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On Self Duality of Pathwidth in Polyhedral Graph Embeddings*

Fedor V. Fomin[†] *Dimitrios M. Thilikos*[‡]

March 14, 2006

Abstract

Let G be a 3-connected planar graph and G^* be its dual. We show that the pathwidth of G^* is at most 6 times the pathwidth of G. We prove this result by relating the pathwidth of a graph with the cutwidth of its medial graph and we extend it to bounded genus embeddings. We also show that there exist 3-connected planar graphs such that the pathwidth of such a graph is at least 1.5 times the pathwidth of its dual.

1 Introduction

In 1990 Golovach [5] observed the following fact: For every Platonic graph G the edge search number of G is equal to the node search number of its dual. Thus for example, the edge search number of the dodecahedral graph is equal to the node search number of icosahedral graph. (See the survey [1] on search games and related parameters.) Later this result was generalized as follows: For every 2-connected plane graph G with maximum vertex degree at most three and each face bordered by at most five edges, the edge search number of G is equal to the node search number of G^* [6]. By the well known relation between search numbers and the pathwidth of a graph, this result implies that for every such graph G, the pathwidth of G is always between pathwidth$(G^*) - 1$ and pathwidth$(G^*) + 1$. Let us note that 2-connectivity condition is important here because trees can have arbitrary large pathwidth while their duals are of pathwidth zero. In fact, the results in [5] triggered the following conjecture of Golovach [5]:

Conjecture 1. *For every 2-connected plane graph G the edge search number of G is equal to the node search number of its dual G^**

For $c \geq 0$, we say that a graph parameter $\mathbf{p}$ is *additively c-self dual on a subclass $\mathcal{G}$ of plane graphs* if for every graph $G \in \mathcal{G}$ and for its geometrical dual G^*, $\mathbf{p}(G^*) \leq \mathbf{p}(G) + c$. So Conjecture 1 would imply that the pathwidth is additively 1-self dual on 2-connected plane graphs.

There are two width parameters, related to pathwidth, namely, branchwidth and treewidth, known to be additively self dual. In case of branchwidth, it follows from the results in [13] that it is additively 1-self dual for planar graphs in general and additively 0-self dual for planar graphs that are not trees. The additively 1-self duality of treewidth was first claimed in [10] for general planar graphs. The first proof of this fact appeared in [8] (see also [3] for a simpler proof.)

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One of the results obtained in this paper (Section 6) implies that for each number $c \geq 0$ pathwidth is *not* additively c-self dual on 2-connected plane graphs. This disproves Conjecture 1. In order to further explore the self duality of pathwidth, we define a weaker version of it: A graph parameter $\mathbf{p}$ is *multiplicatively c-self dual for a subclass $\mathcal{G}$ of plane graphs* if there exists a constant $d \geq 0$ such that for every graph $G \in \mathcal{G}$ and its dual G^*, $\mathbf{p}(G^*) \leq c \cdot \mathbf{p}(G) + d$. It is known that pathwidth is multiplicatively 2-self dual on 2-connected outerplane graphs [2] and 3-self dual on Halin graphs [4]. The main result of this paper is that, for simple 3-connected planar graphs, pathwidth is multiplicatively 6-self dual. Actually we prove a more general result, by showing that for every polyhedral embedding of a graph G in some surface of oriented genus g, the pathwidth of G is at most $6 \cdot (\text{pathwidth}(G^*) + g - 2)$, where G^* is the geometric dual of G (Section 4). On the other side, we show that on 3-connected planar graphs, pathwidth fails to be multiplicatively c-self dual for every $c < 1.5$ (Section 6).

2 Definitions

All graphs in this paper are simple, i.e. without loops and multiple edges except if the opposite is explicitly mentioned. For a graph G we denote by $V(G)$ and $E(G)$ its vertex and edge sets respectively. We use $N_G(v)$ to denote the set of vertices adjacent to v in G. The *degree* of $v \in V(G)$ is the number of vertices in $N_G(v)$. We denote by $K_{2,3}$ the complete bipartite graph with a bipartition of sizes two and three.

