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ON THE COMPUTATION OF BASIC FEASIBLE SOLUTIONS IN LINEAR PROGRAMMING

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ABSTRACT

Given $A \in \mathbf{R}_m^{m \times n}$, $b \in \mathbf{R}^m$ and $c \in \mathbf{R}^n$ consider a primal LP:

$$\max \{ c^T x : Ax = b \ , \ x \geq 0 \} \tag{P}$$

and its dual, written as:

$$\min \{ \tilde{x}^T y : y \in R(A^T) \ , \ y \geq c \} \tag{$\tilde{D}$}$$

where $\tilde{x}$ is <u>any</u> solution of $Ax = b$. We present two algorithms, primal and dual, for the computation of basic feasible solutions (b.f.s.'s). These algorithms use adjacent updates of canonical bases of $N(A)$ and $R(A^T)$ respectively. Starting with any basic (non-feasible) solution, they find a b.f.s. of (P) (with exit to the Simplex Algorithm) or a b.f.s. of $(\tilde{D})$ (exit to the Dual Method) <u>or</u> determine that the problem in question is infeasible. Both algorithms use no artificial variables.

6 References

KEY WORDS:

Linear Programming, Linear Inequalities, Basic Solutions

1. INTRODUCTION

Given $A \in \mathbf{R}_m^{m \times n}$, $b \in \mathbf{R}^m$ and $c \in \mathbf{R}^n$ consider the primal LP:

$$\max \{ c^T x : Ax = b \ , \ x \geq 0 \} \tag{P}$$

In [1] the Simplex Algorithm, [4], for solving (P) was presented in terms of canonical bases $\mathbf{B}$ of $N(A)$ (the null-space of A), and was accordingly called the B-Simplex Algorithm. The $\mathbf{B}$-Simplex Algorithm requires, at the start, that a basic feasible solution (b.f.s.) of (P) be available.

The dual problem of (P) can be written, [1], in the form

$$\min \{ \tilde{x}^T y : y \in R(A^T) \ , \ y \geq c \} \tag{$\tilde{D}$}$$

where $\tilde{x}$ is any solution of $Ax = b$, and $R(A^T)$ is the range of A^T. The Dual Method [5] was similarly implemented in terms of canonical bases $\mathbf{C}$ of $R(A^T)$, and called the C-Dual Method, [1]. It requires, at the start, a b.f.s. of $(\tilde{D})$.

In this paper we consider the problem of finding a b.f.s. of (P) or $(\tilde{D})$. We present two algorithms, primal and dual, using adjacent updates of canonical bases of $N(A)$ and $R(A^T)$ respectively. Both algorithms proceed along basic solutions, reducing their infeasibility [1] until a b.f.s. is produced, or the problem is declared infeasible. Both algorithms use no artificial variables.

Both algorithms use essentially the same data, since a canonical basis of $N(A)$ can be obtained, without computation, from a canonical basis of $R(A^T)$

The primal algorithm reduces the (P)-infesibility, and the dual algorithm reduces the $(\tilde{D})$-infeasibility.

and vice versa [2]. Thus the algorithms can be "mixed", by alternating primal and dual steps.

The paper has 6 sections. The notation, and preliminaries, are given in §§ 2-3. The primal algorithm and examples are presented in §§ 4-5. The dual algorithm is given in § 6.

2. NOTATION

We follow the notation of [1]. In particular:

For any integers j, k :

$$\overline{j,k} = \begin{cases} \{j, j+1, \ldots, k\} & , \text{ if } j \leq k \\ \emptyset & , \text{ if } j > k \end{cases}$$

For any $J \subset \overline{1,n}$:

$\#J$ - the number of elements in J

J^c - the complement of J.

For any two sets J, K :

$J \backslash K = J \cap K^c$, the set-theoretic difference.

