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Characterization and recognition of digraphs of bounded Kelly-width

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Abstract

Kelly-width is a parameter of directed graphs recently proposed by Hunter and Kreutzer as a directed analogue of treewidth. We give several alternative characterizations of directed graphs of bounded Kelly-width in support of this analogy. We apply these results to give the first polynomial-time algorithm recognizing directed graphs of Kelly-width 2. For an input directed graph $G = (V, A)$ the algorithm will output a vertex ordering and a directed graph $H = (V, B)$ with $A \subseteq B$ witnessing either that G has Kelly-width at most 2 or that G has Kelly-width at least 3, in time linear in H.

1 Introduction

The tractability of large classes of NP-complete problems when parameterized by the treewidth of the input graph counts among the strongest results in algorithmic graph theory. The algorithms behind this tractability have two stages: first an algorithm computing treewidth, then an algorithm solving the specific problem using the tree-structure discovered in the first stage. See for example [2] for a recent overview of these algorithms. For directed graphs (digraphs) there have been several proposals for a parameter analogous to treewidth: 'directed treewidth' of Johnson, Robertson, Seymour, Thomas [5], 'D-width' of Safari [7], 'DAG-width' of Berwanger, Dawar, Hunter, Kreutzer [1] and independently Obdržálek [6], and 'Kelly-width' of Hunter and Kreutzer [4]. Which of these proposed parameters is the better analogue of treewidth? In this paper we give evidence in support of the Kelly-width parameter.

The success of a model depends on a balance between the modeling power, which measures how general its domain of application is, and the analytical power, which measures how good it is as an analytical tool. The two are typically in conflict. This is also the case for the above proposals for tree-like parameters of digraphs. The better the modeling power, e.g. the larger the class of digraphs that have bounded parameter value, the worse the analytical power, e.g. the smaller the chance of successfully emulating both stages of the algorithmic results for treewidth. We do not go into details of the modeling and analytical powers of each of the proposed digraph parameters, but simply note that from a purely algorithmic point of view there is no clear winner. How then to choose the digraph parameter which is the most natural directed analogue of treewidth? Note that while some concepts of undirected graphs have unambiguous natural translations to directed graphs, e.g. from paths to directed paths, there are other concepts, e.g. cliques and separators, for which the translation is less clear. The treewidth parameter is known to have many equivalent characterizations. If we start with a characterization of treewidth that uses only concepts that have unambiguous translations to directed graphs then we should arrive at a directed graph parameter which is a natural analogue of treewidth. This is the approach we take in this paper. In Section 3 we give a new characterization of digraphs of Kelly-width at most k arising from a characterization of treewidth that uses the fairly unambiguous concepts of vertex orderings, paths and neighbours.

We also enhance the algorithmic argument in favour of Kelly-width. Digraphs of Kelly-width 1 are the directed acyclic graphs and recognizable by a simple algorithm. For all larger values of k the only algorithms that were known for recognizing digraphs of Kelly-width k had running time exponential in the size of the input digraph [4]. Using the given characterizations we are able to present a fast algorithm recognizing digraphs of Kelly-width 2 in Section 4. For an input digraph $G = (V, A)$ this algorithm will output a vertex ordering and a digraph $H = (V, B)$ with $A \subseteq B$ witnessing either that G has Kelly-width at most 2 or that G has Kelly-width at least 3, in time linear in H. In the positive case the witness can be used to easily find a decomposition of the digraph into a tree-like structure.

2 Graph preliminaries and digraphs of bounded Kelly-width

A *simple finite directed graph* G is a pair of sets, (V, A), where V is finite and A is an irreflexive relation over V. The set V is called the *vertex set* of G, and A is called the *arc set* of G. Since we mostly consider simple finite directed graphs, we shortly call them "digraphs". When we deal with undirected graphs, we will explicitly mention it. For an arbitrary digraph H, $V(H)$ and $A(H)$ denote the vertex and arc set of H, respectively. An arc of graph G is denoted as (u, v) and u is the *start vertex* and v is the *end vertex* of (u, v). Let H be a digraph. We say that G is a *subgraph* of H, if $V \subseteq V(H)$ and $A \subseteq A(H)$. If $V = V(H)$ and G is a subgraph of H then G is a *spanning subgraph* or *partial graph* of H. Further definitions are given when they are needed.

Hunter and Kreutzer introduced the notion of Kelly-width [4]. Kelly-width is a parameter for digraphs, and it is the least width of a so-called *Kelly-decomposition*. We will not define Kelly-decompositions here, since we will not use this notion. The authors gave several alternative characterizations of digraphs of bounded Kelly-width by: elimination process, inductive construction, graph game. We will study graphs of bounded Kelly-width starting from the inductive construction. Let $G = (V, A)$ be a digraph. Let u and v be vertices of G. We call v an *in-neighbour* of u, if (v, u) is an arc of G. The *(open) in-neighbourhood* of u, denoted as $N_G^{\text{in}}(u)$, is the set of in-neighbours of u. The *closed in-neighbourhood* of u, denoted as $N_G^{\text{in}}[u]$, is defined as $N_G^{\text{in}}(u) \cup \{u\}$. Similarly, v is an *out-neighbour* of u, if (u, v) is an arc of G. *Open* and *closed out-neighbourhood* of a vertex are defined respectively. The *out-degree* of a vertex is the number of its out-neighbours. Let X be a set of vertices of G. We define the *common in-neighbourhood* of X, denoted as $\bigcap N_G^{\text{in}}[X]$, recursively:

$$
\begin{aligned}
X = \emptyset \quad &: \quad \bigcap N_G^{\text{in}}[X] =_{\text{def}} \quad V \\
X \neq \emptyset \text{ and } a \in X \quad &: \quad \bigcap N_G^{\text{in}}[X] =_{\text{def}} \quad N_G^{\text{in}}[a] \ \cap \ \bigcap N_G^{\text{in}}[X \setminus \{a\}] \,.
\end{aligned}
$$

The inductive construction characterization of digraphs of bounded Kelly-width by Hunter and Kreutzer started from a basic class of graphs, and the partial graph relation defines the complete class. The basic graphs are called k*-DAGs*. Since certain of our statements become easier, we generalise the definition and define k*-GDAGs*.

Definition 2.1 *Let $k \geq 0$. The class of* generalised k-DAGs*, k-GDAGs, for short, is the class of digraphs inductively defined by the two following construction steps:*

(1) a graph on one vertex is a k-GDAG

(2) let G be a k-GDAG and let u be a vertex that does not appear in G. Let X be a set of at most k vertices of G, called the parent vertices of u*. Then, G' is a k-GDAG where G' emerges from G by adding vertex u and the following arc set:*

$$\Big\{(u, x) : x \in X\Big\} \cup \Big\{(y, u) : y \in \bigcap N_G^{\text{in}}[X]\Big\}.$$

With a k-GDAG, we associate a sequence $\langle x_1, \dots, x_n \rangle$ of vertices, where x_1 is the vertex of the start graph in construction step (1) of Definition 2.1, and x_i, $i \in \{2, \dots, n\}$, is added to the graph on the vertices $x_1, \dots, x_{i-1}$, that has already been constructed, according to construction step (2). Let $k \geq 0$, and let $G = (V, A)$ be a k-GDAG. A vertex sequence $\sigma = \langle x_1, \dots, x_n \rangle$ for G is a *construction sequence* for G, if G can be obtained according to construction steps (1) and (2) adding vertices according to σ and choosing $N_G^{\text{out}}(x_i) \cap \{x_1, \dots, x_{i-1}\}$ as the parent vertices set of x_i, $i \in \{1, \dots, n\}$. Parent vertices are always defined with respect to a vertex sequence. The *child vertices* of a vertex x are those vertices that choose x as a parent vertex.