Let R be a subset of $V(G)$ and S be subset of $E(G)$. In each case, we set $R^c = V(G) \setminus R$ and $S^c = E(G) \setminus S$. If $S \subseteq E(G)$, we use the notation $V(S)$ for the set of endpoints of the edges in S, i.e. $V(S) = \bigcup_{\{u,v\} \in S}(\{u\} \cup \{v\})$. We define

$$\begin{aligned} \partial_G R &= \{e \in E(G) \mid e \cap R \neq \emptyset \text{ and } e \cap R^c \neq \emptyset\}, \text{ and} \\ \delta_G S &= V(S) \cap V(S^c). \end{aligned}$$

In other words, $\partial_G R$ contains all edges that have one endpoint in R and one endpoint in R^c and $\delta_G S$ contains all vertices that are endpoints of edges in S and endpoints of edges in S^c.

A linear ordering (or just an ordering) L of a set S is a bijection $L\colon S \to \{1, \ldots, |S|\}$. Often it will be convenient to denote an ordering by using it to index the set, so that $L(s_i) = i$ for $1 \leq i \leq n$ where i will be referred to as the label of s_i. For a set S we denote by $\mathcal{L}_S$ the set of all linear orderings of S. Let $L = (s_1, \ldots, s_{|S|}) \in \mathcal{L}_S$ be a linear ordering of S. We define

$$\mathbf{pref}(L) = \{\{s_1, \ldots, s_i\} \mid i \in \{1, \ldots, |S|\}.$$

The *cut-width* and the *linear-width* of a graph G (denoted as $\mathbf{lw}(G)$ and $\mathbf{cw}(G)$ respectively) are defined as follows

$$\begin{aligned} \mathbf{cw}(G) &= \min_{L \in \mathcal{L}_{V(G)}} \max_{R \in \mathbf{pref}(L)} |\partial_G R| \\ \mathbf{lw}(G) &= \min_{L \in \mathcal{L}_{E(G)}} \max_{S \in \mathbf{pref}(L)} |\delta_G S|. \end{aligned}$$

A *tree decomposition* of a graph G is a pair (X, U) where U is a tree and $X = (\{X_i \mid i \in V(U)\})$ is a collection of subsets of $V(G)$ such that

1. $\bigcup_{i \in V(U)} X_i = V(G)$,
2. for each edge $\{v, w\} \in E(G)$, there is an $i \in V(U)$ such that $v, w \in X_i$, and
3. for each $v \in V(G)$, the set of nodes $\{i \mid v \in X_i\}$ forms a connected subtree of U.

The *width* of a tree decomposition $(\{X_i \mid i \in V(U)\}, U)$ equals $\max_{i \in V(U)}\{|X_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G.

A *path decomposition* of a graph G is a tree decomposition (X, U) where U is a path. We denote a path decomposition as a sequence $\mathcal{X} = (X_1, \ldots, X_r)$. The *width* of $\mathcal{X}$ equals to $\max_{1 \leq i \leq r}\{|X_i| - 1\}$ and the *pathwidth* of a graph G (we denote it as $\mathbf{pw}(G)$) is the minimum width over all path decompositions of G. Linear-width and pathwidth are closely related as indicated by the following result.

Proposition 1 ([4]). *For any graph G, $\mathbf{pw}(G) \leq \mathbf{lw}(G) \leq \mathbf{pw}(G) + 1$.*

3 Surfaces, duals, medials, and radials

Let Σ be a surface. A *line* in Σ is subset homeomorphic to $(0, 1)$. An O-arc is a subset of Σ homeomorphic to a circle. For a graph G we use the notation (G, Σ) to denote an embedding of G in Σ. A subset of Σ meeting the drawing only at vertices of G is called G-*normal*. If an O-arc is G-normal, then we call it a *noose*. The length of a noose is the number of its vertices.

Given a set $S \subseteq \Sigma$, we use the notation $\overline{S}$ for the closure of S. To simplify notations we do not distinguish between a vertex of an embedded graph and the point of Σ used in the drawing to represent the vertex or between an edge and the open line segment representing it. That way we consider G as the union of the points corresponding to its vertices and edges and a subgraph H of G can be seen as $H \subseteq G$. We call by *face* of G every maximal connected component of $\Sigma - E(G) - V(G)$ (every face is an open set). We use the notation $V(G), E(G)$, and $F(G)$ for the set of the vertices, edges and faces of G. An edge e (a vertex v) is incident with a face r if it belongs to its closure.

