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On the Transportation Paradox

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Abstract

The *transportation paradox* is related to the classical transportation problem. For certain instances of this problem an *increase* in the amount of goods to be transported may lead to a *decrease* in the optimal total transportation cost. Thus this phenomenon has also been named the *more-for-less-paradox.* Even though the paradox has been known since the early days of linear programming, it has got very little attention in the literature and in teaching, and it seems to be almost unknown to the majority of the LP-practitioners. We consider a recent paper where necessary and sufficient conditions for a transportation cost matrix to be *immune* against the paradox are given. These conditions are rather restrictive, supporting results reported from simulations indicating that the paradox may occur quite frequently. Some postoptimal conditions for when the paradox occurs are considered. A simple procedure for modifying an existing instance to exploit the paradox is given and illustrated by examples. Some transportation problem examples from widely used textbooks are presented illustrating that the (potential) appearance of the paradox is ignored.
Keywords: transportation problem; transportation paradox; linear programming; duality

1 Introduction

The classical *transportation problem* is the name of a mathematical model which has a special mathematical structure. The mathematical formulation of a large number of problems conforms (or can be made to conform) to this special structure. So the name is frequently used to refer to a particular form of mathematical model rather than the physical situation in which the problem most natural originates.

The standard problem description is as follows: A commodity is to be transported from each of m sources to each of n destinations. The amounts available at each of the sources are $a_i,\ i = 1, \dots, m$, and the demands at the destinations are $b_j,\ j = 1, \dots, n$. The total sum of the available amounts at the sources is equal to the sum of the demands at the destinations. The cost of transporting one unit of the commodity from source i to destination j is c_{ij}. The goal is to determine the amounts x_{ij} to be transported over all routes (i, j) such that the total transportation cost is minimized.

The mathematical formulation of this standard version of the transportation problem is the following linear program, TP:

$$\begin{aligned}
\text{minimize } & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to } & \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1,\ldots,n \\
& x_{ij} \geq 0,\ i = 1,\ldots,m,\ j = 1,\ldots,n.
\end{aligned}$$

An instance of TP is specified by an $m \times n$ *cost* matrix $C = [c_{ij}]$, an n-dimensional *demand* vector $\mathbf{b} = [b_j]$ and an m-dimensional *supply* vector $\mathbf{a} = [a_i]$. All the data are assumed to be nonnegative real numbers. We will use the notation $z(C, \mathbf{a}, \mathbf{b})$ to denote the optimal objective value of an instance of TP specified by $C, \mathbf{a}$ and $\mathbf{b}$.

The so-called *transportation paradox* is the name of the following behaviour of the transportation problem: Certain instances have the property that it is possible to *decrease* the optimal objective value by *increasing* the supplies and demands. More precisely, let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two other supply and demand vectors, such that $\hat{\mathbf{a}} \geq \mathbf{a}$ and $\hat{\mathbf{b}} \geq \mathbf{b}$. Then the paradox occurs if and only if $z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) < z(C, \mathbf{a}, \mathbf{b})$.

Example 1 (from Appa [4]): Let

$$C = \begin{bmatrix} 50 & 300 \\ 320 & 60 \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} 5 & 10 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 7 & 8 \end{bmatrix}.$$

The optimal solution is

$$X = \begin{bmatrix} 5 & 0 \\ 2 & 8 \end{bmatrix}$$

with $z(C, \mathbf{a}, \mathbf{b}) = 1370$. Now increase a_1 and b_2 by one unit, i.e. let

$$\hat{\mathbf{a}} = \begin{bmatrix} 6 & 10 \end{bmatrix}$$

and

$$\hat{\mathbf{b}} = \begin{bmatrix} 7 & 9 \end{bmatrix}.$$

The optimal solution is then:

$$X = \begin{bmatrix} 6 & 0 \\ 1 & 9 \end{bmatrix}$$

with $z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) = 1160$. So one more unit transported will reduce the optimal cost by 210. □

This seemingly paradoxical situation is readily explained is this small example: To satisfy the demand $b_1 = 7$, one has to transport 2 units on the expensive route (2,1) ($c_{21} = 320$) in addition to the 5 units on the nonexpensive route (1,1). If the supply $a_1 = 5$ could be increased to 6 and the demand $b_2 = 8$ increased to 9, only one unit has to be sent on route (2,1). This more than compensates for the additional unit sent on route (1,1) and on route (2,2) since $c_{21} > c_{11} + c_{22}$.

