ON THE COMPUTATION OF BASIC FEASIBLE SOLUTIONS IN LINEAR PROGRAMMING

BY

ADI BEN-ISRAEL AND SVERRE STORøy

REPORT NO. 19 JUNE 1986

DEPARTMENT OF MATHEMATICAL SCIENCES UNIVERSITY OF DELAWARE
NEWARK, DE 19716, U.S.A.

DEPARTMENT OF COMPUTER SCIENCE UNIVERSITY OF BERGEN

AND

DEPARTMENT OF SCIENCE AND TECHNOLOGY CHR. MICHELSENS INSTITUTE BERGEN, NORWAY

Research supported by the National Science Foundation Grand ECS-8214081 and the Norwegian Council of Fisheries Research.
ABSTRACT

Given $A \in \mathbb{R}_m^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ consider a primal LP:

$$\max \{ c^T x : Ax = b, \ x \geq 0 \} \quad (P)$$

and its dual, written as:

$$\min \{ \tilde{x}^T y : y \in R(A^T), \ y \geq c \} \quad (\tilde{D})$$

where $\tilde{x}$ is any solution of $Ax = b$. We present two algorithms, primal and dual, for the computation of basic feasible solutions (b.f.s.'s). These algorithms use adjacent updates of canonical bases of $N(A)$ and $R(A^T)$ respectively. Starting with any basic (non-feasible) solution, they find a b.f.s. of $(P)$ (with exit to the Simplex Algorithm) or a b.f.s. of $(\tilde{D})$ (exit to the Dual Method) or determine that the problem in question is infeasible. Both algorithms use no artificial variables.

6 References

KEY WORDS:

Linear Programming, Linear Inequalities, Basic Solutions
1. INTRODUCTION

Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ consider the primal LP:

$$\max \{ c^T x : Ax = b, \ x \geq 0 \}$$

In [1] the Simplex Algorithm, [4], for solving $(P)$ was presented in terms of canonical bases $B$ of $N(A)$ (the null-space of $A$), and was accordingly called the $B$-Simplex Algorithm. The $B$-Simplex Algorithm requires, at the start, that a basic feasible solution (b.f.s.) of $(P)$ be available.

The dual problem of $(P)$ can be written, [1], in the form

$$\min \{ \tilde{z}^T y : y \in R(A^T), \ y \geq c \}$$

where $\tilde{z}$ is any solution of $Ax = b$, and $R(A^T)$ is the range of $A^T$. The Dual Method [5] was similarly implemented in terms of canonical bases $C$ of $R(A^T)$, and called the $C$-Dual Method, [1]. It requires, at the start, a b.f.s. of $(\tilde{D})$.

In this paper we consider the problem of finding a b.f.s. of $(P)$ or $(\tilde{D})$. We present two algorithms, primal and dual, using adjacent updates of canonical bases of $N(A)$ and $R(A^T)$ respectively. Both algorithms proceed along basic solutions, reducing their infeasibility until a b.f.s. is produced, or the problem is declared infeasible. Both algorithms use no artificial variables.

Both algorithms use essentially the same data, since a canonical basis of $N(A)$ can be obtained, without computation, from a canonical basis of $R(A^T)$

---

The primal algorithm reduces the $(P)$-infeasibility, and the dual algorithm reduces the $(\tilde{D})$-infeasibility.
and vice versa. Thus the algorithms can be "mixed", by alternating primal and dual steps.

The paper has 6 sections. The notation, and preliminaries, are given in §§ 2-3. The primal algorithm and examples are presented in §§ 4-5. The dual algorithm is given in § 6.

2. NOTATION

We follow the notation of [1]. In particular:

For any integers $j$, $k$:

$$j, k = \begin{cases} \{j, j+1, \ldots, k\}, & \text{if } j \leq k \\ \emptyset, & \text{if } j > k \end{cases}$$

For any $J \subseteq \{1, \ldots, n\}$:

$\#J$ - the number of elements in $J$

$J^c$ - the complement of $J$.

