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Mathematical programming in petroleum production and refining

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1 Introduction

Since the first linear programming computer codes were written, the petroleum industry has been one of the major users of mathematical programming software. Particularly the downstream part of the industry, that is refining and market distribution of petrochemicals and other refined products, is characterized by a very rich variety of product flow and processing possibilities. Combined with the fact that most of the relations between the decisions and the limitations they have to respect are quantitatively describable, often by simple linear equations or inequalities, this has made the refining industry most amenable to mathematical programming. Currently, the oil refining business probably constitutes the largest group of users of mathematical programming software.

Although optimization techniques also have been successfully applied to the upstream part of the industry, that is oil exploration and exploitation of the fields, this is not comparable to the success enjoyed by its downstream counterpart. If we exclude lack of awareness in the business as a possible explanation, we are left with the following plausible reasons. In the operating phase of field development, the production planner does not face a wide specter of distinct products competing for limited capacities. This is in contrast to oil refining, where several crude materials from different sources (oil fields) can be fed into the same process, processes can be operated in a large range of modes, and refined products can go to different inventories, blends or markets external to the refinery. Furthermore, the response from an oil or gas reservoir to operations applied to it is much less predictable than the behavior of a process plant. Reservoir models have for many years been crucial in the analysis of non-developed fields and fields under development, estimations of reserves, and so forth, but are less suitable as components of planning systems, especially when it comes to short term operations. Modeling the reservoir hence has to be done on a very aggregate level, with the risk of sufficient inaccuracy to impair user confidence in the model.

In this work we shall however see that also upstream petroleum production can benefit largely from mathematical programming models. The next section gives an overview of the most relevant models that have been proposed in this vein. Sections 2.1-2.3, where we review the evolution from very simple models to more advanced

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ones, are devoted to deterministic approaches. Then, in Section 2.4 we have a look at attempts to model and solve decision problems where future uncertainty is taken into account, an aspect of enormous relevance in the business under study. In Section 3 we turn to refinery models. We narrow our focus to problems that are particularly challenging from a computational point of view, and care less about problems where state-of-the-art software currently solves most instances of realistic size in reasonable time.

2 Petroleum field exploitation

Faced with the fact that petroleum reserves are non-renewable resources that require enormous investments in order to be utilized commercially, the industry sees a large saving potential, even if improvements of their planning is only marginal. Examples of decisions with such a possibility to be optimized, include both *design* (e.g. platform capacity, platform location, number of wells, well locations), and *operation* (e.g. drilling program, production and injection rates, abandonment time).

Mathematical programming models supporting such decisions have been suggested more or less sporadically since the nineteen fifties, and a first survey of successful applications was given by Garvin et al. [Ga57]. Updates were later given by Durrer and Slater [DS77] and Hallefjord [Hal86]. The survey given in this work is mainly based on the latter, and complemented with papers published later than [Hal86].

2.1 Linear continuous models - from tank models to models incorporating the principles of reservoir simulation

A planning model supporting decisions concerning drilling and production must have some built-in knowledge about how the reservoir reacts to the plan. This can never be done with an accuracy comparable to a reservoir simulator, which task is a what-if analysis rather than to suggest decisions. In parallel with the enormous improvements in computer algorithms and hardware made over the last decades, the planning models' conception of the reservoir has developed from the extreme simplicity to approaches inspired by simulators.

2.1.1 The tank model

The first planning models incorporating some reservoir description were based on the following assumptions:

- At any time t, the production rate $q(t)$ equals the minimum of two bounds: A constant $\overline{q}$, referred to as the platform capacity, and a function proportional to the reservoir pressure, $p(t)$. Hence $q(t) = \min\{\overline{q}, \alpha p(t)\}$ for some $\alpha > 0$.
- The pressure is equal in all parts of the reservoir, and decreases linearly with the cumulative production. That is, $p(t) = p(0) - \beta \int_0^t q(\tau) d\tau$ for some $\beta > 0$.

Due to their resemblance with a tank with an adjustable sink in the bottom, such models are frequently referred to as *tank models*. The assumption actually is that pressure depletion is the only force that leads to production (no water or gas injection), and that pressure drops are immediately propagated to all the wells of the reservoir. Although unrealistic, such aggregate models have the advantage that they are computationally very easy to handle. It is easy to see that the production from a tank model is given by $q(t) = \begin{cases} \overline{q}, & t \leq \overline{t} \\ \overline{q} e^{-\alpha(t-\overline{t})} & t > \overline{t} \end{cases}$ where $\overline{t} = \frac{\alpha p(0) - \overline{q}}{\alpha\beta\overline{q}}$. This

production profile has a shape that is similar to what frequently is observed in practice, with a *plateau phase* $[0, \overline{t}]$ with stable production at the capacity level, and a *decline phase* where production drops exponentially until the net revenue no longer covers fixed costs. A *build-up phase* will be observed if we let the platform capacity be time dependent, for instance by defining $\overline{q}(t) = \gamma \min\{t, \underline{t}\}$, where $\gamma > 0$ reflects the rate at which new wells are drilled, and $\underline{t} > 0$ is the time when drilling stops. The resulting production profile is not hard to find, and this and other similar enhancements of tank models are easily accomplished without introducing much more computational difficulties.