Representativity [11, 12] is the measure of the extent of the local planarity of a graph embedded in a surface. The *representativity* (or *face-width*) $\mathbf{rep}(G, \Sigma)$ of a graph embedding (G, Σ) is the smallest length of a non-contractible noose in Σ. If the oriented genus of Σ is 0, we put $\mathbf{rep}(G, \Sigma) = +\infty$ We call an embedding (Σ, G) *polyhedral* [9] if G is 3-connected and $\mathbf{rep}(\Sigma, G) \geq 3$.

For a given embedding (G, Σ), we denote by (G^*, Σ) its dual embedding. Thus G^* is the geometric dual of G. Each vertex v (face r) in (G, Σ) corresponds to some face v^* (vertex r^*) in (G^*, Σ). Also, given a set $S \subseteq E(G)$, we denote as S^* the set of the duals of the edges in S.

Let (G, Σ) be an embedding and let (G^*, Σ) be its dual. We define *radial graph embedding* (R_G, Σ) of (G, Σ) (also known as vertex-face graph embedding) as follows: R_G is an embedded bipartite graph with vertex set $V(R_G) = V(G) \cup V(G^*)$. For each pair $e = \{v, u\}$, $e^* = \{u^*, v^*\}$ of dual edges in G and G^*, R_G contains edges $\{v, v^*\}, \{v^*, u\}, \{u, u^*\}$, and $\{u^*, v\}$.

The following proposition can be found in [9].

Proposition 2 ([9]). *The following conditions are equivalent:*

- (G, Σ) *is a polyhedral embedding;*
- (G^*, Σ) *is a polyhedral embedding;*
- (R_G, Σ) *has no multiple edges and every 4-cycle of* R_G *is the border of some face.*

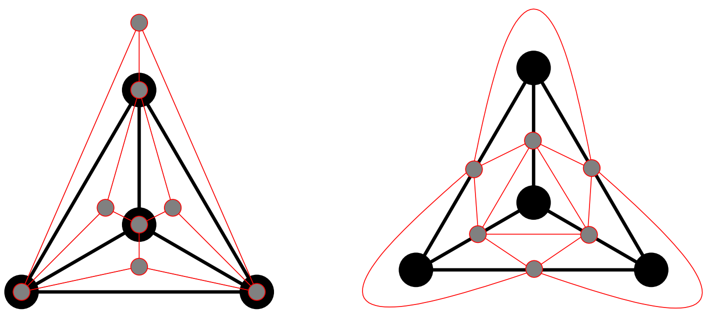

Figure 1: A planar embedding of K_4 along with its radial graph and its medial graph.

The *medial graph embedding* (M_G, Σ) of (G, Σ) is the dual embedding of the radial embedding (R_G, Σ) of (G, Σ). Notice that (M_G, Σ) is a Σ-embedded 4-regular graph. (See Fig. 1 for an example of radial and medial graphs.)

Notice that two dual polyhedral graph embeddings (G, Σ) and (G^*, Σ) have isomorphic radial graphs and medial graphs.

We will also need the following auxiliary result.

Lemma 1. *Let* (G, Σ) *be a polyhedral embedding and let* (M_G, G) *be its medial embedding. Then* $\mathbf{lw}(G) \geq 3$ *and* $\mathbf{cw}(M_G) \geq 6$.

Proof. By the definition of polyhedral embeddings, G is 3-connected and has at least three vertices. Then for every two edges e, f of G, $|\delta_G(\{e\} \cup \{f\})| \geq 3$. Thus $\mathbf{lw}(G) \geq 3$.

Every vertex of M_G is of degree four. Thus for every pair of vertices v, u of M_G, $|\partial_{M_G}(\{v\} \cup \{u\})| \geq 6$ and $\mathbf{cw}(M_G) \geq 6$. □

The definitions of (R_G, Σ) and (M_G, Σ) establish the bijections

$$\rho : E(G) \to F(R_G)$$

mapping the edges of G to the regions of R_G, and

$$\mu : E(G) \to V(M_G)$$

mapping the edges of G to the vertices of M_G. For $S \subseteq E(G)$, we define $\rho(S) = \{\rho(e) \mid e \in S\}$ and $\mu(S) = \{\mu(e) \mid e \in S\}$.