In the following sections we will first look at some historical facts about the discovery and treatment of the transportation paradox. Then in section 3 we consider necessary and sufficient conditions for when the paradox can not occur. In section 4 we study some postoptimal (dual) conditions for the occurrence of the paradox. We also consider how the paradox can be exploited to improve an optimal solution. In section 5 we consider the paradoxical properties of some numerical TP-examples found in widely used textbooks (the paradox is not mentioned in these books). Finally in section 6 we discuss some variants of the TP formulation.

We will assume throughout that the reader is familiar with some basic LP-theory and network flow algorithms (see f.ex. Ahuja [1] and/or Papadimitriou & Steiglitz [12]).

2 Some historical facts

It is not quite clear when and by whom this paradox was first discovered. The transportation problem itself was first formulated by Hitchcock ([11]) in 1941, and was independently treated by Koopmans and Kantorovich (in fact Monge formulated it in 1781 and solved it by geometrical means, see Appendix II in Berge [5] for a description of Monges work). In 1951 Dantzig gave the standard LP-formulation TP in [7] and applied the simplex method to solve it. Since then the transportation problem has become the classical common subject in almost every textbook on operations research and mathematical programming. Very efficient algorithms and corresponding software have been developed for solving it.

The transportation paradox is, however, hardly mentioned at all in any of the great number of textbooks and teaching materials where the transportation problem is treated.

Apparently, several researchers have discovered the paradox independently from each other. But most papers on the subject refer to the papers by Charnes and Klingman [6] and Szwarc [15] as the initial papers. In [6] Charnes and Klingman name it *the more-for-less paradox*, and they write: *The paradox was first observed in the early days of linear programming history (by whom no one knows) and has been a part of the folklore known to some (e.g. A.Charnes and W.W.Cooper), but unknown to the great majority of workers in the field of linear programming.*

According to [4], the transportation paradox is known as *Doigs paradox* at the London School of Economics, named after Alison Doig who used it in exams etc. around 1959 (Doig did not publish any paper on it).

Since the transportation paradox seems not to be known to the majority of those who are working with (or teaching) the transportation problem, one may be tempted to believe that this phenomenon is only an academic curiosity which will most probably not occur in any practical situation. But that seems not to be true. Experiments done by Finke [9], with randomly generated instances of the transportation problem of size 100×100 and allowing additional shipments (postoptimal), show that the transportation costs can be reduced considerably by exploiting the paradoxical properties. More precisely, the average cost reductions achieved are reported to be 18.6% with total additional shipments of 20.5%.

In a recent paper [8], Deineko & al. develop necessary and sufficient conditions for a cost matrix C to be immune against the transportation paradox. As we shall see in the next section, these conditions are rather restrictive, supporting the observations by Finke.

3 When will the paradox not occur?

In [8] Deineko & al. give an exact characterization of all cost matrices C that are *immune* against the transportation paradox. An immune cost matrix satisfies

$$z(C, \mathbf{a}, \mathbf{b}) \leq z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}})$$

for all supply vectors $\mathbf{a}$ and $\hat{\mathbf{a}}$ with $\mathbf{a} \leq \hat{\mathbf{a}}$ and for all corresponding demand vectors $\mathbf{b}$ and $\hat{\mathbf{b}}$ with $\mathbf{b} \leq \hat{\mathbf{b}}$. So regardless of the choice of the supply and demand vectors, the transportation paradox does not arise when the cost matrix C is immune.

The main result in [8] is as follows:

Theorem 1 *A $m \times n$ cost matrix $C = [c_{ij}]$ is immune against the transportation paradox if and only if, for all integers q, r, s, t with $1 \leq q, s \leq m,\ 1 \leq r, t \leq n,\ q \neq s,\ r \neq t$, the inequality*

$$c_{qr} \leq c_{qt} + c_{sr} \tag{1}$$

is satisfied.