For any two sets $J$, $K$:

$J \setminus K = J \cap K^c$, the set-theoretic difference.

For any $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $J \subseteq \{1, \ldots, n\}$:

$A[\cdot, J]$ - the submatrix of columns indexed by $J$,

$x[J]$ - the subvector of components in $J$, in particular,

$x[j]$ - the $j^{th}$ component of $x$, also $x_j$.

\footnote{Using (1),(2) and (3) of § 3.A.}
3. PRELIMINARIES

3.A Canonical bases. Given $A \in \mathbb{R}^{m \times n}$, a subset $J \subset \overline{1,n}$ is basic if $A[.,J]$ is nonsingular.

Given a basic $J$ and a list $\beta$ of the elements of $J^c$,

$$J^c = \{\beta_1, \beta_2, \ldots, \beta_{\overline{n-m}}\}$$

there is a unique basis

$$B = \{b^1, b^2, \ldots, b^{\overline{n-m}}\}$$

of $N(A)$, the null-space of $A$, such that, for any $j \in \overline{1,n-m}$

$$b^j[\beta_j] = 1, \quad b^k[\beta_j] = 0 \quad \text{for all} \quad j \neq k \in \overline{1,n-m}.$$  \hfill (1)

The $J$-canonical basis of $N(A)$ is the pair $\{B, \beta\}$ (or just $B$, if $\beta$ is understood).

Analogously, given a basic $J$ and a list $\gamma$ of the elements of $J$,

$$J = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$$

there is a unique basis

$$C = \{c^1, c^2, \ldots, c^m\}$$

of $R(A^T)$, the range of $A^T$, such that for any $j \in \overline{1,m}$

$$c^j[\gamma_j] = 1, \quad c^k[\gamma_j] = 0 \quad \text{for all} \quad j \neq k \in \overline{1,m}.$$  \hfill (2)

The $J$-canonical basis of $R(A^T)$ is the pair $\{C, \gamma\}$ (or just $C$).

The subspaces $N(A)$ and $R(A^T)$ are orthogonal, hence

$$b^j[\gamma_k] = -c^k[\beta_j] \quad \text{for all} \quad k \in \overline{1,m}, \quad j \in \overline{1,n-m}.$$  \hfill (3)

for any two $J$-canonical bases $\{B, \beta\}$ and $\{C, \gamma\}$. 


3.B Basic solutions. Given a basic $J$, a vector $x$ is the $J$-basic solution of $(P)$ if

\[ Ax = b \quad (4) \]
\[ x[F^c] = 0 \quad (5) \]

If in addition,

\[ x[J] \geq 0 \quad (6) \]

$x$ is the $J$-basic feasible solution ($J$-b.f.s. or just b.f.s.) of $(P),^3$

A vector $y$ is the $J$-basic solution of $(\tilde{D})$ if

\[ y[J] = c[J] \quad (7) \]

If in addition,

\[ y[J^c] \geq c[J^c] \quad (8) \]

$y$ is the $J$-b.f.s. of $(\tilde{D})$.

The $J$-basic solutions $x$ and $y$ satisfy \(^4\)

\[ (y - c)^T x = 0 \quad (9) \]

and

\[ -c^T b^j = y[\beta_j] - c[\beta_j] \quad \text{for all } j \in \bar{1,n-m} \quad (10) \]
\[ \bar{x}^T c^k = x[\gamma_k] \quad \text{for all } k \in \bar{1,m} \quad (11) \]

where $\bar{x}$ is any solution of $(4)$, and \{B, $\beta$\}, \{C, $\gamma$\} are the $J$-canonical bases of $N(A)$, $R(A^T)$.

3.C Adjacent updates. Two basic sets $J$ and $\bar{J}$, with

---

\(^3\) For any basic $J$, there is a unique $J$-basic solution, which need not be feasible. Thus not every basic $J$ has a $J$-b.f.s.