2.1.2 The tank model with delay

Even in the earliest works on linear programming models for oil field development, such as [Ga57] and the papers by Aronofsky and Lee [AL57], Aronofsky and Williams [AW63] and Charnes and Cooper [CC61], it is realized that the tank model as given above is too simple. Indeed, the assumption that late production contributes as much to the pressure drop as early production was considered unrealistic. By extending the correspondence between production and pressure to

$$p(t) = p(0) - \beta \int_0^t w(t-\tau) q(\tau) d\tau, \tag{1}$$

where $w(t)$ is a weight function expressing the impact of today's production on the pressure t time units ahead, this can be accounted for. If $w(t)$ is a monotonously increasing function, like e.g. $w(t) = 1 - e^{-\delta t}$, where δ is a positive constant, then the modified tank model somehow reflects the inertia of the pressure reduction caused by production.

In [AL57], such extended tank models were used in order to schedule the production from several fields sharing a common pipeline capacity. The fact that sufficiently high well pressure is a necessity for oil production is reflected by imposing a lower bound on the pressure. When discretizing time, this is stated as

$$p_{i0} - \sum_{\tau=1}^{t} d_{i,t-\tau} q_{i\tau} \geq \underline{p}_i \tag{2}$$

where $q_{i\tau}$ is the production rate from field i in time period τ (decision variable), $d_{it}, d_{i,t-1}, \ldots, d_{i0}$ are the coefficients in a discretized version of (1) corresponding to field i, and p_{i0} and $\underline{p}_i$ are the initial pressure and the minimum well head pressure of field i, respectively. The other constraints state that the total production from any field cannot exceed total reserves, in any time period the total production from all fields cannot exceed the pipeline capacity of that period.

2.1.3 Modeling with partial differential equations

Having taken the temporal aspects into consideration, it still remains to model spatial pressure variations. This becomes relevant in multi-well models where the goal is to schedule the production from each of the wells on one or more fields. A first attempt was made in [AW63], where a model of a single field with multiple wells is given (in addition to a model similar to the one in [AL57]). Letting q_{it} now denote the production rate from *well* i in period t, the lower well pressure bound is expressed as

$$p_{i0} - \sum_{j \in I} \sum_{\tau=1}^{t} d_{ij,t-\tau} q_{j\tau} \geq \underline{p}_i \tag{3}$$

where I is the set of wells, p_{i0} is the initial pressure in well i, and d_{ijt} says how much the pressure in well i drops t periods ahead in time given one unit of production from well j now.

Assessing all d-coefficients is not straightforward, and is not much addressed in [AW63]. The coefficients should reflect the relation between liquid flow and pressure, which normally is best understood in a continuous model. Under the assumptions:

- the reservoir has a single phase fluid (oil)
- fluid density increases linearly with pressure
- in any direction, flow velocity is proportional to the pressure gradient (Darcy's law applies)
- vertical variation in the reservoir is negligible

pressure and production respect the partial differential equation (non-homogeneous heat equation)

$$k_0 \frac{\partial p}{\partial t} - \frac{\partial}{\partial x_1}\left(k_1(x_1,x_2)\frac{\partial p}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k_2(x_1,x_2)\frac{\partial p}{\partial x_2}\right) = q(x_1,x_2,t) \qquad (4)$$

with initial value conditions $p(x_1,x_2,0) = p^0 \ \forall (x_1,x_2) \in \mathcal{R}$ and boundary conditions $p(x_1,x_2,t) = p^0 \ \ \forall (x_1,x_2) \in \partial\mathcal{R}$
where

- $x = (x_1, x_2)$ are the coordinates of a point in the reservoir
- $\mathcal{R}$ is the set of such coordinates
- $\partial\mathcal{R}$ is the boundary of R
- $p(x_1,x_2,t)$ is the pressure in position (x_1,x_2) at time t
- $q(x_1,x_2,t)$ is the production rate in position (x_1,x_2) at time t
- $k_1(x_1,x_2)$ and $k_2(x_1,x_2)$ are given by the rock properties (permeability and porosity) in position (x_1,x_2) and the fluid properties (viscosity and compressibility)
- k_0 is given by the fluid properties

For a verification of (4), see a textbook on flow in porous media like e.g. Thomas [Th77], p. 4.9.

Approximating the derivatives in (4) by first order backward and second order central differences, we arrive at the linear equations

$$A\vec{p}^t - B\vec{p}^{t-1} = \vec{q}^t \quad t = 1,\ldots,T \qquad (5)$$

$$\vec{p}^0 = (p^0,\ldots,p^0) \qquad (6)$$

where $\vec{p}^t$ and $\vec{q}^t$ are vectors of pressure and production variables corresponding to period t, respectively, and T is the number of time periods. The matrices A and B will have sparsity patterns given by the choice of difference operators, and numerical values given by k_0, k_1 and k_2. This also agrees with (3).

In Wattenbarger [Wa70], a multi-well model of a single reservoir is given. Like in [AW63], the constraints say that the pressure in each well is bounded from below. However, in a preprocessing phase, Wattenbarger solves (5)-(6) such that $\vec{p}^t$ can be expressed linearly in $\vec{q}^1,\ldots,\vec{q}^t$. Hence the result is a set of constraints on the form (3), where the coefficients originate from the partial differential equation (4).

In his preprocessing, Wattenbarger considers $|I|$ different production profiles. Each of these are composed by fixing the production from some well $j \in I$ to unity, and keeping the production from all others to zero. When (5)-(6) is solved, the resulting pressure in well $i \in I$ in periods $1, \ldots, t$ is known, and the coefficients $d_{ij1}, \ldots, d_{ijt}$ are easily calculated.