Proof: To prove the *only if*- part, suppose (1) is not true, i.e. assume $c_{qr} > c_{qt} + c_{sr}$ for some q, r, s, t. Then consider an instance where component q of the supply vector, $a_q = 1$, and where all the other components are zero, i.e. $a_i = 0, i = 1, \ldots, m, i \neq q$. Similarly, let component r of the demand vector, $b_r = 1$, and let all the other components be zero, i.e. $b_j = 0, j = 1, \ldots, n, j \neq r$. Then clearly $z(C, \mathbf{a}, \mathbf{b}) = c_{qr}$. Now let $\hat{\mathbf{a}}$ be a new supply vector which is different from $\mathbf{a}$ only in component s such that $\hat{a}_s = 1$, and similarly let $\hat{\mathbf{b}}$ be different from $\mathbf{b}$ only in component t such that $\hat{b}_t = 1$. Then $\mathbf{a} \leq \hat{\mathbf{a}}$ and $\mathbf{b} \leq \hat{\mathbf{b}}$. In this new instance one unit may be sent directly from source q to destination t, and another unit may be sent from source s to destination r. The total cost of this is $c_{qt} + c_{sr}$. Our assumption then leads to $z(C, \mathbf{a}, \mathbf{b}) > z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}})$, i.e. the paradox has occurred.

For a proof of the *if*-part of the theorem, see the proof of Lemma 3 in [8]. □

Example 2 : Consider the 4×5 cost matrix

$$C = \begin{bmatrix} 4 & 15 & 6 & 13 & 14 \\ 16 & 9 & 22 & 13 & 16 \\ 8 & 5 & 11 & 4 & 5 \\ 12 & 4 & 18 & 9 & 10 \end{bmatrix} \tag{2}$$

Here we see immediately that f.ex. $c_{14} > c_{11} + c_{34}$, which means that (1) is violated for $q = 1, r = 4, s = 3, t = 1$. So C is not immune against the transportation paradox. □

It is rather straightforward to check whether a matrix is immune or not due to the following:

Corollary 1 *It can be checked in $O(mn)$ time whether or not an $m \times n$ cost matrix C is immune against the transportation paradox.*

Proof: To check if (1) is violated for some q, r, s, t, we first determine and store the two smallest elements in each row and each column of C. This can be done in $O(m)$ time for each of the n columns and in $O(n)$ time for each of the m rows. In total this scanning takes $O(mn)$ time.

Now, in case of violation of (1), there must be an element c_{qt} in row q and an element c_{sr} in column r such that the sum of these elements is smaller than c_{qr}. The smallest element in row q is the most actual candidate for c_{qt} (or the second smallest in case c_{qr} is the smallest element in row q). Similarly, the most actual candidate for c_{sr} is the smallest (or second smallest) element in column r. Since these elements have been found and stored already, we may check (1) in constant time for each c_{qr}. So the total running time is $O(mn)$. □

If a route (q,r) do not exist, it is common practise to set $c_{qr} = M$, where M is a big positive number. This prevents x_{qr} from becoming a basic variable (i.e. $x_{qr} = 0$) in the optimal solution. The cost matrix C will, however, then not be immune against the paradox due to the following:

Corollary 2 *A cost matrix C containing one or more big$M$ elements corresponding to nonexisting routes, is not immune against the transportation paradox.*

Proof: Assume for simplicity that $\mathbf{a} > \mathbf{0}$ and $\mathbf{b} > \mathbf{0}$. Then to have consistency, at least one element in each row and at least one element in each column of C must represent the cost of an existing route. Let E be the set of index pairs (i,j) for all existing routes. Suppose route (q,r) do not exist, i.e. $c_{qr} = M$, and assume that $M > 2c_{max}$, where $c_{max} = max_{(i,j)\in E}\{c_{ij}\}$. Then (1) will be violated since there is at least one element c_{qt} in row q where $(q,t) \in E$ and at least one element c_{sr} in column r where $(s,r) \in E$. □