Definition 2.2 *Let $k \geq 0$, and let G be a digraph. G is a* partial k-GDAG *if and only if G is a partial graph of some k-GDAG.*

Note that partial k-GDAGs cannot be associated with a construction sequence in general.

Hunter-Kreutzer k-DAGs are defined analogous to k-GDAGs with the following difference: instead of starting with a graph on a single vertex in construction step (1), k-DAGs start with a complete graph on k vertices. This means that every k-DAG contains a complete subgraph on k vertices, which is not true for k-GDAGs in general. *Partial k-DAGs* are partial graphs of k-DAGs. The following lemma relates k-DAGs and k-GDAGs to each other.

Lemma 2.3 *Let $k \geq 0$, and let $G = (V, A)$ be a digraph.*

(1) If G is a k-GDAG, then G is a partial k-DAG.

(2) If G is a k-DAG, then G is a k-GDAG.

Proof Let G be a k-GDAG with construction sequence $\sigma = \langle x_1, \dots, x_n \rangle$. Obtain H from G by making $G[\{x_1, \dots, x_k\}]$ into a complete directed graph. Note that, with respect to σ, every vertex of H has at most k parent vertices. Hence, a k-DAG H' can be constructed using sequence σ and choosing parent vertices according to H, and G is a partial graph of H'. If G is a k-DAG with construction sequence σ, G is a k-GDAG with construction sequence σ. □

Corollary 2.4 *Let $k \geq 0$, and let G be a digraph. G is a partial k-GDAG if and only if G is a partial k-DAG.*

The Kelly-width of a digraph is a width parameter based on the width of Kelly-decompositions. Kelly-width and Kelly-decomposition were introduced by Hunter and Kreutzer as a decomposition counterpart of tree-decompositions for undirected graphs [4]. The authors showed a strong correspondence between partial k-DAGs and graphs of bounded Kelly-width.

Theorem 2.5 ([4]) *Let $k \geq 0$, and let G be a digraph. G has Kelly-width at most $k+1$ if and only if G is a partial k-DAG.*

We can conclude that Kelly-width also characterises partial k-GDAGs.

Corollary 2.6 *Let $k \geq 0$, and let G be a digraph. G has Kelly-width at most $k+1$ if and only if G is a partial k-GDAG.*

In the following, we will mostly deal with k-GDAGs and partial k-GDAGs. We will also speak of "graphs of bounded Kelly-width". k-DAGs are mentioned to discuss differences between graph classes and with respect to obtained results.

3 Characterizations of graphs of bounded Kelly-width

So far, graphs of bounded Kelly-width have four different characterizations: via elimination process, inductive construction, cops-robber game, decomposition. These many characterizations were the start point for us to consider the concepts of Kelly-width and Kelly-decompositions as a good digraph counterpart of the concepts of treewidth and tree-decompostion of undirected graphs. Treewidth seems a very natural concept, since undirected graphs of bounded treewidth can be characterised by a long list of different statements. In this section, we will add two further results to the list of characterizations for graphs of bounded Kelly-width. We will see that graphs of bounded Kelly-width have a vertex-ordering characterization, and we show that partial k-GDAGs are the same as subgraphs of k-GDAGs. We begin by recalling the elimination process characterization by Hunter and Kreutzer. This characterization will be used later.

3.1 Elimination process characterization

Undirected graphs of bounded treewidth have a nice characterization using an elimination scheme. Let $G = (V,E)$ be an undirected graph on at least two vertices, and let x be a vertex of G. The operation *reducing G by x* yields graph G' that is obtained from G by deleting vertex x and adding the edge set $\{\{u,v\} : u \neq v \text{ and } u,v \in N_G(x)\}$. In words, G' is obtained from G by deleting x and making its neighbourhood (in G) into a clique.

Theorem 3.1 (folklore) *Let $k \geq 0$, and let $G = (V,E)$ be an undirected graph. Then, G has treewidth at most k if and only if G can be reduced to a graph on one vertex by repeatedly reducing by a vertex of degree at most k.*

The characterization of undirected graphs of bounded treewidth in Theorem 3.1 can be translated into the world of digraphs. However, the reduction operation must be adjusted. Let $G = (V,A)$ be a digraph on at least two vertices, and let x be a vertex of G. The operation *reducing G by x* yields graph G' that is obtained from G by deleting vertex x and adding the arc set $\{(u, v) : u \neq v$ and $u \in N_G^{\text{in}}(x)$ and $v \in N_G^{\text{out}}(x)\}$. This definition of the reduction operation is a natural way to translate the completion from the undirected case to the directed case, although it is not the only possibility. Hunter and Kreutzer did this to obtain the following result for digraphs of bounded Kelly-width.

Theorem 3.2 ([4]) *Let $k \geq 0$, and let $G = (V, A)$ be a digraph. Then, G has Kelly-width at most $k + 1$ if and only if G can be reduced to a graph on one vertex by repeatedly reducing by a vertex of out-degree at most k.*

The result of Theorem 3.2 implies an easy algorithm for recognizing graphs of bounded Kelly-width. Unfortunately, this algorithm is not a polynomial-time algorithm. A given graph, partial k-GDAG or not, can have more than one vertex of out-degree at most k. There is no a priori argument or criterion deciding which one to choose.

3.2 Vertex-ordering characterization

In this subsection, we show that graphs of bounded Kelly-width are the graphs whose vertices can be arranged in a linear order to satisfy special conditions. We start with a characterization of k-GDAGs. This characterization is used in most of our proofs about graphs of bounded Kelly-width.

Let $G = (V, A)$ be a digraph. A *path* P in G is a sequence $(x_0, \ldots, x_l)$ of mutually different vertices of G where (x_i, x_{i+1}) is an arc of G for every $i \in \{0, \ldots, l-1\}$. Let σ be a vertex ordering for G. Path P is called *σ-monotone-left*, if $x_l \prec_\sigma \cdots \prec_\sigma x_0$ holds. P starts at vertex x_0; so, if P is σ-monotone-left, it is a σ-monotone-left path starting at x_0. For a vertex u and an arc (x, y) of G, we say that (x, y) *spans over u with respect to σ*, if $x \prec_\sigma u \prec_\sigma y$ or $y \prec_\sigma u \prec_\sigma x$. If the ordering σ is uniquely determined, we shortly say that (x, y) *spans over* u. Let u be a vertex of G. We say that a σ-monotone-left path in G has the *spanning-vertex u property*, if the pair (P, u) satisfies the following condition: if P contains an arc that spans over u, then P contains a vertex $w \prec_\sigma u$ such that $w \in N_G^{\text{out}}(u)$ and the arc of P that spans over u has end vertex w.

Theorem 3.3 *Let $k \geq 0$, and let $G = (V, A)$ be a digraph. G is a k-GDAG if and only if there is a vertex ordering $\sigma = \langle x_1, \ldots, x_n \rangle$ for G such that the pair (G, σ) satisfies the following two conditions:*

(1) for every $i \in \{1, \ldots, n\}$, $|N_G^{\text{out}}(x_i) \cap \{x_1, \ldots, x_{i-1}\}| \leq k$

(2) for every pair u, v of vertices of G where $u \prec_\sigma v$, (u, v) is an arc of G if and only if every σ-monotone-left path starting at v has the spanning-vertex u property.

If G is a k-GDAG, the vertex orderings σ such that the pair (G, σ) satisfies conditions (1) and (2) are exactly the construction sequences for G.