Although the reservoir equations are very simple, this preprocessing procedure actually corresponds to a reservoir simulation. The feasibility of the approach depends however on the linearity of (4), a property that is rarely shared with realistic reservoir models usually attacked by simulators. Especially when the reservoirs contain gas, which is assumed in the above study, the equations turn out to be highly nonlinear.

The approach in [Wa70] is further enhanced by Rosenwald and Green [RG74] who also introduce decisions concerning what wells to drill. To this end a binary variable y_i is introduced for each well $i \in I$, along with the restriction on the number of wells to be drilled ($\sum_{i \in I} y_i = N$ for a given N), and the logic constraints $q_i^t \leq D_i y_i$ for sufficiently large D_i. The authors report that solving cases where $|I| = 12$ and $N = 7$ took about two CPU-hours.

Based on the same principle as in [Wa70] and [RG74], Haugland, Hallefjord and Asheim [HHA88] propose a model with a wider range of decisions. The assumption of linear reservoir equations is still made, but the lower pressure bound is replaced by a constraint that can be written

$$q_i^t \leq J_i(p_i^t - \underline{p}_i), \tag{7}$$

where J_i is the productivity of well i and $\underline{p}_i$ is the smallest well pressure at which production from the well is possible. Like in the above papers, preprocessing is used to replace all pressure variables by linear functions of the production variables. The model in [HHA88] also contains decisions about platform capacity and when the wells should be drilled. A case with 23 binary variables (5 different platform capacities, 6 potential wells, 3 possible periods for drilling) and 60 continuous variables (6 potential wells, 10time periods) was solved in about four CPU-hours.

2.2 Models with discrete decisions

With the paper of Rosenwald and Green [RG74] discrete single field models enter the scene. Such models include decisions on a much more detailed level, such as well location, than did the mixed integer programming models for multiple fields proposed already by Bohannon [Bo70]. Here the goal was to find the number of wells on each of a set of fields, and to schedule the drilling of the wells. A tank model is assumed for each reservoir, and hence the spatial dimension that is necessary for well location is not included. However, the approach in [Bo70] also covers the design of a transportation network utilized by all fields, and contains thus a binary variable for each potential link in a network.

Attempts in the direction of multi-field models for optimizing the number of wells and rigs had already been made in [AW63], where an extension of the model discussed in Section 2.1.1 is given. Decisions considered here are number of rigs and wells. Although these clearly should be integer variables, the authors suggest to consider them as continuous variables, and to apply integer rounding when the LP is solved.

Sizing and location of the rigs were also the topics addressed by Devine and Lesso [DL72], who also considered the problem of assigning wells to platforms. The dynamics of the reservoir is given in terms of tank-type equations, but owing to the large number of discrete variables the model becomes very hard to solve to optimality. Based on this model, Frair and Devine [FD75] suggest an extension

where decisions concerning the timing of well drilling and platform installation are supported. This difficult problem is solved approximately by an iterative two-level heuristic, which at the first level assigns values to platform location variables, and solves the decision problem in remaining variables at the second level.

Aboudi et al. [Ab89] develop a mixed integer programming model intended for use on a large area of oil and gas fields. The problem is to select what fields to develop, and to sequence them optimally. Furthermore, the design of the transportation network, including sequencing of individual links, is to be suggested by the model. Production profiles for all fields are fixed, such that the transportation profiles constitute the only continuous variables in the model. Instances of realistic size are hard to handle by an exact solver, and the authors suggest a number of heuristics, including both general (integer rounding, pivot and complement) and problem-specific techniques.

A field sequencing model is also given by Nygreen et al. [Ny98]. For each field in the model a production profile is given, but for a specified subset of the fields the production can to some extent be modified in order to fit transportation capacities, market demand or resource availability (see below). The model also contains (time dependent) precedence constraints, that is a field can be opened in a given period only if another field is opened (in one of a set of specified periods). Furthermore, opening and producing from a field can be specified to consume various shared resources, and availability of each of the resources are user-supplied functions of time. Hence the model becomes quite flexible and general, and its success is proved by the fact that the model has been in practical use by the Norwegian Petroleum Directorate since the early eighties.

Hansen, Pedrosa and Ribeiro [HPR92] have studied the problem of finding the right size and well assignment for all platforms. Which wells to be drilled is already given, but it is assumed that the drilling cost depends on the platform from which this operation is carried out. Hence the resulting model is a multi-capacitated plant location problem (see Neebe and Rao [NR83]), where the term multi-capacitated stems from the fact that the plant capacity is a decision with a discrete set of possible values. The authors demonstrate how a strong formulation leads to saved computing time when the model is solved to optimality by branch-and-bound, and also suggest a Tabu Search procedure for larger instances. In [HPR94], the authors show that the model also supports decisions concerning what wells to drill.

2.3 Nonlinear reservoir models

The need to add a spatial dimension to the reservoir model, opening for an analysis of where to drill the wells, was covered by the models given in [RG74] and [HHA88]. An evident criticism of these approaches is the linearity assumptions. This applies not only to the linearity of the equations (4) representing the reservoir dynamics, but also to the assumption that well deliverability is a linear function of the state of the reservoir. In [HHA88], for instance, the constraint (7) says that the deliverability increases linearly with the pressure.