4 The main result

Lemma 2. *Let* (G, Σ) *and* (G^*, Σ) *be dual polyhedral embeddings in the surface of oriented genus* g *and let* (G_M, Σ) *be the medial graph embedding. Then for each* $S \subseteq E(G)$, $|S| \geq 2$,

$$\max\{|\delta_{G^*} S^*|, |\delta_G S|\} \leq \frac{|\partial_{M_G} \mu(S)|}{2} \leq 6(\min\{|\delta_{G^*} S^*|, |\delta_G S|\} + g - 2).$$

We postpone the proof of Lemma 2 till the next section. Instead, we present the main result of this paper that follows easily from Lemma 2, and the definitions of linear-width and cutwidth.

Theorem 1. *Let* (G, Σ) *and* (G^*, Σ) *be dual polyhedral embeddings in the surface of oriented genus* g *and let* (G_M, Σ) *be the medial graph embedding. Then*

$$\mathbf{pw}(G^*) \leq \mathbf{cw}(M_G)/2 \leq 6 \cdot (\mathbf{pw}(G) + g - 1).$$

Proof. By Proposition 1, it is enough to prove that

$$\mathbf{lw}(G^*) \leq \mathbf{cw}(M_G)/2 \leq 6 \cdot (\mathbf{lw}(G) + g - 2).$$

Let $K = (v_1, \ldots, v_m)$ be a linear ordering of $V(M_G)$ such that $|\partial_{M_G} R| \leq k$ for every $R \in \mathbf{pref}(K)$. By Lemma 1, $k \geq 6$. The bijection μ maps vertices of M_G to edges of G, and we consider the corresponding linear ordering $L = (\mu^{-1}(v_1), \ldots, \mu^{-1}(v_m))$ of $E(G)$. Let L^* be the ordering of $E(G^*)$ containing the dual edges of L in the same order as they appear in L. Let $S^* \in \mathbf{pref}(L^*)$. If $|S^*| = 1$, then $|\delta_G S^*| = 2 < k/2$. If $|S| \geq 2$, then by Lemma 2, $|\delta_G S^*| \leq |\partial_{M_G} \mu(S)|/2 \leq k/2$.

Let now $L = (e_1, \ldots, e_m)$ be a linear ordering of $E(G)$ with $|\delta_G S| \leq k$ for every $S \in \mathbf{pref}(L)$. By Lemma 1, $k \geq 3$. Consider the linear ordering $K = (\mu(e_1), \ldots, \mu(e_m))$ of $V(M_G)$ and let $R \in \mathbf{pref}(K)$. If $|R| = 1$, then $|\partial_{G_M}(R)|/2 = 2 \leq 6 \cdot (k + g - 2)$. If $|R| \geq 2$, then $|\mu^{-1}(R)| \geq 2$ and by Lemma 2, we have that $|\partial_{M_G} R|/2 \leq 6 \cdot (\delta_G \mu^{-1}(R) + g - 2) \leq 6 \cdot (k + g - 2)$. □

5 Proof of Lemma 2

Let (G, Σ) and (G^*, Σ) be dual polyhedral embeddings in a surface of oriented genus g and let (G_M, Σ) be the medial graph embedding.

For $F \subseteq F(G)$ we define $F^c = F(G) \setminus F$ and the graph

$$\partial_G F = (\cup_{r \in F} \overline{r}) \cap (\cup_{r \in F^c} \overline{r}).$$

In other words, $\partial_G F$ is the graph containing the vertices and the edges that are on the border of faces in F and faces in F^c.

Recall that M_G and R_M are dual graphs and therefore, for every $S \subseteq E(G)$, $|\partial_{M_G}\mu(S)| = |E(\partial_{R_G}\rho(S))|$. By this fact, to prove Lemma 2, it is sufficient to show that

$$\max\{|\delta_{G^*}S^*|, |\delta_G S|\} \leq \frac{|E(\partial_{R_G}\rho(S))|}{2} \leq 6(\min\{|\delta_{G^*}S^*|, |\delta_G S|\} + g - 2) \tag{1}$$

The set of edges S of G corresponds to the set of faces F of R_G, i.e. $F = \rho(S)$. The graph $H = \partial_{R_G}F = \partial_{R_G}\rho(S)$ is a bipartite graph with bipartition $V_1 = \delta_G S$ and $V_2 = \delta_{G^*}S^*$. We define the bijection $\rho_* : E(G^*) \to F(R_G)$ such that for any $e^* \in E(G^*), \rho_*(e^*) = \rho(e)$. Thus $V_1 = \delta_G \rho^{-1}(F)$ and $V_2 = \delta_{G^*}\rho_*^{-1}(F)$. This permits to restate (1) in terms of edges and vertices of H:

$$\max\{|V_1|, |V_2|\} \leq \frac{|E(H)|}{2} \leq 6(\min\{|V_1|, |V_2|\} + g - 2) \tag{2}$$

To proceed with the proof of (2), we need some structural information on H.