For any $A \in \mathbf{R}^{m \times n}$, $x \in \mathbf{R}^n$, $J \subset \overline{1,n}$:

$A[\ ,J]$ - the submatrix of columns indexed by J,

$x[J]$ - the subvector of components in J, in particular,

$x[j]$ - the jth component of x, also x_j.

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[2] Using (1),(2) and (3) of § 3.A.

3. PRELIMINARIES

3.A Canonical bases. Given $A \in \mathbf{R}_m^{m\times n}$, a subset $J \subset \overline{1,n}$ is basic if $A[\ ,J]$ is nonsingular.

Given a basic J and a list β of the elements of J^c,

$$J^c = \{\beta_1, \beta_2, \cdots, \beta_{n-m}\}$$

there is a unique basis

$$\mathbf{B} = \{b^1, b^2, \cdots, b^{n-m}\}$$

of $N(A)$, the null-space of A, such that, for any $j \in \overline{1,n-m}$

$$b^j[\beta_j] = 1, \quad b^k[\beta_j] = 0 \quad \text{for all} \quad j \neq k \in \overline{1,n-m}. \tag{1}$$

The J-canonical basis of $N(A)$ is the pair $\{\mathbf{B}, \beta\}$ (or just $\mathbf{B}$, if β is understood).

Analogously, given a basic J and a list γ of the elements of J,

$$J = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$$

there is a unique basis

$$\mathbf{C} = \{c^1, c^2, \cdots, c^m\}$$

of $R(A^T)$, the range of A^T, such that for any $j \in \overline{1,m}$

$$c^j[\gamma_j] = 1, \quad c^k[\gamma_j] = 0 \quad \text{for all} \quad j \neq k \in \overline{1,m}. \tag{2}$$

The J-canonical basis of $R(A^T)$ is the pair $\{\mathbf{C}, \gamma\}$ (or just $\mathbf{C}$).

The subspaces $N(A)$ and $R(A^T)$ are orthogonal, hence

$$b^j[\gamma_k] = -\ c^k[\beta_j] \quad \text{for all } k \in \overline{1,m},\ j \in \overline{1,n-m} \tag{3}$$

for any two J-canonical bases $\{\mathbf{B}, \beta\}$ and $\{\mathbf{C}, \gamma\}$.

3.B Basic solutions. Given a basic J, a vector x is the J-basic solution of (P) if

$$Ax = b \tag{4}$$

$$x[J^c] = 0 \tag{5}$$

If in addition,

$$x[J] \geq 0 \tag{6}$$

x is the J-basic feasible solution (J-b.f.s. or just b.f.s.) of (P), [3] .

A vector y is the J-basic solution of $(\tilde{D})$ if

$$y[J] = c[J] \tag{7}$$

If in addition,

$$y[J^c] \geq c[J^c] \tag{8}$$

y is the J-b.f.s. of $(\tilde{D})$.

The J-basic solutions x and y satisfy [4]

$$(y - c)^T x = 0 \tag{9}$$

and

$$-c^T b^j = y[\beta_j] - c[\beta_j] \quad \text{for all } j \in \overline{1,n-m} \tag{10}$$

$$\tilde{x}^T c^k = x[\gamma_k] \quad \text{for all } k \in \overline{1,m} \tag{11}$$

where $\tilde{x}$ is any solution of (4), and $\{\mathbf{B}, \beta\}$, $\{\mathbf{C}, \gamma\}$ are the J-canonical bases of $N(A)$, $R(A^T)$.

3.C Adjacent updates. Two basic sets J and $\bar{J}$, with

[3] For any basic J, there is a unique J-basic solution, which need not be feasible. Thus not every basic J has a J-b.f.s.

[4] An orthogonality relation known as complementary slackness.

$$\#(J \cap \bar{J}) = m-1 \tag{12}$$

are called adjacent, and so are called the corresponding canonical bases, basic solutions, and b.f.s.'s. An adjacent update is the transformation of a J-variable (e.g. canonical basis, b.f.s.) to the corresponding $\bar{J}$-variable.