McFarland, Lasdon and Loose [MLL84] argue that in the case of a gas reservoir with water influx, the well deliverability constraint in a continuous-time model becomes more realistic when given as

$$q(t) \leq JK(t) \left(p(t)^2 - \underline{p}^2\right)^{\eta} \tag{8}$$

where J is the (average) productivity constant, $K(t)$ is the number of wells at time t, and η is a reservoir specific constant. The authors combine this constraint with a tank-type model given by the following nonlinear, ordinary, first-order differential

equations:

$$\frac{dV}{dt}(t) = -\alpha_0(p^0 - p(t)), \quad V(0) = V^0 \tag{9}$$

$$\frac{dp}{dt}(t) = \frac{-\alpha_1 q(t) + \alpha_0 p(t)(p^0 - p(t))}{V(t)}, \quad p(0) = p^0 \tag{10}$$

$$\frac{dK}{dt}(t) = I(t) - \left(1 - \frac{V(t)}{V_0}\right) K(t), \quad K(0) = 0 \tag{11}$$

where $V(t)$ is the gas volume at time t, V^0 is the initial volume, $I(t)$ is the well drilling rate at time t, and α_0 and α_1 are reservoir specific constants. Equation (9) is explained by a commonly-held water influx model (see Schiltuis [Sc36]) essentially saying that the water influx is proportional to the decline in reservoir pressure. By assuming that the gas is ideal, and that the temperature in the reservoir is constant, we have that $p(t)V(t) = \alpha_1 m(t)$, where $m(t)$ is the mass of the gas in the reservoir at time t. By combining $\frac{d(m(t))}{dt} = -q(t)$ with (9), we arrive at (10). Finally, (11) reflects the fact that wells are flooded out as water replaces the gas in the reservoir. Here it is assumed that the rate at which this happens is proportional to the gas volume reduction. Note that in (9)-(11) the number of wells is considered as a differentiable function of time.

The authors go on showing how the differential equations above can be handled in an optimization model by discretizing time, and letting $I(t)$ be a consistently defined step-function. An objective function in the parameters of $I(t)$ is defined, and a reduced gradient algorithm [La78] is used to solve the problem. The resulting production profiles are more realistic than those that linear tank models can produce, but their basic shape with a build-up and decline phase are similar.

In [MLL84] a model for oil reservoirs with associated gas is also studied. The authors develop a tank-type model based on differential equations linking pressure and the quantities of liquid and gas in the reservoirs, and optimize over the drilling program in a manner similar to the gas/water case.

In the papers by Iyer et al. [Iy98], van den Heever et al. [He00, He01] and van den Heever and Grossmann [HG00], much more detailed models are given. Like in [MLL84], neither well deliverability nor reservoir dynamics is supposed to be linear, but the nonlinear functions in question are not fully specified in the papers. The suggested models include an arbitrary number of reservoirs, each of which is a tank. Complicating matters further, binary decision variables representing platform installation and linkage between the platforms are defined for each time period. In [He00] and [He01] nonlinearities also occur as a result of a rather detailed model of royalties and taxes.

The solution method suggested in [He01] is a heuristic based on Lagrangian relaxation of coupling constraints. Hence a set of smaller problems, which can be solved independently, is obtained. These subproblems are solved approximately by first assigning feasible values to the binary variables, and then applying a reduced gradient algorithm [Dr94] to the resulting nonlinear program.

In order to limit the computational burden of the models, all the papers cited so far in this section sacrifice the spatial reservoir model for nonlinear constraints. Few papers have suggested to combine the ideas presented in Section 2.1.3 with nonlinear reservoir dynamics, but Lasdon et al. [La86] have given one such model. In the case of a dry gas reservoir, the flow equations become more realistic by replacing (4) with

$$k_0 \frac{\partial \frac{p}{z(p)}}{\partial t} - \frac{\partial}{\partial x_1}\left(k_1(x_1, x_2)\frac{\partial \psi}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k_2(x_1, x_2)\frac{\partial \psi}{\partial x_2}\right) = q(x_1, x_2, t) \tag{12}$$

where $\psi(p) = \int_{p^0}^{p} \frac{\pi}{\mu(\pi)z(\pi)} d\pi$ is called the *gas pseudo potential*, $\mu(p)$ is the viscosity as a function of pressure, and $z(p)$ is a function reflecting to what extent the gas can be considered as ideal ($z(p) \equiv 1$ means an ideal gas). When both $\mu(p)$ and $z(p)$ are known for all relevant values of p, it can also be assumed that $p/z(p)$ and p are available, at least in tabular format, as functions of ψ.

Approximating (12) with finite differences yields a nonlinear system of equations in a finite number of ψ-variables. The nonlinearity comes from the first term due to the correspondence between $p/z(p)$ and ψ The difference equations appear as nonlinear constraints in an optimization model where the objective is to maximize a revenue function of the production variables (q), which in their turn are linked with the reservoir state through deliverability constraints of the form (8). Again, the correspondence between ψ and p must be used in order to give these constraints in terms of ψ.

As realized in [La86], all reservoir state variables are actually given uniquely by the production variables through the flow equations. Hence the deliverability of well i in period t can be written as

$$q_i^t \leq J_i \left(\left(\pi_i^t(q_1^1, \ldots, q_N^1, \ldots, q_1^t, \ldots, q_N^t)\right)^2 - \underline{p}^2 \right)^\eta$$

where π_i^t is the pressure in well i in period t as a function of the production in all wells in periods up to t. All π-functions are given implicitly by (12), or rather the corresponding difference equations. Considering the space of production variables only, we thus face a nonlinear program of the form: $\min_{\vec{q}} \left\{ \vec{f}(\vec{q}) : \vec{g}(\vec{q}, \vec{\pi}(\vec{q})) \leq \vec{0} \right\}$. In the cited paper, a reduced gradient algorithm [La78] is applied to this problem, and evaluation of the gradients requires the solution of the nonlinear difference equations. This represents a great computational challenge, but the authors managed to solve problems with 5 time periods and 19 wells, and a reservoir discretized into 10 horizontal and 18 vertical blocks.

