcal{E}}(b_l, b_r)$. If $\mathrm{d}_{\mathcal{E}}(b_r, w') = 2$ then we are in the long case. Otherwise, if $\mathrm{d}_{\mathcal{E}}(b_r, w') = 1$ then $b_r w' \in E$ and $b_l w' \notin E$ and therefore $d_G(b_l) < d_G(b_r)$ and therefore the slot at position $\mathcal{E}(b_l) - 1$ is empty. Hence, $\mathrm{d}_{\mathcal{F}}(u, v) = \mathrm{d}_{\mathcal{E}}(b_l, b_r)$.

Now, we consider the remaining vertices on the left end of $\mathcal{F}$. Let $y \in A_l$. Let c be the rightmost neighbor of y in $\mathcal{F}$. Then, c is the rightmost X-vertex of degree at least $d_G(c)$ in $\mathcal{F}$. Let c' be the rightmost X-vertex of degree at least $d_G(c)$ in $\mathcal{E}$. All X-vertices to the right of c' in $\mathcal{E}$ have degree at most $d_G(c) - 1$, so that they are to the right also of c in $\mathcal{F}$. Consider the I-vertices to the right of c' in $\mathcal{E}$ in the working interval. We distinguish two cases:

– if $v \prec_{\mathcal{E}} c'$ then all I-vertices to the right of c' in $\mathcal{E}$ (potentially) but one are assigned to vertices to the right of c' in $\mathcal{E}$; in this case, v, that is to the left of c' in $\mathcal{E}$, is to the right of c in $\mathcal{F}$

– if $c' \prec_{\mathcal{E}} v$ then all I-vertices to the right of c' in $\mathcal{E}$ are assigned to vertices to the right of c' in $\mathcal{E}$.

We conclude that the number of vertices between c and v in $\mathcal{F}$ is at least the number of vertices from c' on to the right in the working interval in $\mathcal{E}$. Let l be the number of I-vertices from M between y and u. If $l = 0$ then $\mathrm{d}_{\mathcal{F}}(y, c) \leq \mathrm{d}_{\mathcal{E}}(y, c')$. Let $l \geq 1$. Let z be the close I-vertex from M to the right of y. According to the auxiliary result, there are at least $\mathrm{d}_{\mathcal{E}}(w', z) + 2$ non-neighbours to the left of the leftmost neighbour, d', of z in $\mathcal{E}$. It remains to show that there are enough of them to the right of the rightmost neighbour, d, of z in $\mathcal{F}$. Clearly, all X-vertices to the left of d' in $\mathcal{E}$ are to the right of d in $\mathcal{F}$. If position $\mathcal{E}(b_l) - 1$ does not belong to the working interval then all I-vertices between b_l and d' in $\mathcal{E}$ are assigned to an X-vertex to the left of d'; if position $\mathcal{E}(b_l) - 1$ belongs to the working interval then at most one I-vertex to the left of d' in the working interval in $\mathcal{E}$ is not assigned to an X-vertex to the left of d' (but to d'). In this case, however, $\mathrm{d}_{\mathcal{F}}(y, u) = \mathrm{d}_{\mathcal{E}}(y, b_l) + 2l - 1$, and at least $2l - 1$ vertices to the left of u in $\mathcal{E}$ the working interval are to the right of c in $\mathcal{F}$. For vertices in M it suffices to remember that every such vertex adds a buffer of two vertices to the right end of the working interval. Hence, we conclude that $\mathrm{D}(G, \mathcal{F}) \leq \mathrm{D}(G, \mathcal{E})$, which completes the proof. ∎

With the result of Lemma 4.2 we can construct a simple algorithm for computing the distortion of threshold graphs. The algorithm finds an embedding of smallest distortion among all non-contractive embeddings where the X-vertices are ordered decreasingly by degree. Lemma 4.2 then shows that this actually is a minimum distortion embedding. Let $G = (X, I, E)$ be a connected threshold graph and let $\mathcal{E}$ be an embedding for G where the X-vertices are ordered decreasingly by degree. Let u be the leftmost universal vertex in $\mathcal{E}$. Denote by $R(\mathcal{E})$ the distance between rightmost vertex in $\mathcal{E}$ and its leftmost neighbor. And denote by $L(\mathcal{E})$ the maximum taken over all distances between a vertex to the left of u and its rightmost neighbor in $\mathcal{E}$. Then, $\mathrm{D}(G,\mathcal{E}) = \max\{L(\mathcal{E}), R(\mathcal{E})\}$.

Algorithm `thrg-distortion`
Input threshold graph $G = (X, I, E)$ where $I = \{y_1, \ldots, y_{|I|}\}$ such that $d_G(y_1) \leq \cdots \leq d_G(y_{|I|})$

begin
  let $\mathcal{E}_0$ be the start embedding; **let** u be the leftmost vertex in $\mathcal{E}_0$; **set** $i = 0$;
  while $R(\mathcal{E}_i) \geq L(\mathcal{E}_i) + 2$ **and** $i < |I|$ **do**
    set $i = i + 1$; **let** $\mathcal{E}_i$ be obtained from $\mathcal{E}_{i-1}$ by moving y_i to the left of u
  end while;
  let $\mathcal{E}$ be obtained from $\mathcal{E}_i$ by moving the close I-vertex to the right of u to the right end;
  return $\min\{\mathrm{D}(G,\mathcal{E}), \mathrm{D}(G,\mathcal{E}_i)\}$ and the corresponding embedding;
end.

For completing the algorithm we have to explain three operations. The start embedding is obtained by the following procedure. We only explain the underlying vertex orderings; the embedding then is obtained by adding the necessary empty slots. The X-vertices are ordered decreasingly by degree. The I-vertices are treated separately and in reverse given order, i.e., as $y_{|I|}, \ldots, y_1$, and are placed rightmost between two neighbors as long as possible, and when an I-vertex cannot be placed between two neighbors it is placed at the right end, particularly to the right of the rightmost X-vertex. Note that this embedding has no empty slots between X-vertices.

The second operation is the definition of embedding $\mathcal{E}_i$ inside the **while** loop. The definition depends on whether the picked vertex y_i is between X-vertices or to the right of the rightmost X-vertex in $\mathcal{E}_{i-1}$. If y_i is between X-verticees then y_i is removed, all vertices between u and position $\mathcal{E}_{i-1}(y_i)$ are moved one position to the right, all vertices to the left are moved one position to the left and y_i is placed in the slot previously occupied by u. Note that $\mathcal{E}_i$ is a proper non-contractive embedding for G without unnecessary empty slots and without empty slots between X-vertices. If y_i is to the right of the rightmost X-vertex in $\mathcal{E}_{i-1}$ then y_i is removed, all vertices between u and position $\mathcal{E}_{i-1}(y_i)$ are moved two positions to the right and y_i is placed in the slot at position $\mathcal{E}_{i-1}(u) + 1$.

The third operation defines $\mathcal{E}$ after the **while** loop as follows. If the close I-vertex v to the right of u in $\mathcal{E}_i$ is to the left of the rightmost X-vertex, then remove v, move all vertices to the left of position $\mathcal{E}_i(v)$ one position to the right and place v at the right end at distance 2 to the close vertex to the left.

Theorem 4.3 *There is an $\mathcal{O}(n)$-time algorithm that computes the distortion of a connected threshold graph on n vertices and outputs a corresponding embedding.*

Proof. We prove that Algorithm `thrg-distortion` is exactly such an algorithm. Let $G = (X, I, E)$ be a connected threshold graph with $y_1, \ldots, y_{|I|}$ the I-vertices in increasing degree order. Apply `thrg-distortion` to G. Let r be the number of **while** loop executions. Let $\mathcal{E}_0, \ldots, \mathcal{E}_r, \mathcal{E}$ and u be defined according to `thrg-distortion`. We show that one of the two embeddings $\mathcal{E}$ and $\mathcal{E}_r$ has smallest distortion among all non-contractive embeddings of G with the X-vertices ordered decreasingly by degree. Correctness of `thrg-distortion` then follows directly from Lemma 4.2. We begin by studying $\mathcal{E}_0, \ldots, \mathcal{E}_r$. Let $i \in \{1, \ldots, r\}$. It holds that $R(\mathcal{E}_{i-1}) - 2 \leq R(\mathcal{E}_i) \leq R(\mathcal{E}_{i-1}) - 1$ and $L(\mathcal{E}_{i-1}) + 1 \leq L(\mathcal{E}_i) \leq R(\mathcal{E}_{i-1})$. For the last inequality, note that the distance to the rightmost neighbour of all I-vertices to the left of u and different from y_i in $\mathcal{E}_i$ increases by at most 2 with respect to $\mathcal{E}_{i-1}$, so is bounded above by $R(\mathcal{E}_{i-1})$, and the rightmost neighbour of y_i is to the left of the rightmost X-vertex in $\mathcal{E}_i$. Hence, $\mathrm{D}(G, \mathcal{E}_i) \leq \mathrm{D}(G, \mathcal{E}_{i-1})$ and $L(\mathcal{E}_i) - R(\mathcal{E}_i) \leq 2$. And therefore, $-1 \leq L(\mathcal{E}_r) - R(\mathcal{E}_r) \leq 2$ when there is an I-vertex to the right of u in $\mathcal{E}_r$. Note that $R(\mathcal{E}_i) \leq R(\mathcal{F})$ for any non-contractive embedding $\mathcal{F}$ for G with u the leftmost X-vertex and at least $|I| - i$ I-vertices to the right of u. We distinguish two cases for $\mathcal{E}_r$ after termination of the **while** loop. First, let there be no I-vertex to the right of u in $\mathcal{E}_r$. Then, $r = |I|$. If y_r has exactly one neighbour then all I-vertices have exactly one neighbour. It holds that $R(\mathcal{F}) \geq R(\mathcal{E}_r) + 2$ for any non-contractive embedding $\mathcal{F}$ for G with u the leftmost X-vertex and an I-vertex to the right of u. Note that correctness of this argument relies on the fact that there is an X-vertex without any I-vertex neighbour. Thus, $\mathcal{E}_r$ is a minimum distortion embedding for G. Now, let y_r have at least two neighbours. According to the definition of $\mathcal{E}_0$, y_r is between two neighbours in $\mathcal{E}_{r-1}$, and $R(\mathcal{E}_r) = R(\mathcal{E}_{r-1}) - 1 = |X|$. Since $L(\mathcal{E}_r) \leq R(\mathcal{E}_r) + 1$, we have $L(\mathcal{E}_r) \leq |X| + 1$. ¿From this we conclude that $\mathrm{D}(G, \mathcal{E}_r) = \mathrm{D}(G)$, with similar arguments as above.

As the second case, let there be an I-vertex to the right of u in $\mathcal{E}_r$. Then, $r < |I|$. Let y_r be between X-vertices in $\mathcal{E}_{r-1}$. In particular, all I-vertices are between X-vertices in $\mathcal{E}_{r-1}$. Then, $R(\mathcal{E}_r) = R(\mathcal{E}_{r-1}) - 1$, and $R(\mathcal{E}_r) \leq L(\mathcal{E}_r) \leq R(\mathcal{E}_r) + 1$. Suppose there is a non-contractive embedding $\mathcal{F}$ for G with X-vertex ordered decreasingly by degree and of distortion smaller than $\mathrm{D}(G, \mathcal{E}_r)$. Then, there are at most $R(\mathcal{E}_r) - 1$ vertices to the right of u in $\mathcal{F}$, which means that at least r I-vertices are to the left of u in $\mathcal{F}$. Since we can assume that no I-vertex to the right of u in $\mathcal{F}$ has degree smaller than an I-vertex to the left of u, $y_1, \ldots, y_r$ are to the left of u in $\mathcal{F}$. With the usual exchange argument, it follows that I-vertices to the left of u can be ordered increasingly by degree without increasing the distortion. Hence, $\mathrm{D}(G, \mathcal{E}_r) \leq \mathrm{D}(G, \mathcal{F})$, and $\mathcal{E}_r$ is a minimum distortion embedding for G.

Finally, let y_r be to the right of all X-vertices in $\mathcal{E}_{r-1}$. Then, $R(\mathcal{E}_r) = R(\mathcal{E}_{r-1}) - 2$. Let $i \in \{1, \ldots, r\}$ be smallest such that $\mathrm{d}_{\mathcal{E}_r}(y_i, u) = L(\mathcal{E}_r)$. First, let y_i have exactly one neighbour u. Suppose that there is a non-contractive embedding $\mathcal{F}$ for G with the X-vertices ordered decreasingly by degree, starting with u, and of distortion at most $\mathrm{D}(G, \mathcal{E}_r) - 1 \leq R(\mathcal{E}_r) + 1$. By minimality of $R(\mathcal{E}_{r-1})$, there are at most $|I| - r$ I-vertices to the right of u in $\mathcal{F}$. Thus, $L(\mathcal{F}) \leq L(\mathcal{E}_r)$. If $L(\mathcal{E}_r) \geq R(\mathcal{E}_r)$ then $\mathrm{D}(G, \mathcal{F}) \geq \mathrm{D}(G, \mathcal{E}_r)$ contradicting the distortion assumption for $\mathcal{F}$. So, $L(\mathcal{E}_r) < R(\mathcal{E}_r)$, which means that $R(\mathcal{E}_r) = L(\mathcal{E}_r) + 1$. If there are more than r vertices to the left of u in $\mathcal{F}$ then $L(\mathcal{F}) > L(\mathcal{E}_r)$, which is a contradiction to the choice of $\mathcal{F}$. Hence, there are exactly r vertices to the left of u in $\mathcal{F}$. With the same argument as in the previous case, we can assume that the vertices to the left of u in $\mathcal{F}$ are $y_1, \ldots, y_r$. Then, however, $R(\mathcal{F}) = R(\mathcal{E}_r) - 1$ contradicts the minimality of $R(\mathcal{E}_r)$, so that we can conclude that $\mathcal{E}_r$ is a minimum distortion embeddding for G.