Proof We prove two implications. Let G be a k-GDAG, and let $\sigma = \langle x_1, \ldots, x_n \rangle$ be a construction sequence for G. We show that the pair (G, σ) satisfies the two conditions. By definition of k-GDAGs, every vertex x_i chooses at most k parent vertices among $x_1, \ldots, x_{i-1}$, so that condition (1) is obviously satisfied. We show satisfaction of condition (2) by induction. For x_1 as vertex v, condition (2) is trivially satisfied. Now, consider x_i as vertex v for $i \geq 2$. Let X be the parent vertices set of x_i. Let $u \in \{x_1, \ldots, x_{i-1}\}$. According to construction step (2), (u, v) is an arc of G if and only if u is a vertex in the common in-neighbourhood of X in $G[\{x_1,\ldots,x_{i-1}\}]$. Let X' be the set of vertices x in X for which holds $u \prec_\sigma x$. Applying the induction hypothesis, u is in the in-neighbourhood of $x \in X'$ if and only if every σ-monotone-left path P' starting at x has the spanning-vertex u property.

(a) Let (u,v) be in G. Let P be a σ-monotone-left path starting at v that does not contain u. Let z be the endvertex of P, and let $z \prec_\sigma u$. Note that P contains at least two vertices. Let y be the vertex on P following v. Then, y is a vertex in X. If $y \in X'$, then $P' =_{\text{def}} P - v$ is a σ-monotone-left path starting at y with last vertex $z \prec_\sigma u$ that does not contain u. Since (u,y) is an arc of G, we obtain by applying the induction hypothesis that P' has the spanning-vertex u property, i.e., P' contains a vertex $w \prec_\sigma u$ such that $w \in N_G^{\text{out}}[u]$ and the arc of P' with end vertex w spans over u. Hence, P has the spanning-vertex u property. Let $y \notin X'$. Then, y is an out-neighbour of u and arc (v,y) spans over u, which means that P has the spanning-vertex u property.

(b) Let (u,v) be not in G. Then, there is a parent vertex y of v such that u is not an in-neighbour of y. If $y \in X'$, there is a σ-monotone-left path P starting at y in G that does not have the spanning-vertex u property according to induction hypothesis. We can extend P by adding v as start vertex and obtain a σ-monotone-left path starting at v that does not have the spanning-vertex u property. If $y \notin X'$, then the arc (v,y) spans over u but u is not an in-neighbour of y, i.e., there is a σ-monotone-left path starting at v in G that does not have the spanning-vertex u property.

Hence, the pair (G,σ) satisfies conditions (1) and (2).

We prove the second implication. Let G be a digraph and let $\sigma = \langle x_1,\ldots,x_n\rangle$ be a vertex ordering for G such that the pair (G,σ) satisfies conditions (1) and (2). We show that G is a k-GDAG by showing that G is equal to some k-GDAG. Let H be constructed using σ as construction sequence and choosing the parent vertices according to condition (1), i.e., the parent vertices set of x_i is $X_i =_{\text{def}} N_G^{\text{out}}(x_i) \cap \{x_1,\ldots,x_{i-1}\}$. It is clear that H is a k-GDAG by the choice of the parent vertices and condition (1). We first show that G is a partial graph of H. By definition, for every pair u,v of vertices where $u \prec_\sigma v$, (v,u) is an arc of H if and only if (v,u) is an arc of G. For the remaining arcs of G, we prove by induction over $i \in \{1,\ldots,n\}$ that $\{(u,x_i) \in A(G) : u \prec_\sigma x_i\} = \{(u,x_i) \in A(H) : u \prec_\sigma x_i\}$ holds. The claim obviously holds for $i = 1$. Let $i \geq 2$. We show two inclusions. First, let (u,x_i) be an arc of G, where $u \prec_\sigma x_i$. We show that u is in the common in-neighbourhood of X_i in H. Let y be a vertex in $X_i \setminus \{u\}$. We distinguish two cases.

(a) Let $u \prec_\sigma y$. Since (u,x_i) is an arc of G, every σ-monotone-left path starting at x_i in G has the spanning-vertex u property. In particular, every σ-monotone-left path P starting at x_i and containing y as successor of x_i on P has the spanning-vertex u property. Thus, every σ-monotone-left path starting at y has the spanning-vertex u property, which means that (u,y) is an arc of G due to condition (2). By induction hypothesis, (u,y) is an arc of H.

(b) Let $y \prec_\sigma u$, i.e., $y \prec_\sigma u \prec_\sigma x_i$. Then, there is a σ-monotone-left path P starting at x_i and containing y as the successor of x_i on P, and by assumption, P has the spanning-vertex u property. Note that (x_i,y) spans over u, so that y must be an out-neighbour of u due to the spanning-vertex u property of P. By definition of H, (u,y) is an arc of H.

So, $u \in \bigcap N_H^{\text{in}}[X_i]$, and (u, x_i) is an arc of H according to construction step (2) for k-GDAGs. For the second inclusion, let $u \in \{x_1, \ldots, x_{i-1}\}$ be such that (x_i, u) is not an arc of G. Hence, there is a σ-monotone-left path P starting at x_i which does not have the spanning-vertex u property. Note that P must contain at least two vertices. Let y be the successor of x_i on P. If arc (x_i, y) spans over u, u is not an in-neighbour of y. If $u \prec_\sigma y$, there is a σ-monotone-left path starting at y in G that does not have the spanning-vertex u property. According to condition (2), (u, y) is not an arc of G. For the two cases, it follows by induction hypothesis, that $u \notin \bigcap N_H^{\text{in}}[X_i]$, and (u, x_i) is not an arc of H. We conclude that G and H have the same arc sets, i.e., they are equal. Thus, G is a k-GDAG and σ is a construction sequence for G. □

Also for k-DAGs, a characterization theorem in the flavour of Theorem 3.3 can be formulated. However, it will have a more complex version of condition (1).

We want to extend the characterization result of Theorem 3.3 for k-GDAGs to digraphs of bounded Kelly-width. Since partial k-GDAGs are just the partial graphs of k-GDAGs, there must be some relaxation in the conditions of Theorem 3.3. This relaxation affects condition (2). The following lemma defines a subclass of partial k-GDAGs for which a characterization in the flavour of Theorem 3.3 exists.

Lemma 3.4 *Let $k \geq 0$, and let $G = (V, A)$ be a digraph. The following two statements are equivalent:*

(A) there is a vertex ordering $\sigma = \langle x_1, \ldots, x_n \rangle$ for G such that the pair (G, σ) satisfies the following two conditions:

(1) for every $i \in \{1, \ldots, n\}$, $|N_G^{\text{out}}(x_i) \cap \{x_1, \ldots, x_{i-1}\}| \leq k$

(2) for every pair u, v of vertices of G where $u \prec_\sigma v$, if (u, v) is an arc of G then every σ-monotone-left path starting at v has the spanning-vertex u property

(B) there is a k-GDAG H with construction sequence σ such that the triple (G, H, σ) satisfies the following two conditions:

(3) G is a partial graph of H

(4) $A(G) \cap \{(u, v) : v \prec_\sigma u\} = A(H) \cap \{(u, v) : v \prec_\sigma u\}$

Proof We prove two implications. Let H be a k-GDAG with construction sequence σ such that the triple (G, H, σ) satisfies conditions (3) and (4). Note that the pair (H, σ) satisfies conditions (1) and (2) of Theorem 3.3. Then, satisfaction of condition (1) of Theorem 3.3 and of condition (4) implies satisfaction of condition (1). We prove satisfaction of condition (2) by induction over $i \in \{1, \ldots, n\}$, showing that every arc (u, x_i) for $u \prec_\sigma x_i$ has the property of condition (2). The claim obviously holds for $i = 1$. So, let $i \geq 2$. Let (u, x_i) be an arc of G, where $u \prec_\sigma x_i$, and let P be a σ-monotone-left path starting at x_i in G. Let z be the last vertex of P, let $z \prec_\sigma u$, and let u be not a vertex on P. Then, P has the spanning-vertex u property in H, which means that P contains a vertex $w \prec_\sigma u$ such that $w \in N_H^{\text{out}}(u)$ and the arc of P with end vertex w spans over u. By the choice of H, w is an out-neighbour of u also in G. Hence, path P has the spanning-vertex u property also in G. Thus, the pair (G, σ) satisfies condition (2).