Given the reduced computability provoked by spatial reservoir models and realistic (nonlinear) flow equations, it is not expected that both features can be incorporated in models that also have a considerable amount of discrete decisions like well location, platform sizing, etc. If one of the features must be left out, it seems as if in the case of 'black oil' reservoirs, that is no gas or condensate, no water injection, etc., the linearity assumption is more acceptable. In the other extreme, a dry gas reservoir, pressure drops propagate more quickly, and a tank-type model is not necessarily too inaccurate, whereas the correspondence between pressure and production is highly nonlinear. The problem is that neither of these cases represent realistic petroleum reservoirs, and there is a large research potential in exploring how to devise optimization models that combine sufficient realism with computational efficiency.

2.4 Stochastic field development models

Given the computational difficulties already present in models discussed so far, it is not straightforward how to take into account the fact that most of the data are uncertain. Nevertheless, uncertainty deserves attention in planning models. The time horizon may be stretched to tens of years, the size of the reserves can only be roughly estimated on the basis of seismology and test drilling, the market demand for petroleum is unpredictable, and also the investment and operation costs are hard to assess. Since there is a limit to how much relevant aspects of field development that can possibly be handled in one planning model, it can be argued that accuracy in some of them should be sacrificed for better modeling of the uncertainty.

Much of the work that is done in supply chain management under uncertainty, will also be relevant to the oil and gas industry. In this section we concentrate on

papers that study uncertainty in combination with techniques discussed in the first sections, and refer to Dempster et al. [De00] and Lababidi et al. [La04]. for examples of stochastic logistics models with special emphasis on the petroleum sector.

Jörnsten [Jo93] considers a field sequencing model inspired by [Ab89], where the oil price is considered to be an uncertain parameter. A number of possible *scenarios* with associated probabilities for the future price are given as input. The goal is then to find what fields to be opened today such that the expected profit is maximized. An underlying assumption is that in any future period, the decision maker will face a similar problem, and that this problem also is solved optimally. Therefore, when making decisions today, also decisions to be made in future periods have to be taken into account.

The author applies a technique called *scenario aggregation* to solve this difficult decision problem. We refer to e.g. Section 2.6 in Kall and Wallace [KW94] for a detailed introduction to the method, but in brief it goes as follows. The deterministic problem that arises if we assume one specific scenario is named a *scenario problem*, and its corresponding solution a *scenario solution*. In each iteration we solve all scenario problems. The scenario solutions hence obtained may be inconsistent in the sense that for any two scenarios that coincide up to some period t, the solutions differ at some point $\tau \leq t$. In the next iteration, the inconsistent decision variables in the two scenario problems will have their objective function coefficients modified in such a way that moving their values towards their current average value is encouraged. It can be shown that convergence to a consistent and also optimal solution hence is achievable when the scenario problems are linear programs.

The model in [Jo93] contains integer variables, and the convergence theory developed for the continuous case does not translate easily. Also, the computational work of each scenario problem grows. But as pointed out in [KW94], rapid convergence to a consistent solution can, at least in the case of LP scenario problems, be obtained even if the scenario problems are solved only approximately. In [Jo93], good, but not necessarily optimal, solutions to the stochastic sequencing model are found by applying a simple heuristic to the scenario problems.

Jonsbråten [Jo98a] realizes that the deterministic price assumption in [HHA88] is not realistic, and suggests a modification where the price is given in terms of scenarios. The resulting stochastic program is solved by an aggregation technique where inconsistencies in the binary variables are ignored. By applying the coefficient adjustments to the continuous variables only, he achieves convergence to solutions where also inconsistencies in binary variables gradually disappear. This is accomplished by the close ties between binary and continuous variables in the model.

Haugen [Ha96] considers a set of gas fields with an unknown reserve. He assumes production profiles as discussed in Section 2.1.1, and lets the uncertainty in reserves be represented by a set of possible start dates of the decline phase. One of the goals is to match the total supply to the market demand, and to this end a stochastic dynamic programming model is given. The decisions to be optimized are the sequencing of the fields, and the setting is hence somewhat similar to the one in [Ny98].

Bjorstad, Hefting and Stensland [BHS89] also assume that the production is given by a tank model, and consider four possible decisions for a single field: *Abandon, develop, wait* and *explore*. The size of the reserves is a gamma-distributed stochastic variable which variance is reduced by exploration. Hence the latter decision is an investment in information, and becomes the optimal one if the risk that the irreversible decisions (abandon and develop) turn out to be bad is sufficiently large. Exploration is however an expensive operation, and a key feature of the model is to balance the exploration cost with the value of information. The cost of the wait option is, in addition to fixed holding costs, that future revenues are

discounted, but the option can still be profitable if a more favorable price regime is expected.

Modeling the value of information is also of fundamental importance in the dissertation by Jonsbråten [Jo98b]. Again, the setting is similar to [HHA88], and the problem is to sequence the wells when their deliverabilities are uncertain. Such uncertainties are very relevant, since well productivity is closely connected with reservoir characteristics. The main point is that drilling wells gives more value than increased production, it also reveals much of the reservoir uncertainty. Hence the model should incorporate this mechanism, and reflect the added value of drilling.

The main difficulty with such models is that the random elements become dependent of the decisions, and hence the problem does not fit most of the stochastic programming literature prior to Jonsbråten, Wets and Woodruff [JWW98]. The cited paper provides the methodological foundation for such models, and also an algorithm that is valid even when some or all variables are discrete. In [Jo98b] this is applied to the well sequencing problem.