Now, let y_i have at least two neighbours. Let b be the rightmost neighbour of y_i. Assume that there is a non-contractive embedding $\mathcal{F}$ for G with the X-vertices in decreasing degree order, starting with u, and of distortion at most $R(\mathcal{E}_r) + 1$ and with the smallest number of vertices to the left of u. Without loss of generality, we can assume that I-vertices between X-vertices are rightmost (in the usual sense) in $\mathcal{F}$. Let M be the set of I-vertices between u and

b in $\mathcal{E}_r$ that are not between u and b in $\mathcal{F}$. Let y and y' be two vertices from M where at least one of them, say y, is to the right of u in $\mathcal{F}$. Obtain $\mathcal{F}'$ from $\mathcal{F}$ as follows, where unnecessary empty slots are deleted and necessary empty slots are inserted:

- if y' is to the left of u in $\mathcal{F}$ then move y and y' between two pairs of consecutive X-vertices between u and b
- if y' is to the right of u in $\mathcal{F}$ then move y and y' as in the previous case and additionally move the close vertex to the left of u in $\mathcal{F}$ to the right end.

It holds that $L(\mathcal{F}') = L(\mathcal{F})$ and $R(\mathcal{F}') = R(\mathcal{F})$. Since $\mathcal{F}'$ contains fewer vertices to the left of u than $\mathcal{F}$, this is a contradiction to the choice of $\mathcal{F}$. We conclude that if M contains at least two vertices then all vertices in M are to the left of u in $\mathcal{F}$. By minimality of $R(\mathcal{E}_{r-1})$, it follows that there are at most $|I|-r$ I-vertices to the right of u in $\mathcal{F}$. Without loss of generality, we can assume the following for the vertices to the left of u in $\mathcal{F}$: they contain $y_1,\ldots,y_r$, they appear in increasing degree order, they do not have degree larger than any I-vertex to the right of u. Let p be the number of vertices to the left of u. Suppose that $p > r$. If $L(\mathcal{F}) > L(\mathcal{E}_r)$ then $\mathrm{D}(G,\mathcal{F}) \geq \mathrm{D}(G,\mathcal{E}_r)$, so that $L(\mathcal{F}) \leq L(\mathcal{E}_r)$. Then, $|M| \geq 2(p-r) \geq 2$, which particularly means that all vertices in M are to the left of u in $\mathcal{F}$. However, since, $p-r \geq |M|$, $|M| \geq 2|M|$ implies $|M| = 0$, so that this is a contradiction. Hence, $p = r$ and $|M| \leq 1$. By the definition of the start embedding $\mathcal{E}_0$, the close vertex to the right of u in $\mathcal{E}_r$ is an I-vertex. Obtain $\mathcal{E}$ from $\mathcal{E}_r$ by moving the close vertex to the right of u to the right end. If $|M| = 0$ then $\mathrm{D}(G,\mathcal{F}) = \mathrm{D}(G,\mathcal{E}_r)$, if $|M| = 1$ then $\mathrm{D}(G,\mathcal{F}) = \mathrm{D}(G,\mathcal{E})$. This completes the correctness part of the proof.

For the running time of `thrg-distortion`, observe first that threshold graphs can be represented in $\mathcal{O}(n)$ space where each I-vertex keeps pointers to its leftmost and rightmost neighbor in the decreasingly sorted degree order of the X-vertices. Hence it is sufficient to describe how L- and R-values are computed efficiently. Clearly, R-values are obtained by simple subtraction, since it suffices to remember whether an I-vertex to the right of u is between two neighbours (whose moving results in decreasing the value by 1) or at the end (whose moving results in decreasing the value by 2). For I-vertices we have to distinguish two cases. If the moved I-vertex is at the right end, then the L-value increases by at least and is the maximum over the distance of the moved vertex to its rightmost neighbour and the increased previous L-value. If the moved I-vertex is between neighbours, some distances increase by 2 and some distances increase by 1. Here, we have to find the leftmost I-vertex with rightmost neighbour at maximum distance. This information can be computed in a preprocessing step, when there is no I-vertex between X-vertices and this is only a neighbourhood cardinality problem. Hence, `thrg-distortion` has an $\mathcal{O}(n)$-time implementation. ∎

Although not necessary for an efficient algorithm, we give the following structural result for completeness.

Theorem 4.4 *Let $G = (X, I, E)$ be a connected threshold graph. There is a minimum distortion embedding for G such that the X-vertices are ordered decreasingly and the I-vertices are ordered increasingly by degree.*

Proof. Let $\mathcal{E}$ be a minimum distortion embedding for G without empty slots between X-vertices such that the X-vertices are ordered decreasingly by degree; $\mathcal{E}$ exists due to Lemma 4.2. Denote by b_l and b_r the respectively leftmost and rightmost X-vertex in $\mathcal{E}$. Note that the close vertex to the left of every I-vertex between b_l and b_r is a neighbor. Let a be an I-vertex to the left of b_l and let a' be an I-vertex to the right of b_r. Suppose that $d_G(a) > d_G(a')$. Obtain $\mathcal{E}'$ from $\mathcal{E}$ by exchanging a and a'. Then, $\mathcal{E}'$ is a minimum distortion embedding for G since the rightmost neighbor of a' in $\mathcal{E}$ is a neighbor of a. Iterative application of this operation yields a minimum distortion embedding $\mathcal{F}$ for G without empty slots between b_l and b_r and the X-vertices ordered decreasingly and no I-vertex to the right of b_r has degree smaller than an I-vertex to the left of b_l.

Let a be the rightmost vertex in $\mathcal{F}$ and assume that a is an I-vertex. Let b be the rightmost neighbor of a in $\mathcal{F}$; clearly, $b \prec_{\mathcal{F}} b_r$ since b_r has no neighbors in I. Obtain $\mathcal{G}$ from $\mathcal{F}$ by moving all vertices to the right of b two positions to the right and moving a into the slot at position $\mathcal{F}(b)+1$. Then, $\mathcal{G}$ is a minimum distortion embedding for G since the distance between vertices to the left of b_l and their rightmost neighbors are equal in $\mathcal{F}$ and $\mathcal{G}$ and $\mathrm{d}_{\mathcal{G}}(b_l, a_r) \leq \mathrm{d}_{\mathcal{F}}(b_l, a)$ for a_r the rightmost vertex in $\mathcal{G}$. This inequality is true particularly since the slot to the immediate right of b_r is empty. Repeated application of this construction yields a minimum distortion embedding $\mathcal{G}^*$ for G such that the X-vertices are ordered decreasingly, the rightmost vertex is an X-vertex and all I-vertices between X-vertices have a neighbor as close X-vertex to the left. Without loss of generality, we can assume that the leftmost and rightmost X-vertex in $\mathcal{G}^*$ are b_l and b_r, respectively.

Let a and a' be I-vertices such that $a \prec_{\mathcal{G}^*} a'$ and $d_G(a) > d_G(a')$. Obtain $\mathcal{H}$ from $\mathcal{G}^*$ by exchanging a and a'. If there is a vertex in the slot to the immediate left or right of a in $\mathcal{H}$, this vertex is a neighbor of a' and thus a neighbor of a. For the position of a' it is important to note that all X-vertices to the left of a' in $\mathcal{G}^*$ are neighbors of a'. If the slot to the immediate right of a' in $\mathcal{H}$ is occupied then it is occupied by an X-vertex that is to the left of a' in $\mathcal{G}^*$, thus a neighbor of a'. Hence, $\mathcal{H}$ is non-contractive. For the distortion of $\mathcal{H}$ it suffices to note that the rightmost neighbor of a' in $\mathcal{H}$ is a neighbor of a; leftmost neighbors are at distance less than $\mathrm{d}_{\mathcal{H}}(b_l, b_r) = \mathrm{d}_{\mathcal{G}^*}(b_l, b_r)$. Thus, $\mathcal{H}$ is a minimum distortion embedding for G. Repeated application of the construction yields a minimum distortion embedding of the desired form. ∎

5 Distortion of bipartite permutation graphs

Bipartite permutation graphs are permutation graphs that are bipartite. For the definition and properties of permutation graphs, we refer to [5]. Let $G = (A, B, E)$ be a bipartite graph. A *strong ordering* for G is a pair of orderings (σ_A, σ_B) on respectively A and B such that for every pair of edges ab and $a'b'$ in E with $a, a' \in A$ and $b, b' \in B$, $a \prec_{\sigma_A} a'$ and $b' \prec_{\sigma_B} b$ imply that ab' and $a'b$ are in E. If we denote by $(\sigma_A, \sigma_B)^R$ the pair of the reverse of σ_A and σ_B, then $(\sigma_A, \sigma_B)^R$ is also a strong ordering for G. The following characterization of bipartite permutation graphs is the only property that we will need in this section, and thus we use it as a definition.

Theorem 5.1 ([19]) *A bipartite graph is a bipartite permutation graph if and only if it has a strong ordering.*

Spinrad et al. give a linear-time recognition algorithm for bipartite permutation graphs that produces a strong ordering if the input graph is bipartite permutation [19]. It follows from the definition of a strong ordering that if $G = (A, B, E)$ is a connected bipartite permutation graph then any strong ordering (σ_A, σ_B) satisfies the following. For every vertex a of A, the neighbors of a appear consecutively in σ_B. Furthermore, if $N(a) \subseteq N(a')$ for two vertices $a, a' \in A$ then a is adjacent to the leftmost or the rightmost neighbor of a' in σ_B.

We show two main results about distortion of bipartite permutation graphs. We give a fast algorithm for computing the distortion of bipartite permutation graphs and we give a complete characterization of bipartite permutation graphs of bounded distortion by forbidden induced subgraphs. Both results are obtained simultaneously in the second and third subsection. Before, we consider the relationship of bandwidth and distortion for bipartite permutation graphs. For each vertex v of a bipartite permutation graph, we denote by $cc(v)$ the color class of v and by $\overline{cc}(v)$ the other color class.

5.1 Relationship to bandwidth

As already mentioned, bandwidth and distortion do not always coincide on bipartite permutation graphs, not even on the restricted subclass of complete bipartite graphs. As an example, $\mathrm{bw}(K_{3,4}) = 4$ (two vertices of the second color class are placed first, followed by all three vertices of the first color class, followed by the last two vertices of the second color class) and $\mathrm{D}(K_{3,4}) = 5$. The question arises whether the difference between bandwidth and distortion can be arbitrarily large, like for cycles. We answer this question completely in this subsection. We show that bandwidth gives a 2-approximation for the distortion of bipartite permutation graphs.

As a corollary of a theorem by Fishburn et al. [7], the following normalization result for optimal bandwidth orderings can be obtained. Let $G = (A, B, E)$ be a connected bipartite permutation graph with strong ordering (σ_A, σ_B). We say that a vertex ordering β for G is *normalized* (with respect to (σ_A, σ_B)) if it satisfies the following two conditions:

(C1) for every pair a, a' of vertices from A, $a \prec_{\sigma_A} a'$ implies $a \prec_\beta a'$; and for every pair b, b' of vertices from B, $b \prec_{\sigma_B} b'$ implies $b \prec_\beta b'$

(C2) for every triple u, v, w of vertices of G where $u \prec_\beta v \prec_\beta w$ and $uw \in E$, $uv \in E$ or $vw \in E$.

Condition (C1) requires that β respects the two given orderings. Orderings that respect condition (C2) are called *cocomparability orderings*; hence, condition (C2) requires β to be a cocomparability ordering for G.

Theorem 5.2 ([9]) *Let $G = (A, B, E)$ be a connected bipartite permutation graph, and let $k \geq 0$ be an integer. Let (σ_A, σ_B) be a strong ordering for G. If G has a k-ordering then G has a k-ordering that is normalized with respect to (σ_A, σ_B).*

Theorem 5.3 *Let $G = (A, B, E)$ be a connected bipartite permutation graph. Then,* $\mathrm{D}(G) \leq 2 \cdot \mathrm{bw}(G) - 1$.