For the second implication, let σ be a vertex ordering for G such that the pair (G, σ) satisfies conditions (1) and (2). Let H be the graph obtained from G by adding as many arcs from the

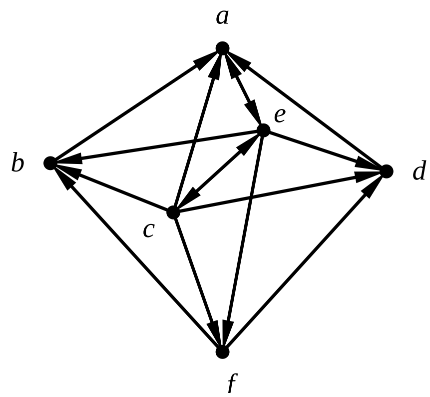

Figure 1: A partial 1-GDAG without the property required by condition (4) of Lemma 3.4.

set $\{(u,v) : u \prec_\sigma v\}$ as possible such that (H,σ) still satisfies conditions (1) and (2). Then, the pair (H,σ) satisfies the two conditions of Theorem 3.3, which means that H is a k-GDAG with construction sequence σ. Furthermore, G is a partial graph of H and condition (4) is satisfied by construction of H. □

The crucial point of the characterization in Lemma 3.4 is condition (4). Informally, the question is whether every partial k-GDAG G can be embedded into a k-GDAG H_G where H_G can be constructed according to the two construction steps such that every vertex chooses only parent vertices that are out-neighbours in G.

For partial 0-GDAGs, the question can immediately be answered positively, since 0-GDAGs do not choose any parent vertex. Interestingly, already for partial 1-GDAGs, the answer is negative. We prove this by giving an example. Consider the graph depicted in Figure 1; let us call it G. Observe that G is a partial 1-GDAG: using construction sequence $\langle e,c,a,f,b,d\rangle$, we can construct a 1-GDAG that contains G as a partial graph. We show that there is no 1-GDAG H and no vertex ordering σ for H such that σ is a construction sequence for H and the triple (G,H,σ) satisfies conditions (3) and (4) of Lemma 3.4. Note that the last vertex of a construction sequence for a 1-GDAG has at most one out-neighbour, which is its parent vertex. G has exactly three vertices with out-degree at most 1, namely a, b and d. Furthermore, vertices c and e have out-degree 5, which means that at least four vertices come after c and e in every possible construction sequence. Hence, c and e are the first two vertices in every construction sequence for a 1-GDAG containing G. We distingusih two cases. Let a be the last vertex. The parent vertex of a is e, and b and d must be in-neighbours of e. Hence, b and d have a new parent vertex. Let b or d be the last vertex. Then, a is the parent vertex, and a must be an out-neighbour of f. Then, a or d can be the predecessor vertex in a construction sequence. If it is a, e must be an out-neighbour of f, which makes e the parent of f. If it is d, either a or e is the parent of f. Hence, there is no 1-GDAG H and construction sequence σ for H such that (G,H,σ) satisfies conditions (3) and (4) of Lemma 3.4.

So, for a characterisation of partial k-GDAGs, we have to relax the conditions a little more.

Theorem 3.5 *Let $k \geq 0$, and let $G = (V,A)$ be a digraph. G is a partial k-GDAG if and only if there are a vertex ordering $\sigma = \langle x_1,\ldots,x_n\rangle$ and a set F of arcs such that the triple (G,F,σ) satisfies the following two conditions, where we set $G' =_{\mathrm{def}} G \cup F$:*

(1) for every $i \in \{1,\ldots,n\}$, $|N^{\mathrm{out}}_{G'}(x_i) \cap \{x_1,\ldots,x_{i-1}\}| \leq k$

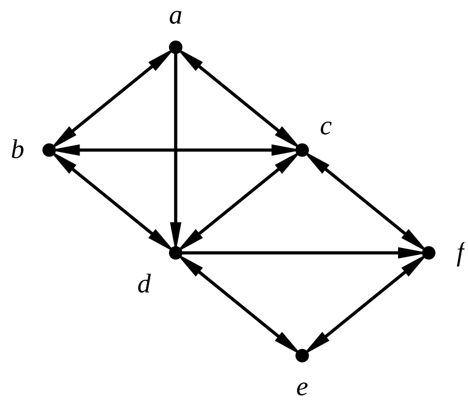

Figure 2: A partial 2-GDAG, which is proved by the vertex sequence $\langle a, b, c, d, e, f \rangle$, without the property required by condition (2) of Lemma 3.4.

(2) for every pair u, v of vertices of G where $u \prec_\sigma v$, if (u, v) is an arc of G then every σ-monotone-left path starting at v in G' has the spanning-vertex u property in G'.

Proof Let G be a partial graph of k-GDAG H, and let σ be a construction sequence for H. Let $F =_{\text{def}} (A(H) \setminus A(G)) \cap \{(u, v) : v \prec_\sigma u\}$. Let $G' =_{\text{def}} G \cup F$. It holds that the triple (G', H, σ) satisfies conditions (3) and (4) of Lemma 3.4, from which follows that the triple (G, F, σ) satisfies the conditions of the theorem. For the converse, let σ be a vertex ordering for G and F a set of arcs such that (G, F, σ) satisfies the two conditions of the theorem, where we set $G' =_{\text{def}} G \cup F$. Hence, the pair (G', σ) satisfies conditions (1) and (2) of Lemma 3.4, from which follows that G' is a partial k-GDAG. Since G is a partial graph of G', G is a partial k-GDAG. □

It is clear that in case $k = 0$ of Theorem 3.5, the set F can be chosen empty. Interestingly, the counterexample in Figure 1 cannot be chosen as counterexample for partial k-GDAGs for $k \geq 2$: let the graph in Figure 1 be G. For constructing a 2-GDAG containing G as partial graph, we can use vertex ordering $\sigma = \langle e, c, a, b, d, f \rangle$. So, the question arises whether the problems for partial 1-GDAGs of satisfying condition (4) in Lemma 3.4 are also problems for partial k-GDAGs for $k \geq 2$. At least for partial 2-GDAGs, Figure 2 gives an example of a graph with problems similar to G from Figure 1.

3.3 Subgraph characterization

From the characterization result of Theorem 3.5, we derive yet another characterization of digraphs of bounded Kelly-width. This characterization is not surprising, rather a necessity. It simply says that "partial graph" in the definition of partial k-GDAGs can be replaced by the more natural term "subgraph". This is an analogue to partial k-trees, which are defined as partial graphs of k-trees and can be characterised as subgraphs of k-trees.