4 When the paradox occurs

The dual problem corresponding to the linear program TP is the following linear program, DP:

$$\text{maximize} \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j$$

subject to

$$u_i + v_j \leq c_{ij},\ i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{3}$$

Here the dual variables u_i and v_j correspond to the m first and the n last equations of TP respectively. It is well known (see f.ex. [12]) that the constraint equations of TP are linearly dependent and that the rank of the constraint matrix is $m + n - 1$. So one equation (any) is redundant and may be omitted. Thus any optimal solution to DP will not be unique. In the following we will assume that the first constraint equation of TP is omitted, and that the corresponding dual variable, u_1, is set to be zero, i.e. $u_1 = 0$ throughout.

Any basic solution of TP has $m + n - 1$ basic variables. Let $X = [x_{ij}]$ be an optimal basic solution (also called an optimal *transportation tableau*) of TP and let B be the set of index pairs (i,j) of all basic variables x_{ij} in X. Then we know from elementary LP-theory that

$$u_i + v_j - c_{ij} = 0$$

for all $(i,j) \in B$, and that

$$x_{ij} = 0$$

for all $(i,j) \notin B$.

The optimal objective value may then be written as:

$$\begin{aligned} z(C, \mathbf{a}, \mathbf{b}) &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - u_i - v_j) x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} (u_i + v_j) x_{ij} \\ &= \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j. \end{aligned}$$

We will now look at some postoptimal conditions which are sufficient for the occurrence of the paradox. We consider only the case where an instance is improved by increasing a single supply a_i and a single demand b_j by the same amount (all the other data are unchanged). A procedure for improving an optimal transportation tableau when the conditions are satisfied will be illustrated.

We have the following result (from [15]):

Theorem 2 *Assume that indexes p and q exist, $1 \le p \le m; 1 \le q \le n$, such that*

$$u_p + v_q < 0. \tag{4}$$

Assume further that a positive number θ exists, such that when supply a_p is replaced by $\hat{a}_p = a_p + \theta$, and demand b_q is replaced by $\hat{b}_q = b_q + \theta$, a basic feasible solution for the new instance can be found which is optimal and has the same set B of basic variables. Then the paradox occurs.

Proof: Since the optimal solution for the new instance has the same set B of basic variables, the optimal dual solution is unchanged. So the new optimal objective value is:

$$z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) = \sum_{i=1}^{m} u_i a_i + u_p\theta + \sum_{j=1}^{n} v_j b_j + v_q\theta = z(C, \mathbf{a}, \mathbf{b}) + \theta(u_p + v_q).$$

From the assumptions we then have that $z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) < z(C, \mathbf{a}, \mathbf{b})$. □

When will a positive θ exist?
We illustrate first by an example (from [15]). A 4×5 instance of TP is given by the cost matrix (2) in Example 2 and the following supply and demand vectors:

$$\mathbf{a} = \begin{bmatrix} 7 & 18 & 6 & 15 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 4 & 11 & 12 & 8 & 11 \end{bmatrix}$$

The optimal transportation tableau for this instance is:

	$v_1 = 1$	$v_2 = -6$	$v_3 = 6$	$v_4 = -2$	$v_5 = 0$
$u_1 = 0$			7		
$u_2 = 15$	4	6		8	
$u_3 = 5$			5		1
$u_4 = 10$		5			10

Here the optimal dual values are written above and on the left of the tableau. The total optimal cost of this solution is 444.
We observe that the set of index pairs for the optimal basic variables is:

$$B = \{(1,3), (2,1), (2,2), (2,4), (3,3), (3,5), (4,2), (4,5)\}.$$

We also observe that

$$u_1 + v_4 = -2 < 0.$$

So let us see if it is possible to increase $a_1 = 7$ and $b_4 = 8$ by a number $\theta > 0$ such that the present optimal basic feasible solution can be modified to become optimal for the new instance with the same set of basic variables:

		$7+\theta$			$a_1 = 7+\theta$
4	$6-\theta$		$8+\theta$		$a_2 = 18$
		$5-\theta$		$1+\theta$	$a_3 = 6$
	$5+\theta$			$10-\theta$	$a_4 = 15$
$b_1 = 4$	$b_2 = 11$	$b_3 = 12$	$b_4 = 8+\theta$	$b_5 = 11$	

Here the supplies and the demands are written to the right and below the tableau. From this tableau we observe that θ may be selected as any number

$$0 < \theta \leq 5.$$

If f.ex. $\theta = 4$ is chosen, the new optimal transportation tableau is:

		11			$a_1 = 11$
4	2		12		$a_2 = 18$
		1		5	$a_3 = 6$
	9			6	$a_4 = 15$
$b_1 = 4$	$b_2 = 11$	$b_3 = 12$	$b_4 = 12$	$b_5 = 11$	

The cost of this solution is $444 + 4(-2) = 436$. So shipping 4 units more will reduce the total transportation cost by 8.

Note that if $\theta = 5$ is chosen (the maximum value for θ), the new optimal transportation tableau will be degenerate (one of the basic variables becomes zero).

We observe that in order to determine the upper bound for θ, a subset $S \subseteq B$ of index pairs has been selected ($S = B \setminus \{(2,1)\}$ in our example). Now suppose we link the elements of S to form a *directed path* DS:

$$DS = \{(1,3), (3,3), (3,5), (4,5), (4,2), (2,2), (2,4)\}.$$

This ordered set defines a directed path which starts at the basic element $(1,3)$ and ends at $(2,4)$. It is *alternating* in the sense that θ is added to the tableau elements corresponding to the odd numbered elements of DS and is subtracted from those corresponding to the even numbered elements of DS. DS consists of an even number of *perpendicular* links.

In general, if indexes p and q exist such that (4) is satisfied, we try to determine an upper bound for θ by constructing a directed alternating path DS, starting at a basic element $(p, -)$ in row p of the optimal transportation tableau and ending at a basic element $(-, q)$ in column q (in our example $p = 1$ and $q = 4$).

DS will consist of index pairs for an odd number of (perpendicular) basic elements of the tableau. Let DS_o and DS_e denote the odd and even numbered elements of DS respectively, such that $DS = DS_o \cup DS_e$. The elements of the cost matrix C corresponding to the index pairs in DS are then related as follows:

Lemma 1

$$CDS = \sum_{(i,j) \in DS_o} c_{ij} - \sum_{(i,j) \in DS_e} c_{ij} = u_p + v_q.$$

Proof: Since the elements of DS are (perpendicular) alternating, we have that

$$CDS = c_{pj_1} - c_{i_1 j_1} + c_{i_1 j_2} - c_{i_2 j_2} + \cdots - c_{i_t j_t} + c_{i_t q}$$

We know that $c_{ij} = u_i + v_j$, $\forall (i,j) \in B$. Thus, since $S \subseteq B$, we have:

$$CDS = (u_p + v_{j_1}) - (u_{i_1} + v_{j_1}) + (u_{i_1} + v_{j_2}) - \cdots - (u_{i_t} + v_{j_t}) + (u_{i_t} + v_q)$$

Summing all this give:

$$CDS = u_p + v_q. \ \square$$

We add θ to the tableau elements corresponding to DS_o and subtract θ from the tableau elements corresponding to DS_e. The upper bound for θ is limited by the smallest basic element of the optimal transportation tableau from which θ is subtracted. So we have the following result:

Corollary 3 *A positive θ exists if and only if*

$$x_{ij} > 0, \ \forall (i,j) \in DS_e.$$

This corollary tells us that if the optimal solution of TP is nondegenerate, and there are components of the optimal dual solution satisfying (4), the paradox will occur. In case of degeneracy the paradox will still occur if the index pairs of DS_e do not include any degenerate elements of the optimal tableau (see Example 7 in section 6).

Repeated use of the process is of course possible if more than one pair of optimal dual values satisfy (4) (the dual solution is unchanged). However, if the maximal value of θ is selected, the new optimal tableau will be degenerate, and this may reduce the possibility of repeated success (as Example 7 also shows).