We call a graph X Σ-*nicely Eulerian* if it satisfies the following properties:

(E1) X is embeddable in Σ;

(E2) All cycles of X are of even length at least four;

(E3) All vertices X are of even degree at least two;

(E4) X does not contain $K_{2,3}$ as a subgraph.

Claim 1. *H is Σ-nicely-Eulerian.*

Proof. (E1) holds because H is a subgraph of R_G. R_G is bipartite graph without (by Proposition 2) multiple edges. Thus all its cycles are of even length at least four and (E2) follows. To verify (E3), notice that every vertex of H is adjacent to at least one edge, so the minimum vertex degree in H is at least one. Let v be a vertex of H. By definition, every edge e of H adjacent to v, is adjacent to a face from F and to a face from F^c. If there is only one face $f \in F$ adjacent to v in R_G, then the degree of v is two. By a simple induction on the number of faces of F adjacent to v in R_G, it follows that the degree of v in H is even.

To prove (E4), assume to the contrary that H contains $K_{2,3}$. Since H is a subgraph of R_G, we have that R_G contains $K_{2,3}$ as well. It means that there are two vertices u, v in R_G connected by three disjoint paths (v, x, u), (v, y, u), and (v, z, u) of length two. The union of every two of these paths is a 4-cycle in R_G, and by Proposition 2, is a border of some face in R_G. This implies that $R_G = K_{2,3}$ (otherwise one of the of the three faces formed by the paths should contain a vertex of R_G). Then G is a triangle. But for each subset of edges S of the triangle, the corresponding graph H contains at most four edges and cannot contain $K_{2,3}$. □

We now come back to the proof of equation (2). Recall that H is bipartite Σ-nicely Eulerian graph and by (E3), each vertex of V_i, $i = 1, 2$ has minimum degree two. This implies that $|E(H)| \geq 2 \cdot |V_i|$, $i = 1, 2$, and the left part of the inequality (2) follows. Finally, the right part of the inequality (2) holds by the following claim.

Claim 2. *Let H be a bipartite Σ-nicely Eulerian graph with bipartition (V_1, V_2) such that $3 \leq |V_1| \leq |V_2|$. Then $|E(H)| \leq 12 \cdot (|V_1| + g - 2)$.*

Proof. For every such a graph H, we set $\theta(H) = \sum_{v \in V_2}(\deg_H(v) - 2)$. For sake of contradiction, let us assume that lemma is not correct. Let H be a counter-example with the smallest number $\theta(H)$. Thus $|E(H)| > 12 \cdot (|V_1| + g - 2)$. First we prove that $\theta(H) = 0$. In fact, if $\theta(H) > 0$ there is a vertex $v \in V_2$ of degree at least four. Since a polyhedral embedding is a 2-cell embedding, a small neighborhood of the point v in Σ is homeomorphic to a disc and it is possible to define a cyclic ordering of the edges incident to v. Let $\{v, v_1\}, \{v, v_2\}, \ldots, \{v, v_r\}$, where $r \geq 4$ and r is even, be a clockwise ordering of edges incident to v. We define the graph H', a Σ-respectful v-splitting of H, as a graph obtained from H by replacing v with two new vertices u and w and by adding edges $\{u, v_1\}, \{u, v_2\}$ and $\{w, v_3\}, , \ldots, \{w, v_r\}$. Notice that H' is also a bipartite Σ-nicely Eulerian graph with bipartition (V_1', V_2') and $|V_1'| = |V_1| \leq |V_2| = |V_2'| - 1 \leq |V_2'|$. Since $|E(H')| = |E(H)|$, we have that $|E(H')| > 12 \cdot (|V_1'| + g - 2)$, which is a contradiction, because $\theta(H') = \theta(H) - 2$. Therefore $\theta(H) = 0$ which means that all vertices in V_2 are of degree two.