Lemma 3.6 *Let $k \geq 0$, and let $G = (V, A)$ be a k-GDAG. Let a be a vertex of G. Then, $G' =_{\text{def}} G - a$ is a partial k-GDAG.*

Proof Let σ be a construction sequence for G. Then, the pair (G, σ) satisfies conditions (1) and (2) of Theorem 3.3. Let $\sigma' =_{\text{def}} \sigma - a$. We show that the triple $(G', \emptyset, \sigma')$ satisfies the three conditions of Theorem 3.5. It is clear that conditions (1) and (2) of Theorem 3.5 are satisfied. So, let (u, v) be an arc of G' where $u \prec_{\sigma'} v$. Let P be a σ'-monotone-left path starting at v in G'. Let z be the last vertex of P, and let $z \prec_{\sigma'} u$. Finally, let u be not a vertex on P. Note that, by construction, P is a σ-monotone-left path in G. Hence, there is a vertex w on P such that $w \in N_G^{\text{out}}(u)$ and the arc of P with end vertex w spans over u. Since $w \neq a$, we conclude that P has the property of condition (3) of Theorem 3.5 also in G'. Hence, G' is a partial k-GDAG. □

Theorem 3.7 *Let $k \geq 0$, and let $G = (V, A)$ be a digraph. Then, G is a partial k-GDAG if and only if G is a subgraph of a k-GDAG.*

Proof If G is a partial graph of a k-GDAG H, then G is also a subgraph of H. Now, let G be a subgraph of k-GDAG H. If G and H have the same vertex sets, G is a partial subgraph of H, hence a partial k-GDAG. Otherwise, let u be a vertex of H that is not contained in G. Then, G is a subgraph also of $H - u$. According to Lemma 3.6, $H - u$ is a partial k-GDAG, i.e., partial graph of a k-GDAG H'. By induction, we conclude that G is a partial graph of a k-GDAG. □

4 A connection between digraphs of bounded Kelly-width and undirected graphs of bounded treewidth

We argued in the introduction that the notion of Kelly-width for digraphs can be considered an appropriate analogue of the notion of treewidth for undirected graphs. We based our argumentation on the many similarities in different characterizations between the two notions. In this section, we illustrate the connection between the two graph notions in a special way: we answer the question whether undirected graphs of bounded treewidth can be embedded into digraphs of bounded Kelly-width. The embedding relation must be defined, and we choose the most natural approach. It will turn out that the basic class, k-GDAGs, does not exactly correspond to the class of $\{0, \ldots, k\}$-trees but to a reasonable bigger class.

For the results in this section, we need further definitions. Let $G = (V, A)$ be a digraph. Two vertices u and v of G are called *strongly adjacent*, if (u, v) and (v, u) are arcs of G. If u and v are strongly adjacent, we call $\{u, v\}$ a *bi-directional arc* of G. Otherwise, if u and v are not strongly adjacent, $\{u, v\}$ is called a *uni-directional arc*. By bi-dir(G), we denote the undirected graph on vertex set V where two vertices are adjacent if and only if they are strongly adjacent in G. Let $G' = (V, E)$ be an undirected graph. G' is *chordal*, if G' does not contain an induced cycle of length greater than 3. A vertex of G' is called *simplicial*, if its neighbourhood is a clique in G'. Every chordal graph has a simplicial vertex [3]. Using these definitions, we can modify Theorem 3.1 to obtain a characterization of chordal graphs of bounded treewidth.

Theorem 4.1 (folklore) *Let $k \geq 0$, and let $G = (V, E)$ be an undirected graph. Then, G is a chordal graph of treewidth at most k if and only if G can be reduced to a graph on one vertex by repeatedly reducing by a simplicial vertex of degree at most k.*

Since chosen vertices in Theorem 4.1 are simplicial, no new edge is added during the elimination process. Vertex orderings for chordal graphs defined by the elimination process are called *perfect elimination schemes*, and the first (leftmost) vertex in the ordering is the first vertex to be eliminated.

Theorem 4.2 *Let $k \geq 0$. An undirected graph G is a chordal graph of treewidth at most k if and only if there is a k-GDAG H such that $G = \text{bi-dir}(H)$.*

Proof Let G be a chordal graph with treewidth at most k, and let $\sigma = \langle x_1, \ldots, x_n \rangle$ be a perfect elimination scheme for G. Let $G_i =_{\text{def}} G[\{x_i, \ldots, x_n\}]$ for every $i \in \{1, \ldots, n\}$. We show by induction that we can construct a k-GDAG H with construction sequence $\langle x_n, \ldots, x_1 \rangle$ (reverse order!) such that $G_i = \text{bi-dir}(H[\{x_i, \ldots, x_n\}])$ for every $i \in \{1, \ldots, n\}$. Let $X_i =_{\text{def}} N_G(x_i) \cap \{x_{i+1}, \ldots, x_n\}$ for every $i \in \{1, \ldots, n\}$. Let H_n be the digraph on vertex x_n. Obviously, $G_n = \text{bi-dir}(H_n)$ and H_n is a k-GDAG. Let $i < n$. Let H_i emerge from H_{i+1} by adding x_i and choosing X_i as the parent vertices set of x_i. Due to Theorem 4.1, X_i contains at most k vertices, so that H_i is a k-GDAG. By definition of σ, X_i is a clique in G_{i+1}, and by induction hypothesis, X_i induces a complete graph in H_{i+1}. In particular, every vertex in X_i is in-neighbour in H_{i+1} of every other vertex in X_i, so that x_i is strongly adjacent with every vertex in X_i in H_i. Hence, $X_i \cup \{x_i\}$ induces a complete subgraph in H_i and G_i, i.e., $G_i = \text{bi-dir}(H_i)$.

For the converse, let $H = (V, A)$ be a k-GDAG. Let $\sigma = \langle x_1, \ldots, x_n \rangle$ be a construction sequence for H, and let X_i be the parent vertices set of x_i with respect to σ, $i \in \{1, \ldots, n\}$. We first show that $\text{bi-dir}(H) = \text{bi-dir}(H')$ where H' is a k-GDAG with construction sequence σ and the following parent vertices sets: $X'_i =_{\text{def}} \{u \in X_i : u \text{ and } x_i \text{ are strongly adjacent in } H\}$. With these definitions, it is an easy induction over the construction steps for H and H' to prove that $\{(u, v) \in A : u \prec_\sigma v\} \subseteq \{(u, v) \in A(H') : u \prec_\sigma v\}$. Then, two vertices of H are strongly adjacent in H if and only if they are strongly adjacent in H', i.e., $\text{bi-dir}(H) = \text{bi-dir}(H')$. Furthermore, X'_i induces a complete subgraph in H', i.e., X'_i is a clique in $\text{bi-dir}(H')$. Then, $\langle x_n, \ldots, x_1 \rangle$ is a perfect elimination scheme for $\text{bi-dir}(H') = \text{bi-dir}(H)$, from which follows that $\text{bi-dir}(H)$ is chordal, and since every set X'_i does not contain more than k vertices, $\text{bi-dir}(H)$ has treewidth at most k due to Theorem 4.1. □

Let us mention that for k-DAGs the statement analogous to Theorem 4.2 becomes: *an undirected graph G is a chordal graph of treewidth $k-1$ or k if and only if there is a k-DAG H such that $G = \text{bi-dir}(H)$.*

Corollary 4.3 *Let $k \geq 0$. An undirected graph G has treewidth at most k if and only if there is a partial k-GDAG H such that $G = \text{bi-dir}(H)$.*

Proof Let G be an undirected graph of treewidth at most k. Then, there is a k-tree G' containing G as partial graph. G' is a chordal graph of treewidth at most k, so there is a k-GDAG H' such that $G' = \text{bi-dir}(H')$ due to Theorem 4.2. Hence, H' has a partial graph H such that $G = \text{bi-dir}(H)$. The converse is analogous. □

Combining the result of Corollary 4.3 and the characterization of Theorem 3.5 provides the following characterization of undirected graphs of bounded treewidth. The definitions of σ-monotone-left paths and spanning-vertex u property for undirected graphs are obtained just by replacing 'arc' by 'edge'. Note that the end vertex of an edge uv with respect to a vertex ordering then is the one vertex preceding the other.