5 Textbook examples

In this section we will present some transportation problem examples found in widely used textbooks. The examples are mainly used for illustrations of modelling and solution procedures in their respective books. The examples confirm that the paradox occurs quite frequently. We also get an illustration of the fact that the paradox is ignored (almost completely) in textbooks of any kind (management science, operations research, mathematical programming).

Example 3: The following TP instance is the introductory example for the study of the transportation problem in the textbook [2] by Anderson & al. In the example there are three origins (plants) and four destinations (distribution centers). The amounts (number of units of the product) available at each of the origins, the demands at each of the destinations and the transportation cost from each origin i to each destination j are given respectively by:

$$\mathbf{a} = \begin{bmatrix} 5000 & 6000 & 2500 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} 6000 & 4000 & 2000 & 1500 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 3 & 2 & 7 & 6 \\ 7 & 5 & 2 & 3 \\ 2 & 5 & 4 & 5 \end{bmatrix}.$$

It is easy to verify that this cost matrix is not immune against the transportation paradox.

The optimal transportation tableau for this instance is:

	$v_1 = 3$	$v_2 = 2$	$v_3 = -1$	$v_4 = 0$
$u_1 = 0$	3500	1500		
$u_2 = 3$		2500	2000	1500
$u_3 = -1$	2500			

The optimal total cost is 39500 for a total load of 13500.

The assumption (4) is satisfied by f.ex. $p = 3$ and $q = 3$ since $u_3 + v_3 = -2$. A directed path DS starting in $(3, 1)$ and ending in $(2, 3)$ can be constructed. Since the optimal solution is non-degenerate, we know from Corollary 2 that a positive number θ can be found, which when added to a_3 and b_3 will give a new optimal tableau with a lower cost.

To determine an upper bound for θ, we consider the following transportation tableau:

$3500-\theta$	$1500+\theta$			$a_1 = 5000$
	$2500-\theta$	$2000+\theta$	1500	$a_2 = 6000$
$2500+\theta$				$a_3 = 2500+\theta$
$b_1 = 6000$	$b_2 = 4000$	$b_3 = 2000+\theta$	$b_4 = 1500$	

We see that any value of θ such that

$$0 < \theta \leq 2500$$

will give a new optimal solution with a lower cost. If $\theta = 2500$ is selected, the total cost is reduced by 5000 (to 34500). Repeated use of the process is then not possible (due to degeneracy, no further positive θ can be found for the other dual combinations satisfying (4)).

The example is thoroughly explained and solved (also by the transportation simplex method) in [2]. The paradoxical properties of this instance are, however, not mentioned.

Example 4: The TP instance below is the main example used by Rao in [14] to illustrate the simplex method for solving TP:

$$\mathbf{a} = \begin{bmatrix} 50 & 50 & 100 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} 30 & 40 & 60 & 70 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 15 & 0 & 20 & 10 \\ 12 & 8 & 11 & 20 \\ 0 & 16 & 14 & 18 \end{bmatrix}.$$

Clearly this matrix is not immune against the paradox.

The optimal transportation tableau for this instance is:

	$v_1 = -8$	$v_2 = 0$	$v_3 = 6$	$v_4 = 10$
$u_1 = 0$		40		10
$u_2 = 5$			50	
$u_3 = -1$	30		10	60

The optimal total cost is 1870. The total load is 200.
We observe that $u_1 + v_1 = -8$ and $u_2 + v_1 = -3$. So we first increase a_1 and b_1:

	40		$10+\theta$	$a_1 = 50+\theta$
		50		$a_2 = 50$
$30+\theta$		10	$60-\theta$	$a_3 = 100$
$b_1 = 30+\theta$	$b_2 = 40$	$b_3 = 60$	$b_4 = 70$	

Here $\theta = 60$ may be selected, and we then get a cost reduction of 480.
The updated tableau may be improved further since $u_2 + v_1 = -3$:

	40		70	110
		$50+\theta$		$50+\theta$
$90+\theta$		$10-\theta$	0	100
$90+\theta$	40	60	70	

Now $\theta = 10$ may be selected, reducing the cost further by 30.
So in total, by exploiting the paradoxical properties, we may reduce the total cost by 510 (from 1870 to 1360) by increasing the total load transported by 70 (from 200 to 270).
This property is not mentioned in [14].