Proof. Let (σ_A, σ_B) be a strong ordering for G. Let β be an optimal bandwidth ordering for G. By Theorem 5.2 we can assume that β is normalized with respect to (σ_A, σ_B). Let $\mathcal{E}$ be the non-contractive embedding for G without unnecessary empty slots and underlying vertex ordering β. We determine $\mathrm{D}(G, \mathcal{E})$ by showing for every pair u, v of adjacent vertices of G that

$d_{\mathcal{E}}(u,v) \le 2 \cdot d_\beta(u,v) - 1$. We prove this claim by induction over the distances in β between adjacent vertices. Let u, v be adjacent vertices such that $d_\beta(u,v) = 1$; then $d_{\mathcal{E}}(u,v) = 1$. Suppose that the claim holds for each pair of adjacent vertices at distance at most s in β. Let u, v be a pair of adjacent vertices such that $u \prec_\beta v$ and $d_\beta(u,v) = s+1$. From condition (C2), it follows for all pairs of vertices between u and v that vertices of the same color class are at distance 2 in G and vertices from different color classes are at distance 1 or 3 in G. We distinguish two cases. First, let there be no pair of consecutive vertices between u and v in $\mathcal{E}$ at distance 3. Then, consecutive vertices between u and v are at distance at most 2 in $\mathcal{E}$. Furthermore, since u and v are from different color classes, there are consecutive vertices between u and v that are adjacent and thus at distance 1 in $\mathcal{E}$. Hence, $d_{\mathcal{E}}(u,v) \le 2 \cdot d_\beta(u,v) - 1$. For the other case, let x, y be consecutive vertices between u and v such that $x \prec_\beta y$ and $d_{\mathcal{E}}(x,y) = 3$. Note that $x \ne u$ and $y \ne v$ and x and y are from different color classes. If $cc(u) = cc(x)$ then $xy \in E$ by condition (C2), so that $cc(u) = cc(y) = \overline{cc}(v) = \overline{cc}(x)$. By condition (C2), $ux, vy \in E$, and since $d_\beta(u,x) \le d_\beta(u,v) - 2$ and $d_\beta(y,v) \le d_\beta(u,v) - 2$, we know $d_{\mathcal{E}}(u,x) \le 2 \cdot d_\beta(u,x) - 1$ and $d_{\mathcal{E}}(y,v) \le 2 \cdot d_\beta(y,v) - 1$ by induction hypothesis. Consequently,

$$\begin{aligned} d_{\mathcal{E}}(u,v) = d_{\mathcal{E}}(u,x) + 3 + d_{\mathcal{E}}(y,v) &\le 2 \cdot d_\beta(u,x) - 1 + 3 + 2 \cdot d_\beta(y,v) - 1 \\ &= 2 \cdot (d_\beta(u,v) - 1) + 1 \\ &= 2 \cdot d_\beta(u,v) - 1 \,. \end{aligned}$$

Hence, $D(G) \le D(G, \mathcal{E}) \le 2 \cdot \mathrm{bw}(G) - 1$. ∎

The bandwidth upper bound on the distortion of connected bipartite permutation graphs in Theorem 5.3 is tight. The star graphs $K_{1,m}$, for $m \ge 2$ and m even, have bandwidth $\frac{m}{2}$, and an optimal bandwidth ordering is obtained by placing the center vertex in the middle of the ordering; whereas by Theorem 3.10, they have distortion $m - 1$. Note that the bandwidth of bipartite permutation graphs can be computed in polynomial time [9].

Now we turn to the main result of this section. We give an algorithm for computing the distortion of bipartite permutation graphs. This algorithm works in a vertex incremental way; it computes minimum distortion embeddings for a sequence of induced subgraphs of the given graph, where each subgraph is obtained from the previous one by adding a new vertex. Although bandwidth and distortion are related parameters, our approach for computing the distortion is completely different from existing bandwidth algorithms. Most (non-trivial) bandwidth algorithms decide for a given graph G and an integer c, whether G has a c-ordering. The bandwidth is at the end obtained by binary search on the possible values of c. As opposed to this approach, our algorithm computes the distortion directly. It is also embedding-based, which means that in every step a minimum distortion embedding is computed. When a new vertex is added, a given embedding for the smaller graph is modified without increasing the distortion if possible, or a subgraph is found that requires larger distortion.

5.2 Lower bound on the distortion of bipartite permutation graphs

We begin by defining and analyzing a special kind of bipartite permutation graphs that we will need for proving lower distortion bounds.

A *clawpath* is a tree such that the set of its vertices that are not leaves induces a path, and each vertex of the path is adjacent to exactly one leaf. Hence, every vertex of the path has degree 3 except the end vertices of the path that have degree 2. The number of edges on this path is called the *length* of the clawpath. Note that the smallest clawpath is $K_{1,1}$, of length 0, and one of the two vertices is chosen to form the path. (Clawpaths are caterpillars where every vertex that is not a leaf has exactly one neighbor that is a leaf.)

Definition 5.4 *We define a* clawpath-like graph *to be a graph obtained from a clawpath by replacing each vertex by a (non-empty) independent set of new vertices.*

When replacing a vertex v with a set of new vertices $v_1, \ldots, v_\ell$ with $\ell \geq 1$, we give each v_i the same neighborhood as v had. Thus we can view this process as iteratively adding to the graph new false twins of chosen vertices. The *underlying clawpath* of a clawpath-like graph is the clawpath from which the graph was obtained according to Definition 5.4. The *length* of a clawpath-like graph is the length of its underlying clawpath.

Clawpath-like graphs are both bipartite and permutation. Hence, they form a subclass of bipartite permutation graphs. Furthermore, they are connected and contain at least one edge. Following Definition 5.4, any clawpath-like graph of length r can be represented by a pair $(x_0, \ldots, x_r)$ and $((C_0, D_0), \ldots, (C_r, D_r))$, where $x_0, \ldots, x_r$ are the path vertices of the underlying clawpath, C_i is the set of vertices path vertex x_i was replaced with, and D_i is the set of vertices the single leaf neighbor of x_i was replaced with. It is in fact sufficient to specify only $((C_0, D_0), \ldots, (C_r, D_r))$, which we will call the *sequence representation*. Thus, every clawpath-like graph has a sequence representation.

Lemma 5.5 *Let G be a bipartite permutation graph. Every induced subgraph of G that is a clawpath-like graph is distance-preserving.*

Proof. We obtain the result in several steps. Let $H = (A, B, E)$ be an induced subgraph of G that is clawpath-like. Let $((C_0, D_0), \ldots, (C_r, D_r))$ be a sequence representation for H. If $r = 0$ then H is a complete bipartite graph and clearly a distance-preserving subgraph of G. So, let $r \geq 1$. According to Lemma 3.6, it suffices to consider the case when $C_0, D_0, \ldots, C_r, D_r$ all contain exactly one vertex each, i.e., we can restrict to clawpaths. For ease of notation, we denote these vertices as $c_0, d_0, \ldots, c_r, d_r$. Note that $c_0, c_1, \ldots, c_r$ correspond to the path vertices. Let (σ_A, σ_B) be a strong ordering for G. Let σ be the union of σ_A and σ_B, so that we do not have to distinguish between color classes. Without loss of generality, we can assume that $d_0 \prec_\sigma c_1$; otherwise use $(\sigma_A, \sigma_B)^R$ as strong ordering.

As an auxiliary result note the following: let (u_1, u_2, u_3, u_4) be an induced path in G. Then, $cc(u_1) = \overline{cc}(u_2) = cc(u_3) = \overline{cc}(u_4)$, and $u_1 \prec_\sigma u_3$ if and only if $u_2 \prec_\sigma u_4$ by the properties of strong orderings. Since (d_0, c_0, c_1, d_1) is an induced path in G, we obtain $c_0 \prec_\sigma d_1$. Assume for $i \in \{1, \ldots, r-1\}$ that we have already shown $c_0 \prec_\sigma d_1 \prec_\sigma \cdots \prec_\sigma e_i$ and $d_0 \prec_\sigma c_1 \prec_\sigma \cdots \prec_\sigma f_i$, where $e_i, f_i \in \{c_i, d_i\}$ appropriately. Note that $(d_{i-1}, c_{i-1}, c_i, c_{i+1})$, $(c_{i-1}, c_i, c_{i+1}, d_{i+1})$ and $(d_i, c_i, c_{i+1}, d_{i+1})$ are induced paths in G. Applying the auxiliary result to the paths and the assumption $d_{i-1} \prec_\sigma c_i$, we obtain $c_{i-1} \prec_\sigma c_{i+1}$, $c_i \prec_\sigma d_{i+1}$ and $d_i \prec_\sigma c_{i+1}$. Hence, c- and d-vertices are ordered by index in the two colour classes.

We show by induction that H is a distance-preserving subgraph of G. Since H is an induced subgraph of G, pairs of vertices at distance 1 in G are at distance 1 in H. Now, let $s \geq 2$ and assume that $\mathrm{d}_H(x,y) = \mathrm{d}_G(x,y)$ for all pairs x, y of vertices of H where $\mathrm{d}_G(x,y) \leq s-1$. Let x, y be a pair of vertices of H such that $\mathrm{d}_G(x,y) = s$. Let $P = (u_0, \ldots, u_s)$ be an x,y-path in G of length s. Without loss of generality, we can assume that $x \prec_\sigma y$ or $x \prec_\sigma u_{s-1}$ depending on whether x and y belong to the same color class or to different color classes. By iterative application of the auxiliary result above we obtain that $u_0 \prec_\sigma u_2 \prec_\sigma u_4 \prec_\sigma \cdots$ and $u_1 \prec_\sigma u_3 \prec_\sigma \cdots$. Let $x \in \{c_j, d_j\}$ for some $0 \leq j \leq r$. Observe that the x,y-path in H contains c_{j+1} and that the index of y in H is at least $j+1$. We distinguish two cases. For the first case, let $x = d_j$. If $y = d_{j+1}$ then $\mathrm{d}_H(x,y) = 3$, and $\mathrm{d}_G(x,y) = 3$ since x and y belong to different colour classes and are not adjcent. Now, let $y \neq d_{j+1}$. If $d_{j+1}u_2 \in E$ then $(d_{j+1}, u_2, u_3, \ldots, u_s)$ is a d_{j+1},y-path in G of length at most $s-1$. We apply the induction hypothesis and obtain $\mathrm{d}_H(d_{j+1},y) = \mathrm{d}_G(d_{j+1},y)$. Since $\mathrm{d}_H(d_{j+1},y) = \mathrm{d}_H(d_j,y) - 1$, we conclude that $\mathrm{d}_H(d_j,y) = s$. Now, let $d_{j+1}u_2 \notin E$. Since $c_{j+1}d_{j+1} \in E$, $u_2 \neq c_{j+1}$. It holds that $u_1 \prec_\sigma d_{j+1}$, since otherwise: $u_1 = d_{j+1}$ means that $u_0 = d_j$ and d_{j+1} are adjacent, $d_{j+1} \prec_\sigma u_1$ means that $u_0 d_{j+1} \in E$ because of $d_j \prec_\sigma c_{j+1}$ and $c_{j+1}d_{j+1} \in E$ and the properties of strong orderings. Since $c_{j+1} \prec_\sigma u_2$ then implies $d_{j+1}u_2 \in E$, we conclude that $x \prec_\sigma u_2 \prec_\sigma c_{j+1}$. Independent of whether $u_1 \prec_\sigma c_j$, $u_1 = c_j$ or $c_j \prec_\sigma u_1$, $u_2 c_j \in E$. Then, $(c_j, u_2, \ldots, u_s)$ is a c_j,y-path in G of length at most $s-1$. We apply the induction hypothesis and conclude with $\mathrm{d}_H(c_j,y) = \mathrm{d}_H(d_j,y) - 1$ that $\mathrm{d}_H(d_j,y) = s$. This completes the first case.

For the second case, let $x = c_j$. If $y = d_{j+1}$ then x and y have a common neighbor, c_{j+1}, in H and G and thus $\mathrm{d}_H(x,y) = \mathrm{d}_G(x,y) = 2$. Let $y \neq d_{j+1}$. If $u_1 \prec_\sigma c_{j+1}$ then $u_2 c_{j+1} \in E$ because of $x \prec_\sigma u_2$. Then, $(c_{j+1}, u_2, u_3, \ldots, u_s)$ is a c_{j+1},y-path in G of length at most $s-1$. Induction hypothesis and $\mathrm{d}_H(c_{j+1},y) = \mathrm{d}_H(c_j,y) - 1$ yield $\mathrm{d}_H(c_j,y) = s$. Finally, let $c_{j+1} \prec_\sigma u_1$ or $c_{j+1} = u_1$. Then, u_1 and d_{j+1} are adjacent (remember that $c_j \prec_\sigma d_{j+1}$), and $(d_{j+1}, u_1, \ldots, u_s)$ is a d_{j+1},y-path in G of length s. Thus, $\mathrm{d}_G(d_{j+1},y) \leq s$, and by induction hypothesis and the first case, we obtain $\mathrm{d}_H(d_{j+1},y) = \mathrm{d}_G(d_{j+1},y) \leq s$. Since $\mathrm{d}_H(d_{j+1},y) = \mathrm{d}_H(c_j,y)$, we conclude $\mathrm{d}_H(c_j,y) = s$. This completes the second case. ∎

Lemma 5.6 *Let $G = (V,E)$ be a clawpath-like graph of length r. Let $k \geq 1$ be an odd integer. If $|V| \geq \frac{1}{2}(rk + r + 2k + 6)$, then $\mathrm{D}(G) \geq k+2$.*

Proof. We show the lemma by induction over the length of the sequence representation. First, let G be a clawpath-like graph of length $r = 0$. Then, G is a complete bipartite graph. If G has at least $\frac{1}{2}(2k+6) = k+3$ vertices, which is an even number, we obtain $\mathrm{D}(G) \geq k+2$ by applying Theorem 3.10. Now, let $r \geq 1$, and assume that the lemma holds for all clawpath-like graphs of length at most $r-1$. We show the lemma for clawpath-like graphs of length r by induction over the number of vertices in set D_r. Let G be a clawpath-like graph of length r with sequence representation $((C_0, D_0), \ldots, (C_r, D_r))$ and let G have at least $\frac{1}{2}(rk + r + 2k + 6)$ vertices. Let $|D_r| \leq \frac{k+1}{2}$. Then, $G[V \setminus D_r]$ is a clawpath-like graph of length $r-1$ on at least $\frac{1}{2}((r-1)k + (r-1) + 2k + 6)$ vertices. By induction hypothesis, $\mathrm{D}(G[V \setminus D_r]) \geq k+2$. Since $G[V \setminus D_r]$ is a distance-preserving subgraph of G due to Lemma 5.5, we obtain $\mathrm{D}(G) \geq k+2$ by Lemma 3.5.