Theorem 4.4 *Let $k \geq 0$, and let $G = (V, E)$ be an undirected graph. G has treewidth at most k if and only if there are a vertex ordering $\sigma = \langle x_1, \ldots, x_n \rangle$ for G and a set F of additional edges such that the triple (G, F, σ) satisfies the following two conditions, where we set $G' =_{\text{def}} G \cup F$:*

(1) for every $i \in \{1, \ldots, n\}$, $|N_{G'}(x_i) \cap \{x_1, \ldots, x_{i-1}\}| \leq k$

(2) for every pair u, v of vertices of G where $u \prec_\sigma v$, if uv is an edge of G then every σ-monotone-left path starting at v in G' has the spanning-vertex u property in G'.

Proof Let G have treewidth at most k. Due to Corollary 4.3, there is a partial k-GDAG H such that $G = \text{bi-dir}(H)$. Applying Theorem 3.5, there are a vertex ordering σ for H and a set F' of arcs such that the triple (H, F', σ) satisfies the two conditions of Theorem 3.5. Without loss of generality, if $(u, v) \in F'$ then $v \prec_\sigma u$. Let $F =_{\text{def}} \{uv : (u, v) \in F'\}$. We show that the triple (G, F, σ) satisfies the two conditions. Condition (1) is clearly satisfied, since every neighbour of x preceding x with respect to σ in $G \cup F$ is a preceding out-neighbour of x in $H \cup F'$. For satisfaction of condition (2), note that if uv, $u \prec_\sigma v$, is an edge of G then (u, v) is an arc of H, and every σ-monotone-left path starting at v in $G \cup F$ is a σ-monotone-left path starting at v in $H \cup F'$. Thus, satisfaction of condition (2) follows from Theorem 3.5.

For the converse, let σ be a vertex ordering for G and F a set of additional edges such that the triple (G, F, σ) satisfies conditions (1) and (2). Let H be an arc-minimal digraph such that $\text{bi-dir}(H) = G$. Let $F' =_{\text{def}} \{(v, u) : uv \in F \text{ and } u \prec_\sigma v\}$. Then, it is easy to verify that the triple (H, F', σ) satisfies conditions (1) and (2) of Theorem 3.5, which means that H is a partial k-GDAG. Applying Corollary 4.3, we conclude that G has treewidth at most k. □

The concept of a σ-monotone-left path having the spanning-vertex u property is unambiguously translated between undirected graphs and directed graphs. Thus, apart from the binary choice of translating 'neighbours' to either 'in-neighbours' or 'out-neighbours', all undirected graph concepts used in Theorem 4.4 to characterize treewidth are unambiguously translated to give Theorem 3.5 characterizing Kelly-width. In our opinion this constitutes a weighty argument that Kelly-width is indeed the natural directed analogue of treewidth.

5 A fast algorithm for recognition of digraphs of Kelly-width 2

Theorem 3.2 gives an algorithm for recognition of digraphs of bounded Kelly-width: a graph has Kelly-width at most $k+1$ if and only if it can be reduced to a graph on a single vertex by repeatedly reducing by a vertex of out-degree at most k (Theorem 3.2). A polynomial-time algorithm does not evolve directly from this result, since it is not clear which of the possible vertices to choose. However, in this section we show that it does give a polynomial-time algorithm for Kelly-width 2. In fact, we will show that every choice of a vertex is then a good choice.

For graphs of Kelly-width 2, vertices of out-degree 0 and 1 can be chosen. We treat the two cases separately. The main difference between both cases is that reducing a graph by a vertex of out-degree 0 does not change the remaining graph, whereas reducing by a vertex of out-degree 1 may add new arcs between vertices in the remaining graph. At first, we consider the out-degree 0 case. We can even show a general result: the Kelly-width of a digraph is not influenced by vertices of out-degree 0.

Theorem 5.1 *Let $k \geq 0$, and let $G = (V, A)$ be a digraph on at least two vertices. Let a be a vertex of G of out-degree 0. Then, G is a partial k-GDAG if and only if the graph obtained from reducing G by a is a partial k-GDAG.*

Proof Let G' be the graph obtained from reducing G by a. Since a does not have any out-neighbours, the reduction operation does not add any new arcs to G'. So, G' is a subgraph of G, which means that G' is a partial k-GDAG due to Theorem 3.7. For the converse, let G' be a partial graph of k-GDAG H'. Let H emerge from H' by adding a according to construction step (2) choosing no parent vertices. By definition, H is a k-GDAG and a is an in-universal vertex in H. Hence, G is a partial graph of H, i.e., a partial k-GDAG. □

The proof of the out-degree 1 case is more complicated. The most natural approach to prove an upper bound for the Kelly-width of a graph is to use the subgraph characterization. Consider the following situation: given a digraph G and a k-GDAG H containing G. Let G' be the result of reducing G by a vertex x of out-degree 1. Certainly, G' does not have to be a subgraph of H. To show that G' still has Kelly-width k, another k-GDAG has to be found that contains G' as subgraph. Such a graph is obtained by performing an operation on H that definitely produces a supergraph of G' and does not increase the Kelly-width. We cannot just perform a reduction on H, since x may have more than one out-neighbour in H. We choose a more selective form of reduction. Let $G = (V, A)$ be a digraph, and let (a, b) be an arc of G. The graph obtained from *in-contracting arc* (a, b) in G, denoted as $G \triangleleft_i (a, b)$, is defined as

$$(G - a) \cup \{(x, b) : x \neq b \text{ and } x \in N_G^{\text{in}}(a)\} .$$

Informally spoken, arc (a, b) is in-contracted by deleting vertex a and making every in-neighbour of a an in-neighbour of b. If a has out-degree 1, in-contracting arc (a, b) is exactly what we mean by reducing by vertex a.

Lemma 5.2 *Let $k \geq 0$, and let $G = (V, A)$ be a k-GDAG. Let σ be a construction sequence for G, and let (b, a) be an arc of G where $a \prec_\sigma b$. Then, $G \triangleleft_i (b, a)$ is a partial graph of a k-GDAG H with construction sequence $\sigma - b$.*

Proof Let $\sigma = \langle x_1, \ldots, x_n \rangle$, and let $\sigma' =_{\text{def}} \sigma - b$. Let $X_i =_{\text{def}} N_G^{\text{out}}(x_i) \cap \{x_1, \ldots, x_{i-1}\}$. We define the following sets:

$$X_i' =_{\text{def}} \begin{cases} X_i & \text{, if } b \notin X_i \\ (X_i \setminus \{b\}) \cup \{a\} & \text{, if } b \in X_i \,. \end{cases}$$

Let H be the k-GDAG with construction sequence σ' and parent vertices sets X_i'. Note that this can be done, since we assumed $a \prec_\sigma b$. We show that $G' =_{\text{def}} G \triangleleft_i (b, a)$ is a partial graph of H. By definition of H, $A(H) \cap \{(v, u) : u \prec_{\sigma'} v\} = A(G') \cap \{(v, u) : u \prec_{\sigma'} v\}$. Note that there may be arcs (u, b) for $a \prec_\sigma u \prec_\sigma b$ in G, which become arcs (u, a) in G', i.e., that change direction with respect to σ. Let (u, b) be such an arc of G. Since (b, a) is an arc of G there is a σ-monotone-left path P starting at b in G that does not contain u and $z \prec_\sigma u$ for z the last vertex of P. Due to condition (2) of Theorem 3.3 and since (b, a) spans over u, a is an out-neighbour of u. Hence, (u, a) is an arc of H by definition.