Let J be a basic set, with a J-canonical basis of $N(A)$,

$$\mathbf{B} = \{b^1, b^2, \cdots, b^{n-m}\}, \quad \beta = \{\beta_1, \beta_2, \cdots, \beta_{n-m}\}$$

and a J-canonical basis of $R(A^T)$,

$$\mathbf{C} = \{c^1, c^2, \cdots, c^m\}, \quad \gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$$

For any $L \subset J^c$ we denote by

$$\beta^{-1}(L) = \{j \in \overline{1,n-m} : \beta_j \in L\}$$

Similarly,

$$\gamma^{-1}(K) = \{j \in \overline{1,m} : \gamma_j \in K\} \text{ for any } K \subset J.$$

Let $\bar{J} \subset \overline{1,n-m}$ satisfy (12), and let $\#\bar{J} = m$. Then $\bar{J}$ is basic if and only if (for proof see [1])

$$b^p[q] \neq 0 \text{ where } p = \beta^{-1}(\bar{J}\backslash J) \text{ and } q = J\backslash\bar{J}, \tag{13}$$

in which case, the $\bar{J}$-canonical basis of $N(A)$,

$$\bar{\mathbf{B}} = \{\bar{b}^1, \bar{b}^2, \cdots, \bar{b}^{n-m}\}, \quad \bar{\beta} = \{\bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_{n-m}\}$$

is given by

$$\bar{b}^p = \frac{1}{b^p[q]} b^p, \quad \bar{\beta}_p = q \tag{14}$$

$$\bar{b}^j = b^j - b^j[q]\bar{b}^p, \quad \bar{\beta}_j = \beta_j \text{ for all } p \neq j \in \overline{1,n-m} \tag{15}$$

Alternatively, $\bar{J}$ is basic if and only if

$$c^s[t] \neq 0 \text{ where } s = \gamma^{-1}(q),\ t = \beta_p, \text{ and } p, q \text{ as in (13)} \tag{16}$$

in which case the $\bar{J}$-canonical basis of $R(A^T)$

$$\overline{\mathbf{C}} = \{\overline{c}^1, \overline{c}^2, \cdots, \overline{c}^m\}, \quad \overline{\gamma} = \{\overline{\gamma}_1, \overline{\gamma}_2, \cdots, \overline{\gamma}_m\}$$

is given by

$$\overline{c}^s = \frac{c^s}{c^s[t]}, \quad \overline{\gamma}_s = t \tag{17}$$

$$\overline{c}^j = c^j - c^j[t]\,\overline{c}^s, \quad \overline{\gamma}_j = \gamma_j \quad \text{for all } s \neq j \in \overline{1,m} \tag{18}$$

4. A PRIMAL ALGORITHM

The algorithm below starts with a J-basic solution x (satisfying (4) and (5)), and tries to find a J-b.f.s. (also satisfying (6)). The algorithm performs a sequence of adjacent updates. Upon exit from the algorithm, either a J-b.f.s. has been found (to be used as the initial b.f.s. of the **B**-Simplex Algorithm) or (P) has been declared infeasible. It is also possible to exit from the algorithm when a J-b.f.s. of $(\tilde{D})$ has been found, which can serve as the initial solution for the **C**-Dual Method.

<u>Primal Algorithm</u>.

<u>Input</u>: $A \in \mathbf{R}_m^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$ (* problem data *)

$J \subset \overline{1,n}$ (* a basic set *)

x (* the J-basic solution of (P) *)

$\mathbf{B} = \{b^1, b^2, \cdots, b^{n-m}\}$, $\beta = \{\beta_1, \beta_2, \cdots, \beta_{n-m}\}$

(* the J-canonical basis of $N(A)$ *)

<u>Step 1</u>: **if** $x \geq 0$ **then exit** (to the **B**-Simplex Algorithm);

if $c^T b^j \leq 0$ for all $j \in \overline{1,n-m}$ **then exit** (to the C-Dual Algorithm); [5]