Let n_b be a number such that $n_b \geq \frac{k+1}{2}$. Assume that the lemma holds for all clawpath-like graphs of length r with at most n_b vertices in their D_r-set. Let G be a clawpath-like graph of length r on at least $\frac{1}{2}(rk+r+2k+6)$ vertices with sequence representation $((C_0, D_0), \ldots, (C_r, D_r))$ where $|D_r| = n_b + 1$. We determine $\mathrm{D}(G)$. Let $\mathcal{E}$ be a minimum distortion embedding for G. We say that a vertex x from D_r has the *compact property* in $\mathcal{E}$ if the close vertex to the left and right of x are both from $D_r \cup C_r$. We distinguish two cases.

Case A
Let all vertices from D_r between vertices from $C_r \cup D_{r-1}$ have the compact property. Let d and d' denote the respectively leftmost and rightmost vertex from D_r in $\mathcal{E}$, and let c and c' denote the respectively leftmost and rightmost vertex from $C_r \cup D_{r-1}$ in $\mathcal{E}$. Since the vertices in D_r are pairwise non-adjacent, $\mathrm{d}_{\mathcal{E}}(d, d') \geq k + 1$. If a vertex from C_r is to the left of d or to the right of d' in $\mathcal{E}$, we obtain $\mathrm{D}(G, \mathcal{E}) = \mathrm{D}(G) \geq k + 2$. Now, let no vertex from C_r be to the left of d or to the right of d', i.e., all vertices from C_r are between d and d' in $\mathcal{E}$. If there is a vertex from D_{r-1} to the left of d then d does not have the compact property in contradiction to the assumption of the case. Hence, $d \prec_\sigma c$. With symmetric arguments, $c' \prec_{\mathcal{E}} d'$. Denote by a and a' the respectively leftmost and rightmost vertex from C_r in $\mathcal{E}$. If $\mathrm{d}_{\mathcal{E}}(d, a') \geq k + 2$ or $\mathrm{d}_{\mathcal{E}}(a, d') \geq k + 2$ then $\mathrm{D}(G) \geq k + 2$. Now, let $\mathrm{d}_{\mathcal{E}}(d, a') \leq k + 1$ and $\mathrm{d}_{\mathcal{E}}(a, d') \leq k + 1$. We determine the number of vertices in $D_r \cup C_r \cup D_{r-1}$. For a pair u, v of consecutive vertices from $D_r \cup C_r \cup D_{r-1}$ in $\mathcal{E}$, note the following:

- if one of u, v is from D_r and the other from C_r then $\mathrm{d}_{\mathcal{E}}(u, v) \geq 1$
- if one of u, v is from D_r and the other from D_{r-1} then $\mathrm{d}_{\mathcal{E}}(u, v) \geq 3$
- in all other cases, $\mathrm{d}_{\mathcal{E}}(u, v) \geq 2$.

From the assumptions, it follows that

$$\begin{aligned}
\mathrm{d}_{\mathcal{E}}(d, d') &= \mathrm{d}_{\mathcal{E}}(d, a') + \mathrm{d}_{\mathcal{E}}(a, d') - \mathrm{d}_{\mathcal{E}}(a, a') \leq 2k + 2 - \mathrm{d}_{\mathcal{E}}(a, a') \\
\mathrm{d}_{\mathcal{E}}(d, d') &= \mathrm{d}_{\mathcal{E}}(d, c) + \mathrm{d}_{\mathcal{E}}(c, c') + \mathrm{d}_{\mathcal{E}}(c', d'),
\end{aligned}$$

which gives

$$\frac{\mathrm{d}_{\mathcal{E}}(d, c) + \mathrm{d}_{\mathcal{E}}(c, c') + \mathrm{d}_{\mathcal{E}}(c', d') + \mathrm{d}_{\mathcal{E}}(a, a')}{2} \leq k + 1 .$$

It follows from the compact property that all vertices from D_r that are between c and c' are between a and a'. Let p be the number of vertices from D_r that are between c and c'. If $p < |C_r|$ then $\mathrm{d}_{\mathcal{E}}(a, a') \geq 2|C_r| - 2$, if $p \geq |C_r|$ then $\mathrm{d}_{\mathcal{E}}(a, a') \geq 2|C_r| - 2 + 2(p - |C_r| + 1) = 2p$. If there is a vertex from D_{r-1} between a and a', both lower bounds increase by 2 since vertices from D_{r-1} are non-adjacent to vertices from D_r as well as C_r. We distinguish two cases with respect to c, c'. First, let $c \in D_{r-1}$ or $c' \in D_{r-1}$. Then, $\mathrm{d}_{\mathcal{E}}(d, c) + \mathrm{d}_{\mathcal{E}}(c', d') \geq 2(n_b + 1 - p)$, and $\mathrm{d}_{\mathcal{E}}(c, c') \geq 2|C_r \cup D_{r-1}| - 2$ or $\mathrm{d}_{\mathcal{E}}(c, c') \geq 2|C_r \cup D_{r-1}| - 2 + 2(p - |C_r| + 1)$. So, for two cases, we obtain with the above inequality:

- if $p \leq |C_r| - 2$ then
  $k + 1 \geq n_b + 1 - p + |C_r| - 1 + |C_r \cup D_{r-1}| - 1$, i.e.,
  $k + 1 \geq n_b + |C_r| - p - 1 + |C_r \cup D_{r-1}| \geq n_b + 1 + |C_r \cup D_{r-1}| = |D_r \cup C_r \cup D_{r-1}|$

– if $p \geq |C_r|$ then
$k+1 \geq n_b + 1 - p + p + |C_r \cup D_{r-1}| - 1 + p - |C_r| + 1$, i.e.,
$k+1 \geq n_b + p - |C_r| + 1 + |C_r \cup D_{r-1}| \geq n_b + 1 + |C_r \cup D_{r-1}| = |D_r \cup C_r \cup D_{r-1}|$.

The case when $p = |C_r| - 1$ requires a more careful analysis. If there is a vertex between a and a' that is not from $D_r \cup C_r$ then there is also an empty slot between a and a' (because of $p = |C_r| - 1$). Thus, $\mathrm{d}_\mathcal{E}(a, a') \geq 2|C_r|$, and we can conclude $|D_r \cup C_r \cup D_{r-1}| \leq k+1$ similar to the case $p \leq |C_r| - 2$ above. If $c, c' \in D_{r-1}$ then $\mathrm{d}_\mathcal{E}(d, c) + \mathrm{d}_\mathcal{E}(c', d') \geq 2(n_b + 1 - p) + 2$, and again we obtain $|D_r \cup C_r \cup D_{r-1}| \leq k+1$ similar to the cases above. Now, let there be only vertices from $D_r \cup C_r$ between a and a' and assume that $c \in C_r$. Note that this particularly means $a = c$. We determine the cardinality of $D_r \cup C_r \cup D_{r-1}$ by partitioning the set into the vertices from D_r to the left of a' and the other vertices. All vertices from $C_r \cup D_{r-1}$ are from a on to the right in $\mathcal{E}$. From $\mathrm{d}_\mathcal{E}(d, a') \leq k+1$, it follows that there are at most $\frac{k+1}{2}$ vertices from D_r to the left of a'. For the other set, observe that there are two slots between c' and the first vertex from D_r to the right of c'. So, the number of vertices in the second set is at most $\lfloor \frac{k+2}{2} \rfloor = \frac{k+1}{2}$. Hence, $|D_r \cup C_r \cup D_{r-1}| \leq k+1$. The case when $c' \in C_r$ is symmetric.

Now, let $c, c' \in C_r$, i.e., $c = a$ and $c' = a'$. Then, $\mathrm{d}_\mathcal{E}(a, a') = \mathrm{d}_\mathcal{E}(c, c')$ and $\mathrm{d}_\mathcal{E}(d, c) + \mathrm{d}_\mathcal{E}(c', d') \geq 2(n_b + 1 - p) - 2$. We analyze analogous to the cases above:

– if $p \leq |C_r| - 2$ then
$k+1 \geq n_b + 1 - p - 1 + 2(|C_r \cup D_{r-1}| - 1)$, i.e.,
$k+1 \geq n_b + |C_r| - p + |C_r \cup D_{r-1}| - 1 \geq n_b + 1 + |C_r \cup D_{r-1}| = |D_r \cup C_r \cup D_{r-1}|$

– if $p = |C_r| - 1$ then $\mathrm{d}_\mathcal{E}(a, a') \geq 2(|C_r \cup D_{r-1}| - 1) + 1$ since a vertex from D_r between a and a' has an empty slot to its left or right. So,
$k+1 \geq n_b + 1 - p - 1 + 2(|C_r \cup D_{r-1}| - 1) + 1$, i.e.,
$k+1 \geq n_b + |C_r| - p + |C_r \cup D_{r-1}| \geq n_b + 1 + |C_r \cup D_{r-1}| = |D_r \cup C_r \cup D_{r-1}|$

– if $p \geq |C_r|$ then
$k+1 \geq n_b + 1 - p - 1 + 2(|C_r \cup D_{r-1}| - 1 + p - |C_r| + 1)$, i.e.,
$k+1 \geq n_b + p - |C_r| + |D_{r-1}| + |C_r \cup D_{r-1}| \geq n_b + 1 + |C_r \cup D_{r-1}| = |D_r \cup C_r \cup D_{r-1}|$.

We have shown that there are at most $k+1$ vertices in $D_r \cup C_r \cup D_{r-1}$. If $r \geq 2$ then $G[V \setminus (D_r \cup C_r \cup D_{r-1})]$ is a clawpath-like graph of length $r-2$ on at least $\frac{1}{2}((r-2)k + (r-2) + 2k + 6) = \frac{1}{2}(rk + r + 2k + 6) - (k+1)$ vertices. Applying the induction hypothesis, $\mathrm{D}(G[V \setminus (D_r \cup C_r \cup D_{r-1})]) \geq k+2$, which gives $\mathrm{D}(G) \geq k+2$ due to Lemmata 5.5 and 3.5. Let $r = 1$. Since $|D_r \cup C_r \cup D_{r-1}| \leq k+1$ and $|D_r| \geq \frac{k+1}{2}$, $|D_{r-1}| \leq \frac{k+1}{2}$, and $G[V \setminus D_{r-1}]$ is a complete bipartite graph on at least $k+3$ vertices. Applying Theorem 3.10 and Lemmata 5.5 and 3.5, we obtain $\mathrm{D}(G) \geq k+2$.

Case B

Let there be a vertex x from D_r that is between vertices from $C_r \cup D_{r-1}$ and does not have the compact property. We will construct a new graph from G with fewer vertices in D_r and without increasing the distortion. Denote by c and c' the respectively leftmost and rightmost vertex from $C_r \cup D_{r-1}$ in $\mathcal{E}$. Note that c is to the left of x and c' is to the right of x in $\mathcal{E}$. Let w be the close vertex to the left or to the right of x in $\mathcal{E}$ such that $w \notin D_r \cup C_r$. Let y be a vertex from D_{r-1}.

Observe that $d_{\mathcal{E}}(w,x) \geq d_G(w,y)+1$, since every shortest w,x-path in G contains a vertex from C_{r-1}, that is at distance 2 from x in G and at distance 1 from y. For $z \in D_r \cup C_r$, $z \neq x$, it holds that $d_G(x,z) = d_{\mathcal{E}}(y,z) - 1$. We obtain graph H from G by deleting x as a D_r-vertex and making it a D_{r-1}-vertex. This particularly means that the sequence representation of H is the following: $(C_0, D_0), \ldots, (C_{r-2}, D_{r-2}), (C_{r-1}, D_{r-1} \cup \{x\}), (C_r, D_r \setminus \{x\})$. Thus, H is a clawpath-like graph with n_b vertices in its D_r-set. We define an embedding $\mathcal{F}$ as follows: take $\mathcal{E}$ and move x one position towards w. By the discussion above, $\mathcal{F}$ is a non-contractive embedding for H. We determine $D(H,\mathcal{F})$. For u,v vertices of H where $u,v \neq x$, $d_{\mathcal{F}}(u,v) = d_{\mathcal{E}}(u,v)$. For $u \in N_H(x) = C_{r-1}$, if u is to the left of x in $\mathcal{E}$ (and $\mathcal{F}$) then $d_{\mathcal{F}}(u,x) < d_{\mathcal{E}}(u,c')$, if u is to the right of x then $d_{\mathcal{F}}(x,u) < d_{\mathcal{E}}(c,u)$. Hence, $D(H,\mathcal{F}) \leq D(G,\mathcal{E})$. Applying the induction hypothesis, $k+2 \leq D(H) \leq D(H,\mathcal{F})$, which gives $D(G) \geq k+2$ by the choice of $\mathcal{E}$. This completes the proof. ∎

Corollary 5.7 *Let $G = (A,B,E)$ be a connected bipartite permutation graph, and let H be an induced subgraph of G that is clawpath-like of length $r \geq 0$. Let $k \geq 1$. If H contains at least $\frac{1}{2}(rk + r + 2k + 6)$ vertices, then $D(G) \geq k+2$.*

Proof. The result directly follows from Lemmas 5.6, 5.5 and 3.5. ∎

5.3 Upper bound on the distortion of bipartite permutation graphs

The goal of this subsection is to give a fast algorithm for computing the distortion of bipartite permutation graphs and to complete the result of Corollary 5.7 by showing the converse. Both results are obtained simultaneously. The algorithm itself will be simple. It takes as input an embedding and tries to modify it by local vertex replacement. We start by formalising these operations.