Example 5: The numerical example used by Vanderbei in Chapter 14 of [16] for illustrating how the simplex method may be implemented to solve the transportation problem, is as follows:

$$\mathbf{a} = \begin{bmatrix} 7 & 11 & 18 & 12 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} 10 & 23 & 15 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 5 & 6 & * \\ 8 & 4 & 3 \\ * & 9 & * \\ * & 3 & 6 \end{bmatrix}.$$

Here $*$ denotes a non-existing route. From Corollary 2 we know that this cost matrix is not immune against the paradox.
The optimal solution is as follows:

	$v_1 = 5$	$v_2 = -3$	$v_3 = 0$
$u_1 = 0$	7		
$u_2 = 3$	3		8
$u_3 = 12$		18	
$u_4 = 6$		5	7

The total cost is 302.
Since $u_1 + v_2 = -3$ and the solution is nondegenerate, the paradox occurs. The reader can easily verify that we may increase a_1 and b_2 by $\theta = 3$. The total cost of the updated solution is 293.
The paradoxical property of this example is not mentioned in [16].

Example 6: The main example in Chapter 7 of Winston [17] is the following TP problem: Three power plants supply the needs for electricity of four cities. The number of kilowatt-hours (in millions) each power plant can supply, and the (peak) power demands at the four cities are respectively:

$$\mathbf{a} = \begin{bmatrix} 35 & 50 & 40 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} 45 & 20 & 30 & 30 \end{bmatrix}.$$

The cost of sending one million kwh from plant i to city j is given by the following matrix:

$$C = \begin{bmatrix} 8 & 6 & 10 & 9 \\ 9 & 12 & 13 & 7 \\ 14 & 9 & 16 & 5 \end{bmatrix}.$$

This matrix is not immune against the transportation paradox. But the optimal solution of this instance is:

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$
$u_1 = 0$		10	25	
$u_2 = 3$	45		5	
$u_3 = 3$		10		30

Here there are not any indexes p and q such that (4) is satisfied. So the paradox do not occur.

However, from Theorem 1 and Lemma 1 we see that if the supplies and demands were such that the optimal set B of index pairs included f.ex. the index pairs (1,3),(3,3) and (3,4), or the index pairs (1,1),(3,1) and (3,4), the paradox may occur. To confirm this, suppose we have another instance where the cost matrix C is the same, but the supplies and the demands are:

$$\mathbf{a} = \begin{bmatrix} 45 & 20 & 60 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 35 & 20 & 40 & 30 \end{bmatrix}.$$

An optimal transportation tableau (not unique) for this instance is:

	$v_1 = 8$	$v_2 = 3$	$v_3 = 10$	$v_4 = -1$
$u_1 = 0$	15		30	
$u_2 = 1$	20			
$u_3 = 6$		20	10	30

The total cost is 1090.
Since $u_1 + v_4 = -1$, the total cost will be reduced by θ if supply a_1 and demand b_4 are both increased by θ, where $0 < \theta \leq 10$.
Another optimal transportation tableau for this instance is:

	$v_1 = 8$	$v_2 = 3$	$v_3 = 10$	$v_4 = -1$
$u_1 = 0$	5		40	
$u_2 = 1$	20			
$u_3 = 6$	10	20		30

Again we see that the total optimal cost will decrease if we increase the same supply and demand as we did in the previous tableau.