Step 2: **let** i be the smallest integer such that $x[i] < 0$; [6]

if $b^j[i] \leq 0$ for all $j \in \overline{1,n-m}$ **then exit** ((P) is infeasible)

else select p such that [7]

$$b^p[i] > 0; \tag{19}$$

Step 3: (* calculation of step size θ_p *)

$$\theta_p := -\frac{x[i]}{b^p[i]} \tag{20}$$

$$\theta := \begin{cases} \min\limits_k \{-\dfrac{x[k]}{b^p[k]} : k \in J,\ b^p[k] < 0,\ x[k] \geq 0\} \\ \infty \text{ if } b^p[J] \geq 0 \end{cases} \tag{21}$$

if $\theta_p \leq \theta$ **then** $q := i$, **go to** Step 4;

else select q such that

$$\theta = -\frac{x[q]}{b^p[q]}, \quad x[q] \geq 0, \quad b^p[q] < 0 \tag{22}$$

let $\theta_p := \theta$ (23)

(* the nonnegative components of x remain nonnegative in $\overline{x}$ below *)

Step 4:

$$\overline{x} := x + \theta_p b^p; \qquad (\text{* update } x \text{ *}) \tag{24}$$

Step 5:

---

[5] If $c^T b^j \leq 0$ for all $j \in \overline{1,n-m}$ then the J-basic solution y of $(\tilde{D})$ is feasible, by (8) and (10).

[6] Any i with $x[i] < 0$ will do, but this choice guarantees that the algorithm stays with i until $x[i] = 0$ or until (P) has been declared infeasible.

[7] Any such p will do, but if $c^T b^p > 0$ then, for $\theta_p > 0$ in (24), the new basic solution $\overline{x}$ will have a higher value of $c^T x$.

$\overline{J} := (J \setminus \{q\}) \bigcup \{\beta_p\}$; (* update J *) (25)

execute (14); (* update **B** *)

execute (15);

Step 6: remove all bars (* $\overline{x}$ becomes x, etc. *)

go to Step 1.

□

The algorithm works by increasing the value of the selected negative component $x[i]$, without violating any of the already satisfied nonnegativity constraints. Indeed, the selection of i in Step 2 guarantees that the algorithm stays with $x[i]$ until it becomes nonnegative, or (P) is declared infeasible. While the algorithm stays with i it solves, by adapting the **B**-Simplex Algorithm of [1], the

Auxilliary Problem

$$\begin{aligned} \max\ & x[i] \\ & Ax = b \\ & x[i] \leq 0 \ . \\ & x[J_+] \geq 0, \ \ x[J^c] \geq 0 \end{aligned} \tag{AP}$$

Here J is the current basic set, and J_+ is the subset of J where the J-basic solution has nonnegtaive components.

Let J be basic, x the J-basic (non-feasible) solution of (P), and let i be the smallest integer with $x[i] < 0$. Since, by (24),

$$\overline{x}[i] := x[i] + \theta_p b^p[i]$$

it follows from (19) that

$$\overline{x}[i] > x[i] \text{ if } \theta_p > 0$$

i.e. if $\theta > 0$ in (21).

Cycling occurs when, following several iterations with $\theta = 0$ (in which the basic solution x does not change), the basic set J is visited again. Let

$$Q = \{\gamma_j \in J : x[\gamma_j] = 0 \ , \ b^p[\gamma_j] < 0 \} \tag{26}$$

If cycling occurs, and the cycle begins and ends in J, then all intermediate basic sets $\overline{J}$ satisfy

$$\overline{J} \backslash J \subset Q \tag{27}$$

since all itermediate pivots $b^p[q]$ have $q \in Q$.

We adapt now a standard argument in LP to show that cycling can be prevented by appropriate selection of q in (22), when $\theta = 0$. We denote by

$$v \underset{\text{lex}}{>} 0$$

the fact that the non-zero vector v is <u>lexicographically positive</u>, i.e. the first non-zero component is positive. Similarly

$$u \underset{\text{lex}}{>} v \quad \text{denotes } u - v \underset{\text{lex}}{>} 0$$

The <u>lexicographic minimum</u> of a set of vectors S is denoted by: lexmin S.