Let $G = (A,B,E)$ be a bipartite permutation graph with strong ordering (σ_A, σ_B). Let a be the leftmost A-vertex in σ_A. An embedding $\mathcal{E}$ for G is called *normalized with respect to* (σ_A, σ_B) if it satisfies the following two conditions:

(D1) $\mathrm{ord}(\mathcal{E})$ is normalized with respect to (σ_A, σ_B)

(D2) for every A-vertex x, $d_{\mathcal{E}}(a,x)$ is even; and for every B-vertex x, $d_{\mathcal{E}}(a,x)$ is odd.

Thus, in a normalized embedding we can partition the slots (containing vertices or being empty) into "$cc(a)$-slots" and "$\overline{cc}(a)$-slots": only $cc(a)$-slots can contain A-vertices, and only $\overline{cc}(a)$-slots can contain B-vertices. It is a simple though important observation that $\mathcal{E}$ is normalized with respect to (σ_A, σ_B) if and only if the reverse of $\mathcal{E}$ is normalized with respect to $(\sigma_A, \sigma_B)^R$, which denotes the pair of reverse orderings. We will show that every connected bipartite permutation graph has a minimum distortion embedding that is normalized with respect to a given strong ordering. Thus, a result analogous to Theorem 5.2 also holds for distortion embeddings.

Our algorithm is based on one single type of operations in embeddings: moving vertices. Vertex moving will appear in three different forms, depending on which vertices are moved into which direction. The three operations are denoted as `RightMove`, `LeftMove` and `DeleteTwo`. The latter operation, `DeleteTwo`, receives an embedding $\mathcal{E}$ and a position p as input and "deletes" the slots at position p and $p+1$ in $\mathcal{E}$, by moving all vertices to the right of position p two positions to the left. Note that the result is a proper embedding if the slots at position p and $p+1$ are empty. When we apply `DeleteTwo`, these two positions are empty.

We give the definition of operation `RightMove` as a small program. For the definition, we introduce the following notation. For an embedding $\mathcal{E}$, a vertex u and a position p, $\mathcal{E} - u$ denotes the embedding obtained from $\mathcal{E}$ by removing u (which leaves an empty slot), and $\mathcal{E} + (u \to p)$ is the embedding obtained from $\mathcal{E}$ placing vertex u in the slot at position p (to obtain a proper embedding, we assume that u is not placed in $\mathcal{E}$ and that the slot at position p in $\mathcal{E}$ is empty). Operation `RightMove` mainly executes a right-shift for vertices of a single color class (if the input embedding is normalized for a bipartite permutation graph). It receives an embedding $\mathcal{E}$ and a vertex u as input and is defined as

Procedure `RightMove`
 begin
  let $p = \mathcal{E}(u) + 2$; **set** $\mathcal{E} = \mathcal{E} - u$;
  while position p in $\mathcal{E}$ is occupied **do**
   let x be vertex at position p in $\mathcal{E}$; **set** $\mathcal{E} = (\mathcal{E} - x) + (u \to p)$; **set** $u = x$; **set** $p = p + 2$
  end while;
  return $\mathcal{E} + (u \to p)$
 end.

Finally, operation `LeftMove` can be considered the counterpart of `RightMove`. It receives an embedding $\mathcal{E}$ and a vertex u as input. The result is the reverse of the result obtained from applying `RightMove` to the reverse of $\mathcal{E}$ and u. The following lemma shows that the three operations are compatible with the notion of normalized embedding.

Lemma 5.8 *Let $G = (A, B, E)$ be a connected bipartite permutation graph, and let $\mathcal{E}$ be a normalized non-contractive embedding for G.*

1. *Let u be a vertex that has a neighbor to its right in $\mathcal{E}$. Let v be the rightmost neighbor of u. Let there be an empty $cc(u)$-slot between u and v in $\mathcal{E}$.*
   Then, `RightMove`$(\mathcal{E}, u)$ *is a normalized non-contractive embedding for G.*
2. *Let v be a vertex that has a neighbor to its left in $\mathcal{E}$. Let u be the leftmost neighbor of v. Let there be an empty $cc(v)$-slot between u and v in $\mathcal{E}$.*
   Then, `LeftMove`$(\mathcal{E}, v)$ *is a normalized non-contractive embedding for G.*
3. *Let u be a vertex such that all $\overline{cc}(u)$-vertices to the right in $\mathcal{E}$ are adjacent to u.*
   Then, `RightMove`$(\mathcal{E}, u)$ *is a normalized non-contractive embedding for G.*
4. *Let the slots at position p and $p+1$ in $\mathcal{E}$ be empty.*
   Then, `DeleteTwo`$(\mathcal{E}, p)$ *is a normalized embedding.*

Proof. First note that the result in all cases is a proper embedding, meaning that every slot is occupied by at most one vertex. Furthermore, vertices that change position move exactly two positions, so that the distance between any pair of vertices from the same color class is even and between any pair of vertices from different color classes is odd. The correctness of statement 4 then is immediate, since vertices are not deleted and the vertex ordering underlying the resulting embedding is equal to $\mathrm{ord}(\mathcal{E})$. For statements 1, 2, 3, the vertex ordering underlying the resulting embedding satisfies condition (C1), since vertices of the same color class do not change order. We show that also condition (C2) is satisfied. Let $\mathcal{F} =_{\mathrm{def}}$ `RightMove`$(\mathcal{E}, u)$. Let a, b, c be three vertices of G where $a \prec_{\mathcal{F}} b \prec_{\mathcal{F}} c$, and let $ac \in E$. If $a \prec_{\mathcal{E}} b \prec_{\mathcal{E}} c$ then $ab \in E$ or $bc \in E$, since $\mathrm{ord}(\mathcal{E})$ satisfies condition (C2). Otherwise, $b \prec_{\mathcal{E}} a \prec_{\mathcal{E}} c$ or $a \prec_{\mathcal{E}} c \prec_{\mathcal{E}} b$. (Note that every vertex moves at most two position for the construction of $\mathcal{F}$, which means it can change relative ordering with at most one vertex.) In the former case, b is a $cc(u)$-vertex, and $u \prec_{\mathcal{E}} a \prec_{\mathcal{E}} v$, and a is a $\overline{cc}(u)$-vertex. Hence, $ua \in E$. And since $u = b$ or $u \prec_{\mathcal{E}} b \prec_{\mathcal{E}} a$, $ba \in E$. In the latter case, $u \prec_{\mathcal{E}} b \prec_{\mathcal{E}} v$ and $ub \in E$, and so $bc \in E$. Hence, $\mathcal{F}$ is normalized. For the non-contractiveness condition, let w be a $\overline{cc}(u)$-vertex between u and v in $\mathcal{E}$. Then, $uw \in E$ by condition (C2), thus w is adjacent to all $cc(u)$-vertices between u and w in $\mathcal{E}$. Hence, the close $cc(u)$-vertex to the left of w in $\mathcal{F}$ is a neighbor, and the close $cc(u)$-vertex x to the right of w in $\mathcal{F}$ is a neighbor or $\mathrm{d}_{\mathcal{F}}(w, x) \geq \mathrm{d}_{\mathcal{E}}(w, x)$. For vertices to the left of u or to the right of v in $\mathcal{E}$, nothing has changed in $\mathcal{F}$. Hence, $\mathcal{F}$ is non-contractive. The correctness of statement 2 immediately follows from the correctness of statement 1.

For statement 3, we distinguish cases with respect to the number of $\overline{cc}(u)$-vertices to the right of u in $\mathcal{E}$. Let $\mathcal{F} =_{\mathrm{def}}$ `RightMove`$(\mathcal{E}, u)$. If there is no $\overline{cc}(u)$-vertex to the right of u in $\mathcal{E}$, then all vertices to the right of u in $\mathcal{E}$ are $cc(u)$-vertices and $\mathrm{ord}(\mathcal{F}) = \mathrm{ord}(\mathcal{E})$, and $\mathcal{F}$ is clearly a normalized non-contractive embedding for G. Let there be exactly one $\overline{cc}(u)$-vertex to the right of u in $\mathcal{E}$, say v, and let $\mathrm{d}_{\mathcal{E}}(u, v) = 1$ and let the slot at position $\mathcal{E}(u) + 2$ in $\mathcal{E}$ be empty. Then, $\mathcal{F}$ differs from $\mathcal{E}$ only in the position of u, and $\mathcal{F}$ is non-contractive. For satisfaction of condition (C2), it suffices to note that adjacent and consecutive vertices exchanged their positions in $\mathrm{ord}(\mathcal{F})$ with respect to $\mathrm{ord}(\mathcal{E})$. Hence, $\mathcal{F}$ is normalized. If there are at least two $\overline{cc}(u)$-vertices to the right of u in $\mathcal{E}$, then $\mathcal{F}$ is the result of at most three consecutive applications of `RightMove` with the following vertices: the $cc(u)$-vertex at distance 1 to the right of the rightmost neighbor of u, then the vertex at distance 1 to the left of the rightmost neighbor of u and finally to u. The last case is captured by statement 1. The cases depend on where the first empty $cc(u)$-slot to the right of u happens to be. ∎

We will always apply the three operations to normalized non-contractive embeddings. Statement 4 of Lemma 5.8 cannot be extended by an unconditional statement about non-contractiveness. However, in all cases when we apply `DeleteTwo`, the two consecutive vertices around the deleted positions never violate the distance condition. Therefore, we assume throughout the subsection that the result of any application of the three operations is a normalized non-contractive embedding, and we will not mention this explicitly again.

To give a first outline, our algorithm for computing the distortion of bipartite permutation graphs iteratively takes a minimum distortion embedding for an connected induced subgraph, add a new vertex to this embedding and determines on that basis the distortion of the extended graph. The new vertex is not an arbitrary vertex, but one with special properties. This process defines a vertex ordering for the given graph, that we formalize in the following. Let $G = (A, B, E)$ be a connected bipartite permutation graph on at least two vertices with strong ordering (σ_A, σ_B). We say that a vertex ordering $\sigma = \langle x_1, \ldots, x_n \rangle$ for G is *competitive* if it has the following properties:

– σ satisfies condition (C1)

– x_1 is the leftmost A-vertex in σ_A, and x_2 is the leftmost B-vertex in σ_B

– for every $i \in \{3, \ldots, n\}$, $N_G(x_i) \cap \{x_1, \ldots, x_{i-1}\} \subseteq N_G(w)$ where w is the $cc(x_i)$-vertex preceding x_i in σ_A or σ_B.

Observe that competitive vertex orderings exist for all connected bipartite graphs and given strong orderings: if the rightmost A-vertex has a neighbor that is not a neighbor of the previous A-vertex then this neighbor has degree 1. Without loss of generality, this neighbor can be chosen as the last B-vertex. And since G is connected the last A-vertex is adjacent to the last two B-vertices, from which follows that all neighbors of the last B-vertex are neighbors of the previous B-vertex. Iteration proves the existence. The following lemma is important for the correctness of the approach of our algorithm. Note that a competitive ordering defines a strong ordering for a connected bipartite permutation graph.

Lemma 5.9 *Let $G = (A, B, E)$ be a connected bipartite permutation graph with competitive ordering $\langle x_1, \ldots, x_n \rangle$. Then, $G[\{x_1, \ldots, x_i\}]$ is connected, for $2 \leq i \leq n$.*

Proof. Suppose the contrary. Let $i \geq 2$ be the smallest value such that $G[\{x_1, \ldots, x_i\}]$ is not connected, which means that x_i has no neighbor among $x_1, \ldots, x_{i-1}$. Note that $i \geq 3$, since x_2 is adjacent to x_1. Let w be the $cc(x_i)$-vertex preceding x_i in σ. Since G is connected, x_i has a leftmost neighbor y in σ, and $x_i \prec_\sigma y$. Let v be the $cc(y)$-vertex preceding y in σ. Since $v \prec_\sigma y$, v is not adjacent to x_i. Then, however, the third condition for competitive orderings is violated by y. Hence, $G[\{x_1, \ldots, x_i\}]$ is connected. ■

We give the first step of our algorithm. We take an induced subgraph and a minimum distortion embedding and extend both by adding a new vertex, which is picked according to a competitive ordering. For a graph $G = (V, E)$, an embedding $\mathcal{E}$ and an integer $k \geq 0$ we say that a vertex x is $(G, \mathcal{E}, k)$-*bad* if x has a neighbor y in G where $y \prec_\mathcal{E} x$ such that $\mathrm{d}_\mathcal{E}(x, y) > k$. In particular, if x is a $(G, \mathcal{E}, k)$-bad vertex then its leftmost neighbor in $\mathcal{E}$ is at distance more than k in $\mathcal{E}$. If the context is clear we will write $(\mathcal{E}, k)$-*bad* or simply k-*bad* vertices.