\(^4\) An orthogonality relation known as complementary slackness.
\[ \#(J \cap \bar{J}) = m - 1 \]  \hspace{1cm} (12)

are called adjacent, and so are called the corresponding canonical bases, basic solutions, and b.f.s.'s. An adjacent update is the transformation of a \( J \)-variable (e.g. canonical basis, b.f.s.) to the corresponding \( \bar{J} \)-variable.

Let \( J \) be a basic set, with a \( J \)-canonical basis of \( N(A) \),

\[ B = \{ b^1, b^2, \cdots, b^{n-m} \}, \quad \beta = \{ \beta_1, \beta_2, \cdots, \beta_{n-m} \} \]

and a \( J \)-canonical basis of \( R(A^T) \),

\[ C = \{ c^1, c^2, \cdots, c^m \}, \quad \gamma = \{ \gamma_1, \gamma_2, \cdots, \gamma_m \} \]

For any \( L \subseteq J^c \) we denote by

\[ \beta^{-1}(L) = \{ j \in 1, n-m : \beta_j \in L \} \]

Similarly,

\[ \gamma^{-1}(K) = \{ j \in 1, m : \gamma_j \in K \} \text{ for any } K \subseteq J. \]

Let \( \bar{J} \subseteq 1, n-m \) satisfy (12), and let \( \#\bar{J} = m \). Then \( \bar{J} \) is basic if and only if (for proof see [1])

\[ b^p[q] \neq 0 \text{ where } p = \beta^{-1}(\bar{J} \setminus J) \text{ and } q = J \setminus \bar{J}, \]  \hspace{1cm} (13)

in which case, the \( \bar{J} \)-canonical basis of \( N(A) \),

\[ \bar{B} = \{ \bar{b}^1, \bar{b}^2, \cdots, \bar{b}^{n-m} \}, \quad \bar{\beta} = \{ \bar{\beta}_1, \bar{\beta}_2, \cdots, \bar{\beta}_{n-m} \} \]

is given by

\[ \bar{b}^p = \frac{1}{b^p[q]} b^p, \quad \bar{\beta}_p = q \]  \hspace{1cm} (14)

\[ \bar{b}^j = b^j - b^j[q] \bar{b}^p, \quad \bar{\beta}_j = \beta_j \text{ for all } p \neq j \in 1, n-m \]  \hspace{1cm} (15)

Alternatively, \( \bar{J} \) is basic if and only if

\[ c^s[t] \neq 0 \text{ where } s = \gamma^{-1}(q), \quad t = \beta_p, \text{ and } p, q \text{ as in (13)} \]  \hspace{1cm} (16)

in which case the \( \bar{J} \)-canonical basis of \( R(A^T) \)
Basic Solutions

\[ \bar{C} = \{ \bar{c}^1, \bar{c}^2, \ldots, \bar{c}^m \} , \quad \bar{\gamma} = \{ \bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_m \} \]

is given by

\[ \bar{c}^s = \frac{c^s}{c^s[t]} , \quad \bar{\gamma}_s = t \] (17)

\[ \bar{c}^j = c^j - c^j[t] \bar{c}^s , \quad \bar{\gamma}_j = \gamma_j \quad \text{for all } s \neq j \in 1,m \] (18)

4. A PRIMAL ALGORITHM

The algorithm below starts with a J-basic solution \( x \) (satisfying (4) and (5)), and tries to find a J-b.f.s. (also satisfying (6)). The algorithm performs a sequence of adjacent updates. Upon exit from the algorithm, either a J-b.f.s. has been found (to be used as the initial b.f.s. of the B-Simplex Algorithm) or (P) has been declared infeasible. It is also possible to exit from the algorithm when a J-b.f.s. of (\( \tilde{D} \)) has been found, which can serve as the initial solution for the C-Dual Method.