We assume initially that

$$c^k \underset{\text{lex}}{>} 0 \quad \text{for all } k \in \gamma^{-1}(Q)$$

referred to below as <u>Assumption 1</u>. It can be guaranteed by rearranging the columns of A so that Q precedes J^c.

Lemma 1. Under Assumption 1, the basic set J cannot be repeated if $q \in Q$ is selected by

$$q = \gamma_s \tag{28}$$

where

$$\frac{c^s}{c^s[\beta_p]} = \text{lexmin}\ \{\frac{c^j}{c^j[\beta_p]} : j \in \gamma^{-1}(Q)\} \tag{29}$$

Proof: We show first that Assumption 1 holds throughout the iterations. From (3) and (26) it follows that

$$c^j[\beta_p] > 0 \quad \text{for all} \quad j \in \gamma^{-1}(Q) \tag{30}$$

and therefore, by (17),

$$\overline{c}^s \underset{\text{lex}}{>} 0 \quad \text{if} \quad c^s \underset{\text{lex}}{>} 0 \tag{31}$$

Similarly, by (18) and (29), for any $s \neq j \in \gamma^{-1}(Q)$

$$\overline{c}^j \underset{\text{lex}}{>} 0 \quad \text{if} \quad c^j \underset{\text{lex}}{>} 0 \tag{32}$$

proving that Assumption 1 is invariant.

We show now that the vector $c^{\gamma^{-1}(i)}$ increases lexicographically at each iteration. Indeed, by (3) and (19),

$$c^{\gamma^{-1}(i)}[\beta_p] = -b^p[i] < 0$$

so that, by (18),

$$\overline{c}^{\gamma^{-1}(i)} := c^{\gamma^{-1}(i)} - c^{\gamma^{-1}(i)}[\beta_p]\overline{c}^s \tag{33}$$

$$\underset{\text{lex}}{>} c^{\gamma^{-1}(i)} \quad \text{since} \quad \overline{c}^s \underset{\text{lex}}{>} 0$$

□

An equivalent anti-cycling rule can be obtained, as in [3], by using the perturbation

$$b(\epsilon) := b + \sum_{j \in Q} A[,j]\epsilon^{\gamma^{-1}(j)}, \qquad 0 < \epsilon << 1 \tag{34}$$

For the perturbed problem

$$\max \{c^T x : Ax = b(\epsilon),\ x \geq 0\}$$

the J-basic solution $x(\epsilon)$ is

$$x(\epsilon) := x + \sum_{j \in Q} \epsilon^{\gamma^{-1}(j)} e_j$$

where e_j is the jth unit vector. Cycling therefore cannot occur in the perturbed problem, a fact which can be translated into an anti-cycling rule for the original problem.

Another anti-cycling rule can be obtained by adapting Bland's rule [2].

5. EXAMPLES

Example 1 (taken from [6])

$$\begin{array}{lll} \max & -2x_1 + x_2 & \\ \text{s.t.} & -2x_1 + x_2 + x_3 + x_4 & = -3 \\ & -x_1 + x_2 - x_3 \qquad\qquad + x_5 & = -2 \\ & x_j \geq 0, \quad j \in \overline{1,5} & \end{array} \tag{P}$$

For the basic set

$$J := \{4,\ 5\},$$

the J-basic solution is

$$x^T = (0,\ 0,\ 0,\ -3,\ -2),$$

which is not feasible. The J-canonical basis of $N(A)$ is

$$b^1 = (1,\ 0,\ 0,\ 2,\ 1)\ ,\quad \beta_1 = 1$$

$$b_2 = (0,\ 1,\ 0,\ -1,\ -1)\ ,\quad \beta_2 = 2$$

$$b_3 = (0,\ 0,\ 1,\ -1,\ 1)\ ,\quad \beta_3 = 3$$

<u>Iteration 1</u>

Step 2: Select $i = 4$ (since $x[4] < 0$) and $p = 1$ ($b^1[4] > 0$)