Lemma 5.10 *Let $G = (A, B, E)$ be a connected bipartite permutation graph on at least three vertices with competitive ordering σ. Let x be the rightmost vertex in σ. Let c be the $cc(x)$-vertex preceding x in σ, and let d be the leftmost neighbor of x in σ. Let $\mathcal{E}$ be a normalized minimum distortion embedding for $G - x$, and let $k =_{\mathrm{def}} \mathrm{D}(G - x, \mathcal{E})$.*

1. *Let $c \prec_\mathcal{E} d$ and $\mathcal{F} =_{\mathrm{def}} \mathcal{E} + (x \to \mathcal{E}(d) + 1)$.*
   Then, $\mathcal{F}$ is a normalized minimum distortion embedding for G of distortion k.

2. *Let $d \prec_\mathcal{E} c$ and $\mathcal{F} =_{\mathrm{def}} \mathcal{E} + (x \to \mathcal{E}(c) + 2)$.*
   Then, $\mathcal{F}$ is a normalized non-contractive embedding for G of distortion k or $k + 2$, and if there is an $(\mathcal{F}, k)$-bad vertex then it is x.

Proof. Note that in either case $\mathcal{F}$ is a normalized embedding: x occupies a $cc(x)$-slot in $\mathcal{F}$ (at odd distance to d or even distance to c), that is empty in $\mathcal{E}$. Hence, $\mathcal{F}$ satisfies condition (D2).

Condition (C1) is satisfied by $\text{ord}(\mathcal{F})$ since x is rightmost among all $cc(x)$-vertices in σ and $\mathcal{F}$. Now, let u, v, w be three vertices of G where $u \prec_{\mathcal{F}} v \prec_{\mathcal{F}} w$. If $u = x$ then $vx \in E$ since $d \prec_{\mathcal{F}} x$ and all $\overline{cc}(x)$-vertices to the right of d are neighbors of x. If $v = x$ then $wx \in E$ by the same argument. If $w = x$ then $vx \in E$ if v is a $\overline{cc}(x)$-vertex and thus to the right of d in $\mathcal{F}$ or $uv \in E$ if v is a $cc(x)$-vertex and equal to c or between c and a neighbor of c. If $u, v, w \neq x$ then $uv \in E$ or $vw \in E$ since $\mathcal{E}$ satisfies condition (C2). Hence, $\mathcal{F}$ satisfies condition (C2), and $\mathcal{F}$ is a normalized embedding for G. For non-contractiveness, note that all vertices to the right of x in $\mathcal{F}$ are neighbors of x and the close vertex to the left of x is a neighbor at distance 1 (cases 1 and 2) or a non-neighbor, namely c, at distance 2 and c and x have a common vertex.

It remains to consider the distortion of $\mathcal{F}$. For the first case, let $c \prec_{\mathcal{E}} d$. By assumption, x has exactly one neighbor to the left, and this neighbor is d, at distance 1. For a neighbor y to the right, y is also a neighbor of c by the properties of competitive orderings. Thus, $\mathrm{d}_{\mathcal{F}}(x, y) \leq \mathrm{d}_{\mathcal{F}}(c, y) \leq k$. Hence, $\mathrm{D}(G, \mathcal{F}) = \max\{1, k\} = k$. Due to Lemma 3.7, $G{-}x$ is a distance-preserving subgraph of G, so that $\mathrm{D}(G{-}u) \leq \mathrm{D}(G)$ due to Lemma 3.5. Hence, $\mathrm{D}(G) = \mathrm{D}(G, \mathcal{F})$ and $\mathcal{F}$ is a minimum distortion embedding for G. For the second case, let $d \prec_{\mathcal{E}} c$. Since d is a neighbor also of c, $\mathrm{d}_{\mathcal{F}}(d, x) = \mathrm{d}_{\mathcal{F}}(d, c) + 2 = \mathrm{d}_{\mathcal{E}}(d, c) + 2 \leq k + 2$. For a neighbor y of x to the right of x, $\mathrm{d}_{\mathcal{F}}(x, y) = \mathrm{d}_{\mathcal{F}}(c, y) - 2$. Hence, $\mathrm{D}(G, \mathcal{F}) = k$ or $\mathrm{D}(G, \mathcal{F}) = k + 2$. Note that $\mathrm{D}(G, \mathcal{F}) \neq k + 1$ since edges join vertices on positions of different parity by condition (D2). And since $\mathcal{E}$ and $\mathcal{F}$ coincide on all vertices that are in $G{-}x$, x can be the only $(\mathcal{F}, k)$-bad vertex. ∎

In the following, we want to solve the question that is raised by the second case of Lemma 5.10, namely we want to decide whether the distortion of the graph in this case is k or $k + 2$. Our main algorithm will do exactly this but requires an input embedding of a special form. The next result shows that this form can be achieved by few modifications or it is easy to decide the distortion question already by looking at a small part of the given embedding. For a connected bipartite permutation graph $G = (A, B, E)$, a number $k \geq 1$ and a normalized non-contractive embedding $\mathcal{E}$ for G, we say that $\mathcal{E}$ has a *nice beginning* if, for b_l and b_r the respectively leftmost and rightmost $(G, \mathcal{E}, k)$-bad vertex in $\mathcal{E}$ and a_r the leftmost neighbor of b_r, all $(G, \mathcal{E}, k)$-bad vertices are $cc(b_r)$-vertices, $\mathrm{d}_{\mathcal{E}}(b_l, b_r) \leq k - 1$, there is no empty $cc(b_r)$-slot between a_r and b_r and there is an empty $\overline{cc}(b_r)$-slot between a_r and b_l in $\mathcal{E}$. Note that b_l is to the right of a_r by the distance conditions.

Lemma 5.11 *Let $G = (A, B, E)$ be a connected bipartite permutation graph on $n \geq 3$ vertices with competitive ordering σ. Let $\mathcal{E}$ be a normalized non-contractive embedding for G of distortion $k+2$, and let there be exactly one $(G, \mathcal{E}, k)$-bad vertex x. Let x be the rightmost $cc(x)$-vertex in σ. Then, one of the following cases holds:*

1. *$\mathrm{D}(G) \leq k$, which is certified by a normalized non-contractive embedding for G*

2. *$\mathrm{D}(G) = k + 2$, which is certified by a normalized non-contractive embedding for G of distortion $k + 2$ and an induced subgraph that is complete bipartite on $k + 3$ vertices*

3. *$\mathrm{D}(G) \leq k + 2$, which is certified by a normalized non-contractive embedding for G of distortion $k + 2$ and with a nice beginning.*

There is an $\mathcal{O}(n)$-time algorithm that identifies a true case and outputs the certificates.

Proof. Let y be the rightmost $\overline{cc}(x)$-vertex in $\mathcal{E}$. If $x \prec_{\mathcal{E}} y$ and there is an empty $\overline{cc}(x)$-slot between x and v, at position p, then $\texttt{DeleteTwo}(\mathcal{E}, p)$ a normalized non-contractive embedding for G that satisfies the assumptions of the lemma. Repeated application deletes all empty $\overline{cc}(x)$-slots between x and y. So, we can assume in the following that there are no empty $\overline{cc}(x)$-slots between x and y in $\mathcal{E}$. Let d be the leftmost neighbor of x in $\mathcal{E}$, and let $\mathcal{F} =_{\text{def}} \texttt{RightMove}(\mathcal{E}, d)$. If there is no $(\mathcal{F}, k)$-bad vertex, then $\mathrm{D}(G) \leq k$, which is certified by normalized non-contractive embedding $\mathcal{F}$. Now, suppose there is an $(\mathcal{F}, k)$-bad vertex. Note that, by definition of $\mathcal{F}$, x is not $(\mathcal{F}, k)$-bad and no other $cc(x)$-vertex is $(\mathcal{F}, k)$-bad. Let w be the rightmost $(\mathcal{F}, k)$-bad vertex in $\mathcal{F}$.

Case A

Let $x \prec_{\mathcal{F}} w$. Since w is not $(\mathcal{E}, k)$-bad, there is no empty $\overline{cc}(x)$-slot between d and w in $\mathcal{E}$, and thus there is no empty $\overline{cc}(x)$-slot between d and y in $\mathcal{E}$. Let c be the leftmost neighbor of w in $\mathcal{F}$. First, let there be an empty $cc(x)$-slot between c and x. Note that, by the choice of w and the definition of c, no $cc(x)$-vertex to the right of c has a right neighbor at distance more than $k-2$ in $\mathcal{E}$. Let $\mathcal{E}' =_{\text{def}} \texttt{LeftMove}(\mathcal{E}, x)$. Since $d \prec_{\mathcal{E}} c \prec_{\mathcal{E}} x$, $\mathcal{E}'$ is normalized and non-contractive, and $\mathrm{D}(G, \mathcal{E}') = k$. Hence, $\mathrm{D}(G) \leq k$.

For the other case, let there not be an empty $cc(x)$-slot between c and x. Denote by C the $cc(x)$-vertices between c and x and denote by D the $\overline{cc}(x)$-vertices between d and w. By the properties of strong orderings, all vertices in C are adjacent to all vertices in D, which means that $G[C \cup D]$ is a complete bipartite graph. We determine the number of vertices in $C \cup D$ based in $\mathcal{E}$. Remember that $\mathrm{d}_{\mathcal{E}}(d, x) = k+2$ and $\mathrm{d}_{\mathcal{E}}(c, w) = k$. From D there are $\frac{k+3}{2}$ vertices between d and x, $\frac{k+1}{2}$ vertices between c and w and $\frac{1}{2}\mathrm{d}_{\mathcal{E}}(c, x)$ vertices between c and x (that have been counted twice), and there are $\frac{1}{2}\mathrm{d}_{\mathcal{E}}(c, x) + 1$ vertices in C. It is important to remember that if w is to the left of x in $\mathcal{E}$ then it is at distance 1. We sum up and obtain:

$$\frac{k+3+k+1+\mathrm{d}_{\mathcal{E}}(c,x)+2-\mathrm{d}_{\mathcal{E}}(c,x)}{2} = \frac{2k+6}{2} = k+3$$

vertices in $C \cup D$. Applying Theorem 3.10, $G[C \cup D]$ has distortion $k+2$. And since H is a distance-preserving subgraph of G due to Lemma 5.5, G has distortion at least $k+2$ according to Lemma 3.5. Since $\mathrm{D}(G) \leq \mathrm{D}(G, \mathcal{E})$, we conclude $\mathrm{D}(G) = k+2$.

Case B

Let $w \prec_{\mathcal{F}} x$. Then, all $(\mathcal{F}, k)$-bad vertices are between d and x, within an interval of length $k-1$ in $\mathcal{F}$. If the slot at position $\mathcal{F}(d)-1$ in $\mathcal{F}$ is not occupied, the two slots at position $\mathcal{F}(d)-2$ and $\mathcal{F}(d)-1$ in $\mathcal{F}$ are not occupied. (Remember that d occupies position $\mathcal{F}(d)-2$ in $\mathcal{E}$.) We obtain a normalized non-contractive embedding $\mathcal{F}'$ for G as $\texttt{DeleteTwo}(\mathcal{F}, \mathcal{F}(d)-2)$. Since all leftmost neighbors of $(\mathcal{F}, k)$-bad vertices are to the left of d in $\mathcal{F}$, $\mathrm{D}(\mathcal{F}') = k$, and thus $\mathrm{D}(G) \leq k$. Now, let the slot at position $\mathcal{F}(d)-1$ in $\mathcal{F}$ be occupied, say by vertex a.

Let there be no empty $cc(x)$-slot between a and x in $\mathcal{E}$. If there is an empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$, then $\mathcal{E}$ is an embedding with a nice beginning. Otherwise, if there is no empty $\overline{cc}(x)$-slot between d and x, let vertex z occupy position $\mathcal{E}(x)-1$ in $\mathcal{E}$. Note that $z \neq d$. According to the properties of $\mathcal{F}$, $\texttt{RightMove}(\mathcal{E}, z)$ is a normalized non-contractive embedding for G of distortion $k+2$ with a nice beginning.