Now, let $u \prec_{\sigma'} v$ and let (u, v) be an arc of G'. Let P be a σ'-monotone-left path starting at v in H that does not contain u and $z \prec_{\sigma'} u$ for z the last vertex of P. If P is a σ-monotone-left path starting at v in G, let $Q =_{\text{def}} P$. Assume that P is not a σ-monotone-left path starting at v in G. This can only be the case, if P in H contains an arc that is not an arc in G. By definition of H, P then contains a and the arc (x, a) for some vertex $x \in N_G^{\text{in}}(b)$. Then, "adding" b between x and a gives a σ-monotone-left path Q starting at v in G. Note that we denote both possible paths by Q. We distinguish two cases. Let (u, v) be an arc of G. Due to condition (2) of Theorem 3.3, u has an out-neighbour $w \prec_\sigma u$ on Q and the arc with end vertex w spans over u. If $w = b$, then a is an out-neighbour of u in H, a is a vertex on Q (by the second case of the definition of Q), $a \prec_{\sigma'} u$ and the arc of Q with end vertex a spans over u. If $w \neq b$, then w is a vertex on Q, and the path has the property of condition (2). Let (u,v) be not an arc of G, i.e., $(u,v) = (u,a)$ and (u,b) is an arc of G. But such arcs do not exist, since every in-neighbour $w \prec_\sigma a$ of b is an in-neighbour also of a according to construction step (2). Hence, G' is a partial graph of H, thus a partial k-GDAG. □

The second lemma that we need for our result above reducing a graph by a vertex of out-degree 1 is stated only for 1-GDAGs.

Lemma 5.3 *Let $G = (V,A)$ be a 1-GDAG with construction sequence σ. Let (a,b) be an arc of G where $a \prec_\sigma b$. Then, $G \triangleleft_i (a,b)$ is a partial 1-GDAG.*

Proof Let $\sigma = \langle x_1, \ldots, x_n \rangle$, and let X_i be the parent vertices sets. Let $G' =_{\text{def}} G \triangleleft_i (a,b)$. Let P be the maximal σ-monotone-left path starting at b in G. We distinguish three cases. Let a be a vertex on P. Let c be the predecessor of $x_s = a$ on P. We define the following sets:

$$X'_i =_{\text{def}} \begin{cases} X_i & \text{, if } a \notin X_i \\ \{c\} & \text{, if } a \in X_i \text{ and } x_i \prec_\sigma c \\ \{b\} & \text{, if } a \in X_i \text{ and } c \prec_\sigma x_i \\ X_s & \text{, if } x_i = c\,, \end{cases}$$

$i \in \{1, \ldots, n\}$. Let F be the digraph on vertex set $V' =_{\text{def}} \{x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n\}$ and with the arcs (x_i, y) for $x_i \in V'$ and $y \in X'_i$. Let $T_1, \ldots, T_k$ be the (weakly) connected components of F, and let $r_1, \ldots, r_k$ be the uniquely defined vertices in $T_1, \ldots, T_k$ with out-degree 0, respectively. Without loss of generality, we can assume $r_1 \prec_\sigma \cdots \prec_\sigma r_k$. For every connected component of F, we define a vertex ordering. Let σ'_i, $i \in \{1, \ldots, k\}$, be a vertex ordering on $V(T_i)$ satisfying the following two conditions:

(1) if $x \in X'_j$ then $x \prec_{\sigma'_i} x_j$

(2) if $X'_j = X'_{j'}$ then $x_j \prec_{\sigma'_i} x_{j'} \Leftrightarrow x_j \prec_\sigma x_{j'}$.

Note that these orderings exist. We obtain vertex ordering σ' by concatenation: $\sigma' =_{\text{def}} \sigma'_1 \circ \cdots \circ \sigma'_k$. Let H' be the 1-GDAG obtained from σ' and the parent vertices sets $X'_1, \ldots, X'_n$; condition (1) and the partition into the connected components of F ensure that H' can be constructed correctly. We show that G' is a partial graph of H'. By construction, for every pair u, v of vertices of G', $u \prec_{\sigma'} v$, if (v,u) is an arc of G' then (v,u) is an arc of H': we added new such arcs only for vertices that are child vertices of a in G (and σ' satisfies condition (1) and the concatenation of $\sigma'_1, \ldots, \sigma'_k$ respects the ordering of $r_1, \ldots, r_k$ with respect to σ). Now, let (u,v) be an arc of G' where $u \prec_{\sigma'} v$. Let P' be a σ'-monotone-left path starting at v in H', and let P' strictly pass u. Note that this means that u and v are in the same connected component of F by definition of σ'. Let P' do not contain c. Then, P' is a σ-monotone-left path starting at v in G. Let $w \prec_\sigma u$ be the out-neighbour of u on P' in G, that exists due to Theorem 3.3. Let w' be the predecessor of w on P'; due to Theorem 3.3; $w \prec_\sigma u \prec_\sigma w'$. Hence, according to condition (2) of the definition of σ', $w \prec_{\sigma'} u \prec_{\sigma'} w'$, and the arc (w, w') spans over u, i.e., (u,v) is an arc of H'. Let P' contain c. Then, the maximal σ-monotone-left path Q starting at v in G contains a. Let x be the predecessor of a in Q in G. We distinguish the following three cases:

- $x \prec_\sigma c$: then x is the predecessor of c on P' in H', and Q is obtained from P' by replacing c by a. Furthermore, Q does not contain u, so contains an out-neighbour $w \prec_\sigma u$ of u. Then, P' contains w, and the arc of P' with end vertex w spans over u, i.e., (u, v) is an arc of H'.

- $c \prec_\sigma x$: then x is a child vertex of b in H', and Q is obtained from P' by replacing the subpath of P' from b to c by a. If u is a child vertex of a in G, then u is a child vertex of b or c in H', and according to condition (2) of the definition of σ', we conclude that (u, v) is an arc of H'.

- $c = x$: then, Q is obtained from P' by adding a as the successor of c on Q. We conclude that (u, v) is an arc of H'.

In particular, $N^{\text{in}}_G(a) \subseteq N^{\text{in}}_{H'}(b)$. Hence, G' is a partial graph of H', and therefore a partial 1-GDAG.

Let P pass a. Let w be the out-neighbour of a on P in G such that $w \prec_\sigma a$ and the arc of P with end vertex w spans over a. Consider $G'' =_{\text{def}} G \triangleleft_i (a, w)$. According to Lemma 5.2, there is a 1-GDAG H with construction sequence $\sigma - a$ that contains G'' as partial graph. In particular, $N^{\text{in}}_G(a) \subseteq N^{\text{in}}_H(w) \subseteq N^{\text{in}}_H(b)$, where the second inclusion is an immediate consequence of the characterization of Theorem 3.3. Hence, G' is a partial graph of 1-GDAG H, i.e., a partial 1-GDAG.