Step 3: $\theta_1 := \dfrac{3}{2},\quad \theta := \infty,\quad q := i = 4$

Step 4: $\overline{x}^T := (0,\ 0,\ 0,\ -3,\ -2) + \dfrac{3}{2}(1,\ 0,\ 0,\ 2,\ 1) = (\dfrac{3}{2},\ 0,\ 0,\ 0,\ -\dfrac{1}{2})$

Step 5: $\overline{J} := (J\backslash\{4\}) \bigcup \{1\} = \{5,\ 1\}$

$$\overline{b}^1 := \frac{1}{2}(1,\ 0,\ 0,\ 2,\ 1) = (\frac{1}{2},\ 0,\ 0,\ 1,\ \frac{1}{2}),\ \overline{\beta}_1 := 4$$

$$\overline{b}^2 := b^2 - b^2[4]\overline{b}^1 = (\frac{1}{2},\ 1,\ 0,\ 0,\ -\frac{1}{2}),\ \overline{\beta}_2 := 2$$

$$\overline{b}_3 := b^3 - b^3[4]\overline{b}^1 = (\frac{1}{2},\ 0,\ 1,\ 0,\ \frac{3}{2}),\ \overline{\beta}_3 := 3$$

<u>Iteration 2</u>

Step 2: Select $i = 5$ and $p = 3$.

Step 3: $\theta_3 := -\dfrac{x[5]}{b^3[5]} = \dfrac{1}{3}\ ,\quad \theta := \infty,\quad q := i = 5$

Step 4: $\overline{x}^T := (\dfrac{3}{2},\ 0,\ 0,\ 0,\ -\dfrac{1}{2}) + \dfrac{1}{3}\ (\dfrac{1}{2},\ 0,\ 1,\ 0,\ \dfrac{3}{2}) = (\dfrac{5}{3},\ 0,\ \dfrac{1}{3},\ 0,\ 0)$

Step 5: $\overline{J} := (\{5,\ 1\}\backslash\{5\}) \bigcup \{3\} = \{1,\ 3\}$

$$\overline{b}^3 := \frac{2}{3}(\frac{1}{2},\ 0,\ 1,\ 0,\ \frac{3}{2}) = (\frac{1}{3},\ 0,\ \frac{2}{3},\ 0,\ 1)\ ,\ \overline{\beta}_3 := 5$$

$$\overline{b}^1 := b^1 - b^1[5]\overline{b}^3 = (\frac{1}{3},\ 0,\ -\frac{1}{3},\ 1,\ 0)\ ,\ \overline{\beta}_1 := 4$$

$$\overline{b}^2 := b^2 - b^2[5]\overline{b}^3 = (\frac{2}{3},\ 1,\ \frac{1}{3},\ 0,\ 0)\ ,\ \overline{\beta}^2 := 2$$

Iteration 3

Step 1: $x \geq 0$, **exit** (to the **B**-Simplex Algorithm) with $J = \{1,\ 3\}$.

Example 2

$$\begin{array}{lll} \max & 3x_1 + x_2 & \\ \text{s.t.} & x_1 + 2x_2 + x_3 & = 3 \\ & -x_1 - x_2 \qquad\quad + x_4 & = -5 \\ & x_j \geq 0,\ j \in \overline{1,4} & \end{array} \qquad \text{(P)}$$