Let there be an empty $cc(x)$-slot between a and x in $\mathcal{E}$. Let v be the leftmost $cc(x)$-vertex such that there is no empty $cc(x)$-slot between v and x in $\mathcal{E}$. Let $\mathcal{G} =_{\text{def}}$ `LeftMove`$(\mathcal{E}, x)$. If there is no $(\mathcal{G}, k)$-bad vertex, then $\mathcal{G}$ is a normalized non-contractive embedding certifying $\text{D}(G) \leq k$. So, let there be a $(\mathcal{G}, k)$-bad vertex. Let u be the leftmost $(\mathcal{G}, k)$-bad vertex in $\mathcal{G}$. Since x is not $(\mathcal{G}, k)$-bad, all $(\mathcal{G}, k)$-bad vertices are $\overline{cc}(x)$-vertices and $x \prec_{\mathcal{G}} u$ and $x \prec_{\mathcal{E}} u$ (the second relationship follows from the fact that $\text{d}_{\mathcal{E}}(a, x) = k + 1$ and $\text{d}_{\mathcal{E}}(v, x) \leq k - 3$) and $\text{d}_{\mathcal{G}}(x, u) \geq 5$. If there is an empty $\overline{cc}(x)$-slot between v and u in $\mathcal{G}$ then `LeftMove`$(\mathcal{G}, y)$ is a normalized non-contractive embedding of distortion k for G. Remember that there is no empty $\overline{cc}(x)$-slot between u and y in $\mathcal{E}$ by the discussion at the beginnging of the proof. If there is no empty $\overline{cc}(x)$-slot between v and u in $\mathcal{G}$ then $\mathcal{G}$ is a normalized non-contractive embedding with a nice beginning, particularly since there is an empty $cc(x)$-slot between x and u in $\mathcal{G}$. ■

Only two more definitions, and we are ready for presenting the central routine of our algorithm. Let $G = (A, B, E)$ be a bipartite permutation graph and let $\mathcal{E}$ be a normalized embedding for G. Let d and x be vertices of G from different color classes where $d \prec_{\mathcal{E}} x$. We call a pair (v, w) of vertices for v a $cc(x)$-vertex and w a $\overline{cc}(x)$-vertex a *blocking pair* if $v \prec_{\mathcal{E}} w$, $\text{d}_{\mathcal{E}}(v, w) = 3$ and $vw \notin E$. We call vertex w for $d \prec_{\mathcal{E}} w \prec_{\mathcal{E}} x$ a *breakpoint vertex* between d and x if (v, w) is a blocking pair for some vertex v, there is no empty $cc(x)$-slot between d and v and no empty $\overline{cc}(x)$-slot between w and x in $\mathcal{E}$. The algorithm then is the following:

Algorithm `bpg-distortion`
Input embedding $\mathcal{E}$ and number k
begin
  while there is an $(\mathcal{E}, k)$-bad vertex **do**
    let x be the rightmost $(\mathcal{E}, k)$-bad vertex in $\mathcal{E}$;
    let d be the leftmost neighbor of x in $\mathcal{E}$;
    if there is no empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$ **then** **reject** **end if**;
    let $\mathcal{F} =$ `RightMove`$(\mathcal{E}, d)$;
    if slot at position $\mathcal{F}(d) - 1$ is not occupied in $\mathcal{F}$ **then** **accept** **end if**;
    if there is no breakpoint vertex between d and x in $\mathcal{F}$ **and**
      there is an empty $cc(x)$-slot between d and x in $\mathcal{F}$ **then** **accept** **end if**;
    set $\mathcal{E} = \mathcal{F}$
  end while;
  accept
end.

The input of the algorithm is a normalized non-contractive embedding of distortion $k + 2$ with a nice beginning. With the results of Lemma 5.8 it is clear that all embeddings during the execution of `bpg-distortion` are normalized non-contractive. If the execution of the **while** loop stops since there is no bad vertex in $\mathcal{E}$, $\mathcal{E}$ has distortion at most k, and the algorithm accepts correctly. In the following, we show that the algorithm always stops with the correct answer, which means that it accepts if the distortion of the input graph is at most k and it rejects if the distortion of the input graph is at least $k + 2$. This correctness proof is partitioned in three lemmata. We begin with properties about the intermediate embeddings. A **while** loop execution is called a *round* of the algorithm.

Lemma 5.12 *Let $G = (A, B, E)$ be a connected bipartite permutation graph with normalized non-contractive embedding $\mathcal{G}$ of distortion $k + 2$ with a nice beginning. Apply* `bpg-distortion` *to $(\mathcal{G}, k)$. Let $\mathcal{E}$, $\mathcal{F}$, c and x have the values according to* `bpg-distortion` *at the end of a round, where we assume that there is an empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$. Denote by x_l and x_r the respectively leftmost and rightmost $(\mathcal{F}, k)$-bad vertex.*

(W1) $\mathrm{D}(G, \mathcal{F}) \leq k + 2$

(W2) *$d \prec_{\mathcal{F}} x_l$ or $d = x_l$, and $x_r \prec_{\mathcal{F}} x$*

(W3) *the slot at position $\mathcal{F}(d) - 2$ in $\mathcal{F}$ is empty*

(W4) *all $(\mathcal{F}, k)$-bad $cc(x_r)$-vertices are to the right of all $(\mathcal{F}, k)$-bad $\overline{cc}(x_r)$-vertices*

(W5) *if there is an empty $cc(x)$-slot between d and x in $\mathcal{F}$ then there is an empty $cc(x)$-slot between d and the leftmost $(\mathcal{F}, k)$-bad $cc(x)$-vertex in $\mathcal{F}$.*

Proof. We prove satisfaction of the conditions by induction over the number of rounds. If the current round is not the first one, we assume that $\mathcal{E}$ satisfies the properties. If the current round is the first round, $\mathcal{E}$ is an embedding with a nice beginning. For the definition of $\mathcal{F}$, the following holds. Let u be the rightmost $\overline{cc}(x)$-vertex such that there is no empty $\overline{cc}(x)$-slot between d and u in $\mathcal{E}$; note that $u \prec_{\mathcal{E}} x$ by the assumption about the existence of an empty slot. Then, the $\overline{cc}(x)$-vertices between d and u are exactly the vertices that change position from $\mathcal{E}$ to $\mathcal{F}$. It follows that all $(\mathcal{F}, k)$-bad $cc(x)$-vertices are $(\mathcal{E}, k)$-bad, since they are not moved and there leftmost neighbors are not moved (the leftmost neighbors are to the left of d). An $(\mathcal{F}, k)$-bad $\overline{cc}(x)$-vertex is $(\mathcal{E}, k)$-bad or is between d and u. We first show that u is at distance at least 3 to the left of the leftmost $(\mathcal{E}, k)$-bad vertex in $\mathcal{E}$. For first-round $\mathcal{E}$, this is clear from the fact that there is an empty $\overline{cc}(x)$-slot between d and the leftmost k-bad vertex by definition of nice beginning. For the current round not being the first round, $\mathcal{E}$ is the result of a `RightMove` operation, applied to some vertex d'. By assumption, $\mathcal{E}$ satisfies condition (W3), so that the slot at position $\mathcal{E}(d') - 2$ in $\mathcal{E}$ is empty. If d' is a $\overline{cc}(x)$-vertex then u is clearly to the left of d' at distance at least 4 and no $(\mathcal{E}, k)$-bad vertex is to the left of d' in $\mathcal{E}$ by condition (W2). Let d' be a $cc(x)$-vertex. We show that there is no empty $\overline{cc}(x)$-slot between d' and x. Let $\mathcal{E}'$ the input embedding to the previous round, and let x' be the rightmost $(\mathcal{E}', k)$-bad vertex. Since $d'x' \in E$ and $dx \in E$, all $\overline{cc}(x)$-vertices between d and x' are adjacent to all $cc(x)$-vertices between d' and x. Hence, there is no breakpoint vertex between d' and x. If there is an empty $\overline{cc}(x)$-slot between d' and x' then the algorithm would have accepted in the previous round. Therefore, there can be no empty $\overline{cc}(x)$-slot between d' and x in $\mathcal{E}$. Thus, all empty $\overline{cc}(x)$-slots between d and x are to the left of d', and u is at distance at least 3 to d' in $\mathcal{E}$.

(W1)

No $\overline{cc}(x)$-vertex between d and u is $(\mathcal{E}, k)$-bad. Hence, no $(\mathcal{F}, k)$-bad vertex has a neighbor at distance more than $k + 2$ in $\mathcal{F}$, which means that $\mathrm{D}(G, \mathcal{F}) \leq k + 2$.

(W2)

Since no vertex to the right of x is moved for defining $\mathcal{F}$ or $(\mathcal{E}, k)$-bad, and since d is leftmost neighbor of x, all left neighbors of x in $\mathcal{F}$ are at distance at most k. Thus, $x_r \prec_{\mathcal{F}} x$. For x_l, remember from the beginning that u is at distance at least 3 to the left of the leftmost $(\mathcal{E}, k)$-bad vertex in $\mathcal{E}$, so that no $(\mathcal{F}, k)$-bad vertex is to the left of d in $\mathcal{F}$.

(W3)
This is immediately clear from the fact that d is the leftmost moved vertex.

(W4)
Vertices that are $(\mathcal{F}, k)$-bad but not $(\mathcal{E}, k)$-bad are $\overline{cc}(x)$-vertices between d and u. Since $\mathcal{E}$ satisfies condition (W4), which is clear for $\mathcal{G}$ by the definition of nice beginning, no $(\mathcal{F}, k)$-bad $\overline{cc}(x)$-vertex is to the right of a $(\mathcal{F}, k)$-bad $cc(x)$-vertex in $\mathcal{F}$.

(W5)
For this condition, we partition the sequence of rounds into intervals. A new interval always starts when x changes color class with respect to the previous round, and the first interval starts with the first round. Note that during the rounds of a single interval, new bad vertices are only from the same color class. Consider the first round, which is the first round of the first interval. Denote by x_l and x_r the respectively leftmost and rightmost k-bad vertex. By definition of nice beginning, there is no empty $cc(x_r)$-slot between x_l and x_r in $\mathcal{G}$. Hence, all empty $cc(x_r)$-slots are to the left of x_l. Now, consider the beginning of an arbitrary but later interval. Let $\mathcal{E}$ be the input embedding of the first round of the interval, and denote by x_l and x_r the respectively leftmost and rightmost $(\mathcal{E}, k)$-bad vertex. Let $\mathcal{E}'$ be the input embedding of the previous round, which is the last round of the previous interval. Denote by x' the rightmost $(\mathcal{E}', k)$-bad vertex in $\mathcal{E}'$ and denote by d' its leftmost neighbor. Then, $d' = x_l$ or $d' \prec_{\mathcal{E}} x_l$ according to condition (W2). And the slot at position $\mathcal{E}(d') - 2$ is empty in $\mathcal{E}$. And since the leftmost neighbor of x_r, denoted as d_r, is at distance $k+2$ to the left of x_r in $\mathcal{E}$, which means at distance at least 3 to the left of d' in $\mathcal{E}$, there is an empty $cc(x_r)$-slot between d_r and x_l in $\texttt{RightMove}(\mathcal{E}, d_r)$. This completes the proof. ∎

Let $G = (A, B, E)$ be a bipartite permutation graph, and let $\mathcal{E}$ be an embedding for G. Let b, x be two vertices of G of the same color class, where $b \prec_{\mathcal{E}} x$. Let H be an induced subgraph of G that is clawpath-like. We say that H *has a proper connection on* (b, x) if H satisfies the following conditions in $\mathcal{E}$:

(P1) $x \in V(H)$, the slot at position $\mathcal{E}(x) - 1$ is occupied, say by vertex c, and $bc \in E$

(P2) H contains no $cc(x)$-vertex to the left of b and no $\overline{cc}(x)$-vertex to the left of c

(P3) no $\overline{cc}(x)$-vertex to the left of x has a neighbor in H to the right of x

(P4) the $cc(x)$-vertices between b and x in H correspond to a last path vertex of the clawpath underlying H.

We use such clawpath-like graphs to extend them on their proper connections.

Lemma 5.13 *Let $G = (A, B, E)$ be a connected bipartite permutation graph with normalized non-contractive embedding $\mathcal{E}$ of distortion $k+2$ with a nice beginning. Apply* `bpg-distortion` *to $(\mathcal{E}, k)$. If the algorithm accepts then $\mathrm{D}(G) \le k$, if the algorithm rejects then G contains a clawpath-like graph of length r on $\frac{1}{2}(rk + r + 2k + 6)$ vertices as induced subgraph.*

Proof. We show the lemma by induction over the number of iterations of the **while** loop of `bpg-distortion`. We begin with the first iteration. Let x and d be the vertices chosen according to the algorithm. By definition of nice beginning, there is an empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$. Let $\mathcal{F} =_{\text{def}}$ `RightMove`$(\mathcal{E}, d)$. According to the properties of $\mathcal{E}$, the slot at position $\mathcal{F}(d)-1$ in $\mathcal{F}$ is occupied, say by vertex u, and there is no empty $cc(x)$-slot between d and x in $\mathcal{F}$. Let c be any $\overline{cc}(x)$-vertex between d and x in $\mathcal{F}$ such that there is no empty $\overline{cc}(x)$-slot between d and c in $\mathcal{F}$. Then, G contains a clawpath-like graph of length 0 on $\frac{k+5}{2}$ vertices as induced subgraph with a proper connection on (d, c), as we show in the following. Let b be the vertex occupying the slot at position $\mathcal{F}(c) - 1$ in $\mathcal{F}$; note that b indeed exists due to the properties of $\mathcal{E}$. Then, $bc \in E$. Let $H_{d,c}$ be the subgraph of G induced by the $\overline{cc}(x)$-vertices between d and c and the $cc(x)$-vertices between b and x in $\mathcal{F}$. To show satisfaction of the conditions (P1–4), it remains to show satisfaction of condition (P4); the other conditions are clearly satisfied by the definition of $H_{d,c}$. Since $dx \in E$ and $bc \in E$, all $cc(x)$-vertices in $H_{d,c}$ are adjacent to all $\overline{cc}(x)$-vertices in $H_{d,c}$ by the properties of normalized embeddings and strong orderings. Thus, $H_{d,c}$ is a complete bipartite graph, i.e., a clawpath-like graph of length 0, and the $cc(c)$-vertices correspond to a last path vertex of the underlying clawpath. For the number of vertices in $H_{d,c}$, note that $\mathrm{d}_{\mathcal{F}}(d, x) = k$ and there are $\frac{1}{2}(\mathrm{d}_{\mathcal{F}}(d, c) + 1)$ $cc(c)$-vertices and $\frac{1}{2}(\mathrm{d}_{\mathcal{F}}(b, x) + 1)$ $\overline{cc}(c)$-vertices in $H_{d,c}$, which sums up to $\frac{k+5}{2}$ vertices (note that $\mathrm{d}_{\mathcal{F}}(d, c) + \mathrm{d}_{\mathcal{F}}(b, x) = k + 1$).