6 Variants of the transportation problem

Consider the formulation which in the following will be named GTP:

$$\begin{aligned} \text{minimize} & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} \geq a_i, \ i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} \geq b_j, \ j = 1, \ldots, n \\ & x_{ij} \geq 0, \ i = 1, \ldots, m, \ j = 1, \ldots, n. \end{aligned}$$

A number of variants of GTP, including TP, have been studied by several authors (e.g. Appa [3], Charnes and Klingman [6], Hadley [10], Proll [13], Szhwarc [15]). In [3] 54 different variants are treated in detail. The main concern in [3] is to study which variants are equivalent (have the same optimal solution) and which are not. This effort was partly motivated by the incorrect treatment of some of these aspects in [10] (Exercise 9-26, p.327). As pointed out by Proll in [13], the erroneous conclusion reached by Hadley was independently also recognized and commented on by Charnes and Klingman ([6]) and by Szwarc ([15]).

In this section we shall not consider all the possible variants of GTP. We will only look at the relation between TP and the variant of GTP where the assumption $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ is valid. The following important result has been proved in [15]:

Theorem 3 *If the optimal dual variables corresponding to an optimal solution of TP satisfy $u_i + v_j \geq 0$ for all i and j, the problems TP and GTP are equivalent.*

From the examples in the foregoing sections we see immediately that TP and GTP are not equivalent when the paradox occurs in TP: The improvement procedure applied on an optimal TP solution, where in each step a single a_i and a single b_j are increased by the same amount, lead to feasible solutions of GTP with *lower* objective values than the optimal TP value.

One may ask if repeated use of the improvement procedure (until no further improvements are possible) will always lead to an optimal solution of GTP. The following example shows that the answer is negative.

Example 7 :

$$\mathbf{a} = \begin{bmatrix} 10 & 6 & 15 & 4 \end{bmatrix},$$
$$\mathbf{b} = \begin{bmatrix} 5 & 14 & 10 & 6 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 5 & 8 & 7 & 6 \\ 6 & 10 & 5 & 5 \\ 7 & 15 & 3 & 16 \\ 15 & 21 & 8 & 18 \end{bmatrix}.$$

The optimal TP-tableau for this instance is:

	$v_1 = 5$	$v_2 = 8$	$v_3 = -4$	$v_4 = 3$
$u_1 = 0$		10		
$u_2 = 2$		0		6
$u_3 = 7$	5	4	6	
$u_4 = 9$			4	

The total cost of this solution is 255.
Since $u_1 + v_3 = -4$ and $u_2 + v_3 = -2$, we have two possible starting points for improvements. If we start with the first alternative, we increase a_1 and b_3 (by $\theta = 4$) and get the following solution:

	14		
	0		6
5	0	10	
		4	

The total cost of this solution is $255 - 4 * 4 = 239$. This solution can not be further improved (trying to increase a_2 and b_3 yields $\theta = 0$). The solution is in fact optimal for GTP.

But if we start to improve the optimal TP tableau by first increasing a_2 and b_3 (again by $\theta = 4$), we get the following solution:

	10		
	4		6
5	0	10	
		4	

The total cost of this solution is $255 - 2 * 4 = 247$. This solution can not be improved further (trying to increase a_1 and b_3 yields $\theta = 0$), so we do not reach an optimal solution of GTP from this starting point.

7 Conclusions

We have considered the classical transportation problem and studied the occurrence of the so-called *transportation paradox* (also called *the more-for-less paradox*). Even if the first discovery of the paradox is a bit unclear, it is evident that it has been known since the early days of LP. It has, however, got very little attention in the literature, and it is almost completely ignored in textbooks all over the world, both elementary and more advanced books. The main reason for this may be that it is considered as a rather odd phenomenon which hardly occurs in any practical situation.

The simulation research reported by Finke in [9] indicates, however, that the paradox may occur quite frequently. The rather restrictive conditions for a cost matrix to be immune against the paradox (see Theorem 1) point in the same direction.

We therefore urge that the transportation paradox should be given much more attention both in practical applications and in teaching. The paradox should be taught as an important part of the numerical properties of this classical problem.

In addition we hope that a lot of the existing excellent software for TP will be extended to include at least a preprocessing routine for deciding whether the cost matrix is immune or not against the paradox. If the cost matrix is not immune, and there are optimal dual variables satisfying (4), an option allowing postprocessing of the optimal solution should be available. The cost of these additional computations is modest. Such postprocessing may provide valuable new insight in the problem from which the data for the actual TP-instance originates.

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