Let P do not contain a and do not pass a. Reconsider the definition of graph F above. Using the parent vertices sets $X_1, \ldots, X_n$, we construct an analogous graph and define an analogous 1-GDAG H'. It holds that a and b are not contained in the same connected component of F, and by definition of 1-GDAGs, the vertices in the same connected component of F with b are out-neighbours of every vertex in the same connected component of F as a. Hence, G' is a subgraph of H', i.e., a partial 1-GDAG due to Theorem 3.7. □

Theorem 5.4 *Let $G = (V, A)$ be a partial 1-GDAG, and let (a, b) be an arc of G. Then, $G \triangleleft_i (a, b)$ is a partial 1-GDAG.*

Proof Let G be a partial graph of 1-GDAG H, and let σ be a construction sequence for H. Let $G' =_{\text{def}} G \triangleleft_i (a, b)$ and $H' =_{\text{def}} H \triangleleft_i (a, b)$. Obviously, G' is a partial graph of H'. If $b \prec_\sigma a$, then H' is a partial graph of a 1-GDAG due to Lemma 5.2. If $a \prec_\sigma b$, then H' is a partial graph of a 1-GDAG due to Lemma 5.3. Hence, G' is a partial 1-GDAG. □

Corollary 5.5 *Let $G = (V, A)$ be a digraph, and let a be a vertex of out-degree 1 of G. Then, G is a partial 1-GDAG if and only if the graph obtained from reducing G by a is a partial 1-GDAG.*

Proof Let G' be the digraph obtained from reducing G by a. Let b be the out-neighbour of a in G. Then, $G' = G \triangleleft_i (a, b)$, and G' is a partial 1-GDAG due to Theorem 5.4. For the converse, let G' be a partial graph of 1-GDAG H'. We add vertex a to H' according to construction step (2) choosing b as the parent vertex of a and obtain 1-GDAG H. Since $N^{\text{in}}_G(a) \subseteq N^{\text{in}}_{G'}(b) \subseteq N^{\text{in}}_{H'}(b)$, $N^{\text{in}}_G(a) \subseteq N^{\text{in}}_H(a)$, which means that G is a partial graph of H, i.e., a partial 1-GDAG. □

The result of Theorem 5.4 is stronger than the corresponding implication of Corollary 5.5: reducing a graph by a vertex of out-degree 1 can be simulated by an appropriate in-contraction operation. Equivalence only holds, if the start vertex of the in-contracted arc has out-degree 1. It is a natural question to ask whether Corollary 5.5 holds for partial k-GDAGs for $k \geq 2$, which means to ask whether a generalised version of Lemma 5.3 can be proved. The comparison with reduction results for undirected graphs of bounded treewidth shows that there is little hope to generalise Corollary 5.5 to reducing vertices of out-degree more than 2.

Using the two main results about reducing a graph, we obtain a characterization of graphs of Kelly-width 2 that is stronger than Theorem 3.2.

Theorem 5.6 *Let $G = (V, A)$ be a digraph. Then, G is a partial 1-GDAG if and only if G can be reduced to a graph on one vertex by repeatedly reducing by an arbitrary vertex of out-degree at most 1.*

Proof We show the statement by induction over the set of vertices. Clearly, a digraph on one single vertex is a partial 1-GDAG (construction step (1)). Let G have $n \geq 2$ vertices, and let the statement be true for all digraphs on $n - 1$ vertices. Let u be a vertex of G of out-degree at most 1, and let G' be the digraph obtained from reducing G by u. Due to Theorem 5.1 and Corollary 5.5, G' is a partial 1-GDAG if and only if G is a partial 1-GDAG, and due to induction hypothesis, G' is a partial 1-GDAG if and only if G' can be reduced to a digraph on one vertex by always choosing an arbitrary vertex of out-degree at most 1. Since u was chosen arbitrarily, we conclude the statement. □

The result of Theorem 5.6 implies a fast algorithm for recognition of graphs of Kelly-width at most 2. The reduction sequence can even be used to construct a witness for being a Kelly-width-2 graph: a 1-GDAG containing the input graph G. In the negative case, our algorithm outputs a graph H' and a vertex sequence $\sigma' = \langle x_r, \ldots, x_n \rangle$ of the following kind: let $G_n =_{\text{def}} G$, and let G_i be obtained from G_{i+1} by reducing by x_{i+1}, $i \in \{r, \ldots, n-1\}$. Then, x_{i+1}, $i \in \{r, \ldots, n-1\}$, has out-degree at most 1 in G_{i+1}, and G_r does not contain any vertex of out-degree at most 1. Due to Theorem 5.6, G can therefore not be a partial 1-GDAG. The output witness H' is defined as follows: let $H'_r =_{\text{def}} G_r$, and obtain H'_{i+1} from H'_i by adding x_{i+1} according to construction step (2) of Definition 2.1 choosing as parent vertex the out-neighbour of x_{i+1} in G_{i+1}, if there is one. Then, $H' =_{\text{def}} H'_n$. Using σ' it is easy to verify by the user of the algorithm that H' arises from G and thus the algorithm worked correctly. Moreover, H' is independent of σ' in the sense that on input H' any reduction sequence on vertices of out-degree at most 1 will result in G_r.

Theorem 5.7 *There is an algorithm that, given a digraph G, decides whether G has Kelly-width at most 2, and if so, outputs a 1-GDAG H that contains G as partial graph. If G has Kelly-width at least 3, the algorithm outputs a witness for this case, which is a graph containing G as partial graph and a vertex sequence. The running time and the working space of the algorithm are linear in the size of the output graph.*

Proof The algorithm is simple: choose an arbitrary vertex of out-degree at most 1 and reduce the graph by this vertex. This elimination process can be carried out until the graph contains only one vertex if and only if G has Kelly-width at most 2 (Theorem 5.6). Furthermore, the proofs of Theorem 5.1 and Corollary 5.5 show that the reverse order in which vertices are eliminated is a construction sequence of a 1-GDAG H that contains G as partial graph, and H can be constructed in time linear in its size. If G is not a graph of Kelly-width at most 2, there will be an elimination step in which no vertex of out-degree at most 1 can be chosen. So, let H' be the obtained reduced graph, and let $\langle x_i, \ldots, x_n \rangle$ be the already generated construction sequence. We add the vertices $x_i, \ldots, x_n$ to H' in the sense of construction step (2) for k-GDAGs choosing one or no parent vertex. We obtain a graph that contains G as subgraph, and we output this graph and the sequence $\langle x_i, \ldots, x_n \rangle$.

For the running time of the reduction algorithm, it is first to note that, after every reduction step, the resulting graph is a subgraph of H. Reducing by a vertex takes time linear in the number of is neighbours, since the in-neighbours of the reduced vertex become in-neighbours of the single out-neighbour, if there is one. Let a be eliminated and let b be its only out-neighbour. The crucial point is to find the in-neighbours of a that are in-neighbours also of b. These are exactly the vertices whose out-degree is decreased by 1. We assume that the adjacency lists of the vertices are ordered with respect to some ordering. The intersection of the two in-neighbourshoods is computed by just scanning the two lists. The reason that this is linear time even if the in-neighbourhood list of b is larger than that of a is that the in-degree of a in H is not smaller than the in-degree of b. This gives linear running time and working space in the size of input and output graph. □

For deciding whether a graph has Kelly-width exactly 2, it suffices to run the algorithm only for non-acyclic graphs. Acyclic graphs are exactly the graphs of Kelly-width 1.

6 Final remarks

Since Kelly-width of a directed graph is a new concept, a lot of problems can still be solved. The most important question, however, affects the status of Kelly-width: does it really capture the notion of treewidth for undirected graphs in the directed setting? We gave good reasons to answer positively: we presented two new characterizations of digraphs of bounded Kelly-width, and we gave an easy algorithm for recognition of digraphs of Kelly-width 2. This recognition algorithm can be considered a directed version of the undirected counterpart: a reduction algorithm for graphs of treewidth at most 1. Recognition algorithms for graphs of bounded Kelly-width are of great interest, since one can expect that they also compute a Kelly-decomposition, which is important for the design of algorithms solving optimization problems on digraphs of bounded Kelly-width.

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