We now consider an arbitrary **while** loop execution. Let $\mathcal{E}$, x and d be defined according to the algorithm. We assume that there is a $cc(x)$-vertex b, where $b = x$ or $b \prec_{\mathcal{E}} x$, such that there is no empty $cc(x)$-slot between b and x in $\mathcal{E}$ and (b, x) is a proper connection for a clawpath-like graph $H_{b,x}$ of length r on $|V(H_{b,x})| = \frac{1}{2}(rk + r + k + 5)$ vertices. We consider cases according to `bpg-distortion`.

No empty slot
Suppose there is no empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$. Let c be the vertex occupying position $\mathcal{E}(x) - 1$ in $\mathcal{E}$. Since $dx \in E$ and $bc \in E$, all $\overline{cc}(x)$-vertices between d and x are adjacent to all $cc(x)$-vertices between b and x. Because of conditions (P1–4), subgraph H of G induced by the $\overline{cc}(x)$-vertices between d and x and $V(H_{b,x})$ is a clawpath-like graph of length r. We determine the number of vertices of H. There are $\frac{k+3}{2}$ $\overline{cc}(x)$-vertices between d and x in $\mathcal{E}$ and at least all $\frac{k+1}{2}$ $\overline{cc}(x)$-vertices to the left of c are not contained in $H_{b,x}$. Hence, $|V(H)| \geq \frac{1}{2}(rk+r+k+6)$.

For the remaining cases, let there be an empty $\overline{cc}(x)$-slot between d and x in $\mathcal{E}$. Let $\mathcal{F} =_{\text{def}}$ `RightMove`$(\mathcal{E}, d)$.

Position $\mathcal{F}(d) - 1$ not occupied
Let the slot at position $\mathcal{F}(d) - 1$ not be occupied in $\mathcal{F}$. Then, the slots at position $\mathcal{F}(d) - 2$ and $\mathcal{F}(d) - 1$ are not occupied in $\mathcal{F}$. Let $\mathcal{G} =_{\text{def}}$ `DeleteTwo`$(\mathcal{F}, \mathcal{F}(d) - 2)$. Due to Lemma 5.12, all $(\mathcal{F}, k)$-bad vertices are between d and x in $\mathcal{F}$. And since $\mathrm{d}_{\mathcal{F}}(d, x) = k$, the leftmost neighbor of every $(\mathcal{F}, k)$-bad vertex is to the left of d in $\mathcal{F}$. If there is no vertex to the left of d in $\mathcal{F}$, then there are no $(\mathcal{F}, k)$-bad vertices, and $\mathrm{D}(G, \mathcal{F}) = \mathrm{D}(G, \mathcal{G}) = k$. Otherwise, let w be the close vertex to the left of d in $\mathcal{F}$. Then, w is the close vertex to the left of d also in $\mathcal{E}$, and $\mathrm{d}_{\mathcal{G}}(w, d) = \mathrm{d}_{\mathcal{F}}(w, d) - 2 = \mathrm{d}_{\mathcal{E}}(w, d)$. Hence, $\mathcal{G}$ is a normalized non-contractive embedding for G. And since $\mathrm{D}(G, \mathcal{F}) \leq k + 2$, it follows that $\mathrm{D}(G, \mathcal{G}) = k$. Equality follows from $\mathrm{d}_{\mathcal{G}}(d, x) = k$.

Position $\mathcal{F}(d) - 1$ occupied
Let u be the vertex occupying position $\mathcal{F}(d) - 1$ in $\mathcal{F}$. As the first case, let there be no empty $cc(x)$-slot between d and x. Let c be a $\overline{cc}(x)$-vertex between d and x such that there is no empty

$\overline{cc}(x)$-slot between d and c in $\mathcal{F}$, and let b be the close vertex to the left of c. Analogous to the beginning of the proof, the $\overline{cc}(x)$-vertices between d and c and the $cc(x)$-vertices between c and x define a clawpath-like graph of length 0 on $\frac{k+5}{2}$ vertices with a proper connection on (d, c). As the second case, let there be an empty $cc(x)$-slot between d and x in $\mathcal{F}$; let p be the position of the leftmost empty $cc(x)$-slot between d and x. Let there be no breakpoint vertex between d and x in $\mathcal{F}$. We want to move u two positions to the right to obtain an embedding without vertex occupying the slot at position $\mathcal{F}(d)-1$. Suppose there is a blocking pair (v, w) such that v is a $cc(x)$-vertex and w is a $\overline{cc}(x)$-vertex and $d \prec_{\mathcal{F}} w$ and $\mathcal{F}(v) < p$. When we move u then v has to move and would come too close to the non-neighbor w. Note that $vw \notin E$ implies that no vertex to the right of w is adjacent to v. In particular, no $\overline{cc}(x)$-vertex between w and x has a left neighbor at distance more than k. Remember that $\mathrm{d}_{\mathcal{F}}(u, x) = k + 1$. By assumption, there is an empty $\overline{cc}(x)$-slot between w and x. Since $wx \in E$, $\texttt{RightMove}(\mathcal{F}, w)$ is a normalized non-contractive embedding without (v, w) being a blocking pair. If there are further blocking pairs with vertices to the left of position p, repeat the described procedure. If there are no further blocking pairs, $\mathcal{F}' =_{\mathrm{def}} \texttt{RightMove}(\mathcal{F}, u)$ is a normalized non-contractive embedding of distortion at most $k+2$ with $(\mathcal{F}', k)$-bad vertices only between d and x. We obtain a normalized non-contractive embedding of distortion at most k by deleting the two empty slots to the left of d, similar to the case above.

Finally, let there be a breakpoint vertex w between d and x; let v be the vertex such that (v, w) is a blocking pair. By definition, there is no empty $cc(x)$-slot between d and v and there is no empty $\overline{cc}(x)$-slot between w and x. Note that $v \prec_{\mathcal{F}} b$ by condition (W5) of Lemma 5.12. Let a be a $(\mathcal{F}, k)$-bad $\overline{cc}(x)$-vertex that is not $(\mathcal{E}, k)$-bad. This particularly means that there are no empty $\overline{cc}(x)$-slots between d and a in $\mathcal{F}$. Note that $a \prec_{\mathcal{F}} w$. Let c be the vertex occupying the slot at position $\mathcal{F}(a) - 1$. Now, let $H_{d,a}$ be the subgraph of G induced by $V(H_{b,x})$ and the $\overline{cc}(x)$-vertices between d and a and between w and x and the $cc(x)$-vertices between c and v. Similar to the beginning of the proof, all $\overline{cc}(x)$-vertices between d and x are adjacent to all $cc(x)$-vertices between b and x. And since $dx \in E$ and a has a neighbor to the left of u in $\mathcal{F}$, all $cc(x)$-vertices between u and v are neighbors of all $\overline{cc}(x)$-vertices between d and a. And no $cc(x)$-vertex between u and v is adjacent to a vertex from w on to the right. Hence, $H_{d,a}$ is a clawpath-like graph of length $r + 1$. It remains to determine the number of vertices in $V(H_{d,a}) \setminus V(H_{b,x})$:

- $\frac{1}{2}\mathrm{d}_{\mathcal{F}}(c, v) + 1$ $cc(x)$-vertices between c and v
- $\frac{1}{2}\mathrm{d}_{\mathcal{F}}(d, a) + 1$ $\overline{cc}(x)$-vertices between d and a
- $\frac{1}{2}(\mathrm{d}_{\mathcal{F}}(w, x) - 1)$ $\overline{cc}(x)$-vertices between w and x, where the vertex occupying the slot at position $\mathcal{F}(x) - 1$ in $\mathcal{F}$ is not counted,

which sums up to $\frac{1}{2}(\mathrm{d}_{\mathcal{F}}(c, v) + \mathrm{d}_{\mathcal{F}}(d, a) + \mathrm{d}_{\mathcal{F}}(w, x) - 1) + 2$ new vertices. With the definition of the selected vertices, it holds $\mathrm{d}_{\mathcal{F}}(c, a) = 1$ and $\mathrm{d}_{\mathcal{F}}(v, w) = 3$, so that $\mathrm{d}_{\mathcal{F}}(c, v) + \mathrm{d}_{\mathcal{F}}(d, a) + \mathrm{d}_{\mathcal{F}}(w, x) = k + 1 - 3 = k - 2$. Hence, $H_{d,a}$ contains $|V(H_{b,x})| + \frac{k+1}{2} \geq \frac{1}{2}(rk + r + k + 6)$ vertices.

We have seen that in case `bpg-distortion` stops the decision is correct in sense of our definitions, if it does not stop every k-bad vertex is associated with a clawpath-like graph of special properties. This completes the proof. ∎

So far, there is a third possible case for `bpg-distortion` that is not covered by Lemma 5.13, namely the algorithm might not terminate on an input. We actually have already proven that this certainly cannot happen, as conditions (W2) in Lemma 5.12: in every round of the algorithm, the number of vertices to the right of k-bad vertices increases. Now, we are ready for presenting the two main results of this section.

Theorem 5.14 *Let $G = (A, B, E)$ be a connected bipartite permutation graph, and let $k \geq 1$ be an odd integer. Then, $\mathrm{D}(G) \leq k$ or G contains a clawpath-like graph of length r on $\frac{1}{2}(rk + r + 2k + 6)$ vertices as induced subgraph.*

Proof. We show the statement by induction over the number of vertices of G. If G contains at most two vertices, then $\mathrm{D}(G) \leq 1$. So, let G have $n \geq 3$ vertices. Assume that the claim holds for all graphs on at most $n-1$ vertices. Let σ be a competitive ordering for G, and let x be the last vertex in σ. If $\mathrm{D}(G-x) \geq k+2$, then $G-x$ contains a clawpath-like graph of length r on $\frac{1}{2}(rk + r + 2k + 6)$ vertices as induced subgraph, and thus G. Now, let $\mathrm{D}(G-x) \leq k$, and let $\mathcal{F}$ be the embedding obtained as in Lemma 5.10 on input $\mathcal{E}$, σ and x. Assume that $\mathrm{D}(\mathcal{F}) = k+2$. Then, Lemma 5.11 can be applied to $\mathcal{F}$, and in connection with Lemma 5.13, we obtain the claim. ∎

Corollary 5.15 *Let $k \geq 1$ be an odd integer. Then, a connected bipartite permutation graph G has distortion at most k if and only if G does not contain a clawpath-like graph of length r on $\frac{1}{2}(rk + r + 2k + 6)$ vertices as induced subgraph.*

Proof. The statement directly follows from Theorem 5.14 and Corollary 5.7. ∎

In addition, Theorem 5.14 gives a simple algorithm for computing the distortion of a bipartite permutation graph.

Theorem 5.16 *There is an $O(n^2)$-time algorithm that computes the distortion of a connected bipatite permutation graph on n vertices. The algorithm certifies the computed distortion by a normalized non-contractive embedding as an upper bound and an induced clawpath-like subgraph as a lower bound.*

Proof. The algorithm is clear from the discussion so far. Let $G = (A, B, E)$ be a bipartite permutation graph with competitive ordering $\sigma = \langle x_1, \ldots, x_n \rangle$. Let $G_i =_{\mathrm{def}} G[\{x_1, \ldots, x_i\}]$ for $1 \leq i \leq n$. Iteratively, normalized minimum distortion embeddings for $G_1, \ldots, G_n$ are computed applying the algorithms of Lemmata 5.10 and 5.11 as preprocessing and `bpg-distortion` (in Lemma 5.13) as the main procedure. If the distortion of G_{i+1} is larger than the distortion of G_i then the algorithms even output an induced clawpath-like graph as certificate. The computed minimum distortion embedding for G_i serves as input for computing the distortion of G_{i+1}. For the running time, it mainly suffices to observe that `bpg-distortion` does not move a vertex twice. Storing the information about the number of vertices to the right of a position, it can be checked in constant time whether there are empty slots between two vertices. Consecutive vertices in the embeddings are at distance at most 3, so that at most $3n$ slots are used. Hence, `bpg-distortion` has an $\mathcal{O}(n)$-time implementation. The two preprocessing algorithms require

$\mathcal{O}(n)$ time, and a competitive ordering is obtained in linear time. Since the main algorithm has $\mathcal{O}(n)$ iterations, the $\mathcal{O}(n^2)$ running time follows.

∎

6 Final remarks

Our algorithm for computing the distortion of bipartite permutation graphs has running time $\mathcal{O}(n^2)$. It seems possible that this running time can be even improved to linear time by using information about the embedding of a previous round. Our algorithm basically works with any normalized minimum distortion embedding for the smaller graph.

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