We now construct the graph J from H as follows: for every vertex v in V_1 we contract one of its two incident edges. The resulting graph J has no loops because of property (E2), however, it can have multiple edges. But by property (E4), H excludes $K_{2,3}$, and therefore the multiplicity of the edges in J is at most two. We further reduce J to a graph J' by replacing multiple edge by simple ones. By the construction of J, $|V_1| = |V(J)| = |V(J')|$ and $|V_2| = |E(J)| \leq 2|E(J')|$. Since J' is a minor of H, we have that J' is Σ-embeddable. As $|V(J')| = |V_1| \geq 3$, by Euler's formula, $|E(J')| \leq 3(|V(J')| + g - 2)$. This implies that $|V_2| \leq 6(|V_1| + g - 2)$. Finally, as all vertices of V_2 are of degree two in H, we have that $|E(H)| = 2 \cdot |V_2|$, which implies that $|E(H)| \leq 12 \cdot (|V_1| + g - 2)$, a contradiction to the status of H as a counter-example. □

6 Counterexamples

For every positive integer i, we define graphs G_i recursively, using the graphs C, M, and L in Fig. 2 as ingredients. Notice that the vertices of these graphs that are not grey are partitioned into white and black triples. Given some graph G with l white triples $W_1, \ldots, W_l$ and a graph $H \in \{M, L\}$ we denote as $G \clubsuit H$ the graph obtained if we take the disjoint union of G and l copies of H, say $H_1 \ldots H_l$, and then identify W_i with the black triple of $H_i, i = 1, \ldots, l$. We use the notation $J_1 \clubsuit J_2 \clubsuit J_3$ to denote the graph $(J_1 \clubsuit J_2) \clubsuit J_3$ and the notation $J_1 \clubsuit J_2^{(\alpha)} \clubsuit J_3$ to denote the graph $J_1 \clubsuit J_2 \clubsuit \ldots \clubsuit J_2 \clubsuit J_3$ where the operation $\clubsuit J_2$ is repeated α times for some $\alpha \geq 0$. Using this notation, and given a positive integer i, we define $G_i = C \clubsuit M^{(i-1)} \clubsuit L$. Thus the graph G_i is obtained from one copy of C, $\frac{3}{2} \cdot \frac{3^i - 1}{2} = \sum_{\ell=1}^{i-1} 3^\ell$ copies of M and 3^{i-1} copies of L. This process is illustrated in Fig. 3 for the graph $G_2 = C \clubsuit M \clubsuit L$.

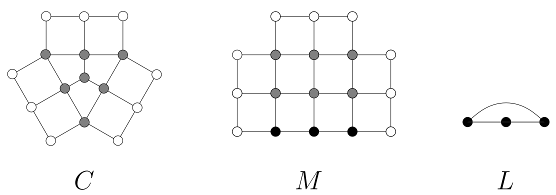

Figure 2: Graphs C, M, and L

In what follows, we prove that for every $i \geq 1$, $\mathbf{pw}(G_i) \geq 3i + 1$ (Lemma 3) and $\mathbf{pw}(G_i^*) \leq 2i + 5$ (Lemma 4).

The proof of the first inequality is by induction which will be based on a series of observations. Before proceeding, it is perhaps appropriate to recall a few definitions. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is a subgraph of a contraction of G.

Observation 1. *The removal from G_i of the grey vertices in C creates three isomorphic connected components G_i^j, $1 \leq j \leq 3$. For every $i \geq 2$ and $1 \leq j \leq 3$, the graph G_{i-1} is the minor of G_i^j.*

The proof of the next fact is easy, although tedious to write out, and we omit it.

Observation 2. $\mathbf{pw}(G_1) \geq 4$.

Lemma 3. *For each $i \geq 1$, $\mathbf{pw}(G_i) \geq 3i + 1$.*

Proof. We proceed by induction. For $i = 1$ the lemma follows from Observation 2. Assume that the statement of the lemma is correct for all graphs G_k, $1 \leq k < i$.