Goel and Grossmann [GG04] also consider a problem where information depends on decisions. To a simplified version of [HG00] (e.g. linear deliverability functions), they add uncertainty in the reserves and deliverabilities. Once a reservoir is decided to be developed, all its uncertainty is assumed eliminated, and hence development has an added information value. This is modeled by use of a decision-dependent scenario tree, and solved by an approximation algorithm based on decomposition.

3 Oil Refining

Optimization of oil refining operations has been a well-established discipline since the nineteen fifties. For reasons pointed out in the introduction, the business is very suitable for automated decision support, and consequently a number of software packages dedicated to oil refining have been developed. In this section we first point out the major characteristics of the underlying models common to most of these, and later we have a closer look at one specific challenge frequently occurring in refinery planning.

Refined petroleum products are made in mainly two types of processes: *refining* and *blending*. In the refining or separation processes, crude or partly refined materials are turned into a set of intermediate products. The proportion and quality of each such output is normally a fixed proportion of the input. This contrasts the blending or mixing operations, where the desired output determines or restricts the intermediates to be input. In the case of *recipe blending*, one unit of a given product is composed by a set of intermediates in specified proportions. In *specification blending* the input proportions are not fixed, but have to be chosen such that the output satisfies certain quality requirements. Recipe blending occurs mainly in the petrochemical industry, whereas specification blending is present in most refinery planning models.

Typical examples of refining and separation processes include crude oil distillation, where the oil feed is separated into a range of components with varying boiling point and density. Other examples are various purifying processes where contaminants like sulfur are eliminated. A recipe typically corresponds to a chemical reaction, whereas blend specifications may originate from governmental pollution restrictions, for gasoline it may be minimum octane number, or other user requirements. The blending process is often accomplished by pouring all the components in a common container in a ship tanker.

We consider a simplified setting where m different crude oils can be input to a single separation process, n different intermediate products are output and fed into p parallel recipe blending processes. The decisions to be assessed by the model are

the input quantities $x_1, \ldots, x_m$, the flow y_{jk} of intermediate product j to blend (or end product) k, and the quantity z_k of end product k ($i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$). Assume the unit costs and revenues are $c_1^-, \ldots, c_m^-$ and $c_1^+, \ldots, c_p^+$, let the process capacities be B (separation) and $b_1, \ldots, b_p$ (blending), and assume each unit of crude oil i supplied to the separation process results in an output of a_{ij} units of intermediate product j. Finally, by letting d_{jk} denote the required quantity of intermediate product j to produce one unit of blend k, the planning model takes the form:

$$\begin{aligned}
\max \quad & \textstyle\sum_{k=1}^{p} c_k^+ z_k - \sum_{i=1}^{m} c_i^- x_i && && (13)\\
& \textstyle\sum_{k=1}^{p} y_{jk} - \sum_{i=1}^{m} a_{ij} x_i \le 0 && j = 1, \ldots, n && (14)\\
& y_{jk} - d_{jk} z_k = 0 && j = 1, \ldots, n; k = 1, \ldots, p && (15)\\
& \textstyle\sum_{i=1}^{m} x_i \le B && && (16)\\
& 0 \le z_k \le b_k && k = 1, \ldots, p && (17)\\
& x_i \ge 0 && i = 1, \ldots, m && (18)\\
& y_{jk} \ge 0 && j = 1, \ldots, n; k = 1, \ldots, p && (19)
\end{aligned}$$

If we assume p specification rather than recipe blends, q quality attributes are defined for each blend, T_k^ℓ is an upper bound on the value of the quality attribute ℓ in blend k, t_j^ℓ is the value of the quality attribute ℓ in intermediate product j, constraint (15) is replaced by

$$\textstyle\sum_{j=1}^{n} y_{jk} - z_k = 0 \quad k = 1, \ldots, p; \ell = 1, \ldots, q \tag{20}$$

and

$$\textstyle\sum_{j=1}^{n} t_j^\ell y_{jk} - T_k^\ell z_k \le 0 \quad k = 1, \ldots, p; \ell = 1, \ldots, q \tag{21}$$

In the infancy of LP models built upon the principles (13)-(21) were presented and discussed in numerous papers and also some books (a curious example of which is the one by Symonds [Sy55]). The historically interested reader is referred to Charnes, Cooper and Mellon [CCM52], Davie [Da55], Symonds [Sy56], and also Aronofsky, Dutton and Tayyabkhan [ADT78], but these contributions are not discussed further here.

Although the combination of LP-algorithms and increasing computing capabilities gave an enormous added value to the refining industry, it was soon realized that even small but very realistic modifications of the assumptions above seriously complicate the planning model. Assume for instance that the refinery does not have storage tanks with free capacity for all n intermediate products, and therefore chooses to *pool* two of them in the same tank. Unfortunately, the resulting decision problem can in general no longer be modeled as a linear program, and the rest of this section is devoted to a study of how the computational difficulties hence implied can be handled.