For the basic set

$$J = \{3,\ 4\}$$

the basic solution

$$x^T = (0,\ 0,\ 3,\ -5)$$

is not feasible. The J-canonical basis of $N(A)$ is

$$b^1 = (1,\ 0,\ -1,\ 1)\ ,\ \beta_1 = 1$$

$$b^2 = (0,\ 1,\ -2,\ 1)\ ,\ \beta_2 = 2$$

Iteration 1

Step 2: Select $i := 4$, $p := 1$

Step 3: $\theta_1 := 5$, $\theta := 3$, $q := 3$

Step 4: $\overline{x}^T := (0, 0, 3, -5) + 3\,(1, 0, -1, 1) = (3, 0, 0, -2)$

Step 5: $\overline{J} := (\{3, 4\}\backslash\{3\}) \bigcup \{1\} = \{4, 1\}$

$$\overline{b}^1 := \frac{1}{-1} b^1 = (-1, 0, 1, -1)\ ,\ \overline{\beta}_1 := 3$$

$$\overline{b}^2 := b^2 - b^2[3]\overline{b}^1 = (-2, 1, 0, -1)\ ,\ \ \overline{\beta}_2 := 2$$

<u>Iteration 2</u>

Step 2: Select $i := 4$, but $b^1[4] < 0$, $b^2[4] < 0$. Therefore (P) is infeasible, **exit.**

6. A DUAL ALGORITHM

The primal algorithm of § 4 uses adjacent updates of canonical bases of $N(A)$ to improve the (P)-feasibility of the current basic solutions. Analogously, a dual algorithm can be given which uses canonical bases of $R(A^T)$ to improve the $(\tilde{D})$-feasibility.

<u>Dual Algorithm.</u>

<u>Input</u>: $A \in \mathbf{R}_m^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$ (* problem data *)

$J \subset \overline{1,n}$ (* a basic set *)

y (* the J-basic solution of $(\tilde{D})$ *)

$\mathbf{C} = \{c^1, c^2, \cdots, c^m\}$, $\gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$

(* the J-canonical basis of $R(A^T)$ *)

$\tilde{x}$ any solution of $Ax = b$

Step 1: **if** $y[J^c] \geq c[J^c]$ **then exit** (to the C-Dual Algorithm);

if $\tilde{x}^T c^k \geq 0$ for all $k \in \overline{1,m}$ **then exit** (to the B-Simplex Algorithm); [8]

Step 2: **let** j be the smallest integer such that $y[j] < c[j]$; [9]

if $c^k[j] \leq 0$ for all $k \in \overline{1,m}$ **then exit** ($(\tilde{D})$ is infeasible)

else select s such that

$$c^s[j] > 0;$$

Step 3: (* calculation of step size θ_s *)

$$\theta_s := -\frac{y[j] - c[j]}{c^s[j]}$$

$$\theta := \begin{cases} \min\limits_k \{-\dfrac{y[k] - c[k]}{c^s[k]} : k \in J^c,\ c^s[k] < 0,\ y[k] \geq c[k]\} \\ \infty \text{ if } c^s[J^c] \geq 0 \end{cases}$$

if $\theta_s \leq \theta$ **then** $t := j$, **go to** Step 4;

else select t such that [10]

$$\theta = -\frac{y[t] - c[t]}{c^s[t]}, \quad y[t] \geq c[t], \quad c^s[t] < 0$$

let $\theta_s := \theta$

Step 4:

---

[8] If $\tilde{x}^T c^k \geq 0$ for all $k \in \overline{1,m}$ then the J-basic solution x of (P) is feasible, by (6) and (11).

[9] In analogy with the primal algorithm, j can be chosen arbitrarily. This particular choice guarantees that the algorithm stays with j until $y[j] = c[j]$, or $(\tilde{D})$ is declared infeasible.

[10] As in the primal algorithm, an anti-cycling rule (such as given in Lemma 1) should be used in this selection.

$\bar{y} := y + \theta_s c^s$; (* update y *)

Step 5:

$\bar{J} := (J \backslash \{\gamma_s\}) \bigcup \{t\}$; (* update J *)

execute (17); (* update C *)

execute (18);

Step 6: remove all bars (* $\bar{y}$ becomes y, etc. *)

go to Step 1.

□

Finiteness can be proved here analogously to the finiteness proof in Lemma 1.

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