For a sake of contradiction, suppose that $\mathbf{pw}(G_i) \leq 3i$. Let $(X_1, \ldots, X_t)$ be a path decomposition of G_i of width at most $3i$. Let G_i^j, $j = 1, 2, 3$ be the three isomorphic connected components of the graph obtained from G_i by removing the grey vertices of C. Path-width is a parameter closed under the operation of taking minors, hence by Observation 1, $\mathbf{pw}(G_i^j) \geq \mathbf{pw}(G_{i-1})$. Then, by induction hypothesis, $\mathbf{pw}(G_i^j) \geq 3i - 2$.

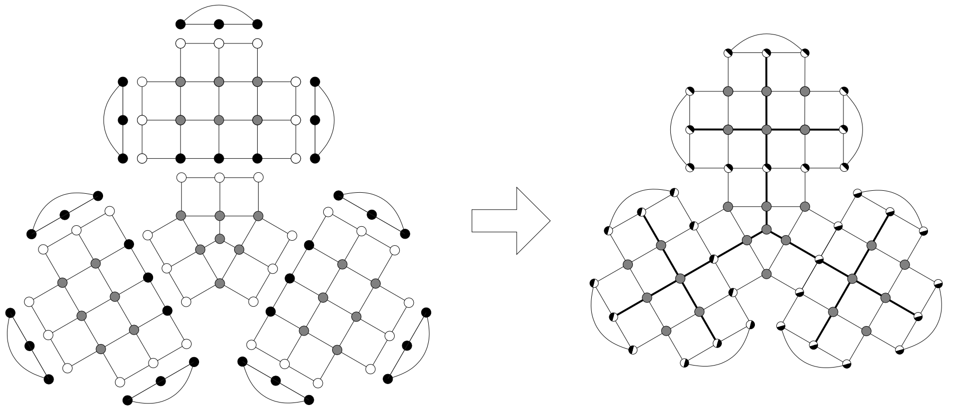

Figure 3: The construction of G_2.

We claim that for each $1 \leq j \leq 3$, there exists an h_j, $1 \leq h_j \leq t$, such that $|X_{h_j}|$ contains at least $3i - 1$ vertices from G_i^j. In fact, if the claim does not hold, then $(X_1 \cap V(G_i^j), \ldots, X_1 \cap V(G_i^j))$ is the path decomposition of G_i^j of width at most $3i - 3$, which is a contradiction to the fact that $\mathbf{pw}(G_i^j) \geq 3i - 2$.

Thus we can choose three indices h_j, $1 \leq j \leq 3$, such that $|X_{h_j} \cap V(G_i^j)| \geq 3i - 1$. Notice that the indices h_1, h_2 and h_3 are pairwise distinct because otherwise there exists an index h, $1 \leq h \leq t$, such that $|X_h| \geq 6i - 2 > 3i + 1$, which is a contradiction to the assumption $\mathbf{pw}(G_i) \leq 3i$.

Let us assume w.l.o.g, that $h_1 < h_2 < h_3$. Then

$$\text{for } j = 1, 3, \ (X_{h_j} \setminus X_{h_2}) \cap V(G_i^j) \neq \emptyset \tag{3}$$

To prove (3) observe that, if (3) does not hold, then X_{h_2} contains at least $3i - 1$ vertices of G_i^j, for some $j \neq 2$, and thus $|X_{h_2}| \geq 6i - 2 > 3i + 1$, which is a contradiction to the assumption that $\mathbf{pw}(G_i) \leq 3i$. By (3), we can choose the vertices v_1 and v_3 such that $v_j \in (X_{h_j} \setminus X_2) \cap V(G_i^j)$, $j = 1, 3$. By the construction of G_i, the subgraph of G_i induced by the vertex set $V(G_i) \setminus V(G_i^2)$ is 3-connected and therefore v_1 is connected with v_3 via three vertex disjoint paths that avoid the vertices of $V(G_i^2)$. Let P_1, P_2, P_3 be such paths. The removal of X_{h_2} from G_i separates v_1 and v_3 into different connected components. Therefore, X_{h_2} contains at least one internal vertex of each of the paths P_1, P_2, and P_3. But since none of these internal vertices is in $V(G_i^2)$, the size of X_{h_2} (which contains at least $3i - 1$ vertices of G_i^2) is at least $3i + 2$. This is a contradiction to the assumption that $\mathbf{pw}(G_i) \leq 3i$ and concludes the proof of $\mathbf{pw}(G_i) \geq 3i + 1$. □

Now we turn to the proof of $\mathbf{pw}(G^*) \leq 2i + 5$. We need some definitions.