3.1 The pooling problem

From now on we assume that the blending is of specification-type. In order to establish a working model for the problem indicated above, we replace the refining process with a set of flow streams entering storage tanks (see Figure 1). The number of streams entering tank $j \in \{1, \ldots, n\}$ is m_j, and the quality of these streams are not necessarily equal. We let $\vec{\lambda_{ij}} = \left(\lambda_{ij}^1, \ldots, \lambda_{ij}^L\right) \ge \vec{0}$, where $i \in \{1, \ldots, m_j\}, j \in$

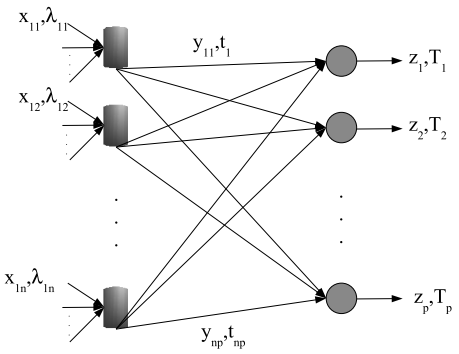

Figure 1: The pooling problem

$\{1,\ldots,n\}$ denote the quality of the ith input flow to tank j. Hence the quality $\vec{t_j}$ of the product in tank j no longer is a given constant, but a variable depending on the input flow to the tank.

The final blends are composed by mixing the products in the tanks in selected proportions. This must be done in such a way that the resulting quality is no larger than $\vec{T_k}$ for blend $k = 1,\ldots,p$. We define the variables x_{ij} and y_{jk} as the ith input flow to tank j and the flow from tank j to blend k, respectively. Let constants Z_k denote the demand for blend k, and let c_{ij}^- and c_k^+ denote cost of the ith input flow to tank j and revenue of blend k, respectively. Hence we arrive at the model:

$$\max_{x,y,t} \quad \sum_{k=1}^{p} c_k^+ \sum_{j=1}^{n} y_{jk} - \sum_{j=1}^{n} \sum_{i=1}^{m_j} c_{ij}^- x_{ij} \tag{22}$$

$$\sum_{i=1}^{m_j} x_{ij} - \sum_{k=1}^{p} y_{jk} = 0 \qquad j = 1,\ldots,n \tag{23}$$

$$\sum_{i=1}^{m_j} \lambda_{ij}^{\ell} x_{ij} - t_j^{\ell} \sum_{k=1}^{p} y_{jk} = 0 \qquad j = 1,\ldots,n; \ell = 1,\ldots,L \tag{24}$$

$$\sum_{j=1}^{n} t_j^{\ell} y_{jk} - T_k^{\ell} \sum_{j=1}^{n} y_{jk} \leq 0 \qquad k = 1,\ldots,p; \ell = 1,\ldots,L \tag{25}$$

$$\sum_{j=1}^{n} y_{jk} \leq Z_k \qquad k = 1,\ldots,p; \tag{26}$$

$$x_{ij} \geq 0 \qquad j = 1,\ldots,n; i = 1,\ldots,m_j \tag{27}$$

$$y_{jk} \geq 0 \qquad j = 1,\ldots,n; k = 1,\ldots,p \tag{28}$$

This formulation subsumes the one given for $L = 1$ by Foulds et al. [FHJ92], but is not as general as the formulation based on the minimum cost flow problem proposed by Haugland [Ha91]. Clearly, the *quality balance* (24) and the *quality constraint* (25) are non-linear (actually *bilinear*) since they contain products of variables. In fact, no linear programming formulation of the problem has to the best of our knowledge ever been given, and it is not likely that such formulations exist.

3.1.1 Local optimization methods

The computational challenge offered by (22)-(28) was realized already by Haverly [Ha78, Ha79, Ha80]. He applied a successive linear programming (SLP) technique where in every iteration each bilinear term $t_j^{\ell} y_{jk}$ is replaced by $\hat{t}_j^{\ell} y_{jk}$ for some guessed quality value $\hat{t}_j^{\ell}$. The resulting linear program is solved, and new values

of the quality variables are obtained. This process continues until convergence. In [Ha78] it is however demonstrated that even for very small instances ($n = 2$, $m_1 = 2$, $m_2 = 1$ and $p = 2$) the final solution depends on the initial guess.

Better SLP techniques, where each bilinear term is replaced by its first order Taylor approximation, were suggested already by Griffith and Stewart [GS61]. That is, first some values $\hat{t}_j^\ell$ and $\hat{y}_{jk}$ of t_j^ℓ and y_{jk}, respectively, are guessed, and $\hat{t}_j^\ell y_{jk} + \hat{y}_{jk} t_j^\ell - \hat{t}_j^\ell \hat{y}_{jk}$ substitutes $t_j^\ell y_{jk}$ in (24)-(25). Again we get an LP that suggests new linearization points. Improvements of the SLP technique have been suggested by e.g. Palacios-Gomez et al. [PLE82], Baker and Lasdon [BL85] and Sarker and Gunn [SG97]. Another local optimization technique based on Benders decomposition was given by Floudas and Aggarwal [FA90]. None of these methods guarantee more than a local optimum, but the SLP technique is fast, and is implemented in most refinery optimization software.

Practical applications of SLP-approaches to the pooling problem are reported in e.g. DeWitt et al. [De89], Rigby et al. [RLW95] and Amos et al. [ARG97]. The latter considers a problem where the pools are the distillation units, and the flow entering this is crude oil. The quality (e.g. relative sulfur content) of the refined products leaving a distillation process hence depends on the composition of the crude oils input to the process.

Audet et al. [Au04] suggest a variable neighborhood heuristic method where the local search procedure is identical to the method in [Ha78]. Given a solution to (22)-(28) for fixed values of all t-variables. Assume that either all t-variables or all y-variables in (22)-(28) are fixed, and consider the optimal solution to the corresponding LP. Its adjacent and feasible extreme points in the LP constitute its *first neighborhood*, all feasible extreme points where two basic variables are replaced make up the *second neighborhood*, and so on. In each iteration of the heuristic some neighbor of the current solution is used as a starting point of the local search. First a starting point is drawn randomly from the first neighborhood, and if this does not improve the solution we move on to the next neighborhood, until the maximum number of neighborhoods is reached or improvement is achieved.

By varying the initial solution, the risk of avoiding the global solution is reduced. This is confirmed by the numerical experiments reported in [Au04], which show that the proposed heuristic found the global optimum in all cases that were solvable by exact methods.

3.1.2 Global optimization methods

Rather than Taylor approximations as replacements for the non-linear terms, Foulds et al. [FHJ92] introduce new variables, and add linear constraints that tie these close to the bilinear terms that they replace. For any function f with domain Ω, the *convex* and *concave envelopes* of f over Ω, $Vex_\Omega f$ and $Cav_\Omega f$, are defined as the pointwise supremum (infimum) of all convex (concave) functions that for all argument values in Ω yield a smaller (larger) value than f. Letting $f(t, y) = ty$ (we drop sub- and superscripts for a moment) and $\Omega = [\underline{t}, \overline{t}] \times [\underline{y}, \overline{y}]$, it can be shown (see Al-Khayyal and Falk [AF83]) that $Vex_\Omega f(t, y) = \max\{\underline{t}y + \underline{y}t - \underline{t}\underline{y}, \overline{t}y + \overline{y}t - \overline{t}\overline{y}\}$ and $Cav_\Omega f(t, y) = \min\{\underline{t}y + \overline{y}t - \underline{t}\overline{y}, \overline{t}y + \underline{y}t - \overline{t}\underline{y}\}$. Hence the variable replacing ty, call itη, can be bounded between these envelopes through linear constraints.

In [FHJ92] a tree-search method exploiting the above idea is given. The linear relaxation obtained by the suggested replacement is solved, and if we in the solution obtained, $(\hat{t}, \hat{y}, \hat{\eta})$, have $\hat{\eta} \neq \hat{t}\hat{y}$, the above is repeated recursively on each of the four new rectangles $[\underline{t}, \hat{t}] \times [\underline{y}, \hat{y}]$, $[\underline{t}, \hat{t}] \times [\hat{y}, \overline{y}]$, $[\hat{t}, \overline{t}] \times [\underline{y}, \hat{y}]$ and $[\hat{t}, \overline{t}] \times [\hat{y}, \overline{y}]$. The authors are hence able to solve problems of size up to $n = 8, m_1 = \cdots = m_8 = 4, p = 16, L = 1$. In [Ha91] it is proved that this algorithm converges to a global optimum.

Floudas and Visweswaran [FV93, VF93, FV96] developed an algorithm that iteratively improves upper and a lower bounds, and hence converges to a solution arbitrarily close to the optimum in a finite number of iterations. The lower and upper bounds are based on the solution to primal and relaxed dual subproblems. Androulakis et al. [AVF96] managed to solve pooling problems of size up to $n = 5$ and $L = 30$ by applying this algorithm.

Adhya et al. [AS99] suggest an algorithm based on Lagrangian relaxation of (23)-(26), that yields better lower bounds than those of the relaxation used in [FHJ92]. Thus the search tree generated by their method is smaller, and they manage to solve the problems solved in [FHJ92] and also others, e.g. problems of size $n = 3, m_1 = 2, m_2 = 3, m_3 = 3, p = 4, L = 5$.

Clearly, the pool qualities are given uniquely by the flow proportions $\frac{x_{ij}}{\sum_{i=1}^{m_j} x_{ij}}$ $= \frac{x_{ij}}{\sum_{k=1}^{p} y_{jk}}$ Ben-Tal et al. [BEG94] gave a formulation where this is used to eliminate all quality variables, resulting in a model where the number of bilinear terms is independent of L. This *proportion formulation* and the *flow formulation* (22)-(28) are exploited by Audet et al. [Au04], who apply the Branch-and-Cut algorithm suggested in [Au00] to both formulations. Their tests are rather inconclusive when it comes to what formulation to use, but the hardest problems reported in [AS99] were easiest to solve with the proportion formulation.

Although much progress on solution approaches to the pooling problems has been seen over the past thirty years, solving large scale instances to optimality is still not realistic. Hopefully, future research will further advance the size of problems for which this is possible, and will also contribute to better heuristic methods for problems of size beyond this bound.

4 Conclusions

This survey demonstrates that there is a great variety of mathematical programming applications in the oil and gas industry. In the upstream part, that is operations concerned with development of and production from petroleum fields, the practical use of mathematical programming does not correspond very well with the richness in model variety presented in academic work. Therefore it seems as if scientific progress in the area is somewhat hampered by the lack of feedback from the intended users. Shortcomings and points of weakness in existing methodology are hard to identify without the user interaction, and a major challenge faced by the researchers is to bring more industry experience into their work.

Looking at the downstream section, we see an industry that already has had an enormous benefit of mathematical programming techniques. Solving refinery planning problems by use of Linear Programming has been well established for decades, and is in many respects straightforward. In this survey we have chosen to focus on one problem related to this that was recognized early after LP was first used, and still causes difficulties for the refinery planners. Only small instances of the pooling problem can be solved by exact methods, and most heuristic methods seem to be very much under the influence of Successive Linear Programming, which was introduced already in the sixties. One of the best opportunities for research contributions to the downstream part of the oil business hence lies in devising better methods, both exact and approximate, for solving the frequently-occurring pooling problem.

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