A planar embedding of a simple 3-connected planar graph G is called *extended Halin graph* if the graph obtained after removing all vertices incident to its exterior face is a tree T, which is called the *skeleton* G. Thus, for example, for each $i \geq 1$, the graph G_i is an extended Halin graph (in Figure 3 the skeleton of G_2 is depicted by the the bold edges). An *outerplane graph* is a planar embedding of an outerplanar graph with every vertex on the exterior face. The *weak dual* of a plane graph G is the graph obtained from the dual G^* by deleting the vertex corresponding to the exterior face of G.

The following proposition can be found in [2].

Proposition 3 ([2]). *Let G be a 2-connected outerplane graph and T be its weak dual. Then* $\mathbf{pw}(G) \leq 2 \cdot \mathbf{pw}(T) + 2$.

The next observation is crucial for our arguments.

Observation 3.

- *The weak dual of an extended Halin graph G is a 2-connected outerplane graph H such that the weak dual of H is the skeleton of G.*

The following results are also easy known exercises.

Observation 4.

- *Let a graph H' be obtained from a graph G by placing vertices of degree two on some edges of H. Then $\mathbf{pw}(H') \leq \mathbf{pw}(H) + 1$.*
- *The pathwidth of a tree of radius k is at most k.*

Lemma 4. *For each $i \geq 1$, $\mathbf{pw}(G_i^*) \leq 2i + 5$.*

Proof. Let T_i be the skeleton of G_i, and let H_i be the weak dual of G_i. Since H_i is obtained from G_i^* by removing one vertex, $\mathbf{pw}(G_i^*) \leq \mathbf{pw}(H_i) + 1$ (the removal of a vertex cannot decrease pathwidth by more than one). By Observation 4, $\mathbf{pw}(T_i) \leq i+1$. By Observation 3, T_i is the weak dual of H_i. By Proposition 3, $\mathbf{pw}(H_i) \leq 2(\mathbf{pw}(T_i) + 1)$. Putting all together we obtain that

$$\mathbf{pw}(G_i^*) \leq \mathbf{pw}(H_i) + 1 \leq 2(\mathbf{pw}(T_i) + 1) + 1 \leq 2i + 5.$$

□

By Lemmata 4 and 3 we arrive at the following conclusion.

Theorem 2. *For every $c < 1.5$ there is an infinite set of 3-connected planar graphs that are not multiplicatively c-self dual.*

7 Discussions

Let $\mathcal{G}$ be a subclass of the class of planar graphs. We define

$$\mathbf{thres}(\mathbf{pw}, \mathcal{G}) = \inf\{c \mid \text{pathwidth is multiplicatively } c\text{-self dual on } \mathcal{G}\}.$$

By Theorems 1 and 2, if $\mathcal{G}$ is the class of simple 3-connected planar graphs, then $3/2 \leq \mathbf{thres}(\mathbf{pw}, \mathcal{G}) \leq 6$. Using the same machinery as in Lemma 3, it is possible to prove that for each $i \geq 1$, the 2-connected outerplane graph H_i which is the weak dual of G_i, is of pathwidth at least $2i + 1$. However, by Observation 4 and the fact that the removal of a vertex cannot decrease pathwidth by more than one, we obtain that $\mathbf{pw}(H_i^*) \leq i + 2$. Therefore, if $\mathcal{G}$ is the class of 2-connected planar graphs, then $2 \leq \mathbf{thres}(\mathbf{pw}, \mathcal{G})$. We conjecture that for the same class, $\mathbf{thres}(\mathbf{pw}, \mathcal{G}) = 2$.

Notice that for each $i \geq 1$, $\mathbf{tw}(H_i) = 2$ and $\mathbf{tw}(G_i) = 3$. An interesting question is whether any counterexample to the multiplicatively c-self duality of pathwidth on simple 3-connected graphs is of small treewidth. In fact, H_i and G_i can be naturally extended to examples of graphs with higher treewidth. However, the constants c, are getting smaller and closer to 1. This encourages us to conclude with the following conjecture.

Conjecture 2. *If $\mathcal{G}$ is the class of 2-connected planar graphs of treewidth at least m, then* $\mathbf{thres}(\mathbf{pw}, \mathcal{G}) = \frac{m}{m-1}$.

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