Primal Algorithm.

**Input:** \( A \in \mathbb{R}_m^{n \times n}, \quad b \in \mathbb{R}^m, \quad c \in \mathbb{R}^n \quad \text{(* problem data *)} \)

\( J \subset 1,n \quad \text{(* a basic set *)} \)

\( x \quad \text{(* the J-basic solution of (P) *)} \)

\( B = \{ b^1, b^2, \ldots, b^{n-m} \}, \quad \beta = \{ \beta_1, \beta_2, \ldots, \beta_{n-m} \} \)

\( \quad \text{(* the J-canonical basis of } N(A) \text{ *)} \)

**Step 1:** if \( x \geq 0 \) then exit (to the B-Simplex Algorithm);
if $c^T b^j \leq 0$ for all $j \in \overline{1,n-m}$ then exit (to the C-Dual Algorithm);  

Step 2: let $i$ be the smallest integer such that $x[i] < 0$;  

if $b^j[i] \leq 0$ for all $j \in \overline{1,n-m}$ then exit (($P$) is infeasible)

else select $p$ such that

$$b^p[i] > 0;$$

(19)

Step 3: (* calculation of step size $\theta_p$ *)

$$\theta_p := -\frac{x[i]}{b^p[i]}$$

(20)

$$\begin{aligned}
\theta &:= \\
&= \min_{k} \{-\frac{x[k]}{b^p[k]}: k \in J, b^p[k] < 0, x[k] \geq 0\}
\end{aligned}$$

(21)

if $\theta_p \leq \theta$ then $q := i$, go to Step 4;

else select $q$ such that

$$\theta = -\frac{x[q]}{b^p[q]}, \quad x[q] \geq 0, \quad b^p[q] < 0$$

(22)

let $\theta_p := \theta$

(* the nonnegative components of $x$ remain nonnegative in $\overline{x}$ below *)

Step 4:

$$\overline{x} := x + \theta_p b^p; \quad \quad \quad \quad \quad (\text{update } x \text{ *)}$$

(24)

Step 5:

---

6 If $c^T b^j \leq 0$ for all $j \in \overline{1,n-m}$ then the J-basic solution $p$ of ($\overline{D}$) is feasible, by (8) and (10).

6 Any $i$ with $x[i] < 0$ will do, but this choice guarantees that the algorithm stays with $i$ until $x[i] = 0$ or until ($P$) has been declared infeasible.

7 Any such $p$ will do, but if $c^T b^p > 0$ then, for $\theta_p > 0$ in (24), the new basic solution $\overline{x}$ will have a higher value of $c^T x$. 

---
Basic Solutions

\( \overline{J} := (J \setminus \{q\}) \cup \{\beta_p\} \); 

(* update \( J \) *)

execute (14);

(* update \( B \) *)

execute (15);

Step 6: remove all bars

(* \( \overline{x} \) becomes \( x \), etc. *)

go to Step 1.

The algorithm works by increasing the value of the selected negative component \( x[i] \), without violating any of the already satisfied nonnegativity constraints. Indeed, the selection of \( i \) in Step 2 guarantees that the algorithm stays with \( x[i] \) until it becomes nonnegative, or \((P')\) is declared infeasible. While the algorithm stays with \( i \) it solves, by adapting the B-Simplex Algorithm of [1], the Auxilliary Problem

\[
\text{max } x[i] \\
Ax = b \\
x[i] \leq 0 . \\
x[J_+] \geq 0, \ x[J^p] \geq 0 
\]

Here \( J \) is the current basic set, and \( J_+ \) is the subset of \( J \) where the \( J \)-basic solution has nonnegative components.

Let \( J \) be basic, \( x \) the \( J \)-basic (non-feasible) solution of \((P)\), and let \( i \) be the smallest integer with \( x[i] < 0 \). Since, by (24),

\[
\overline{x}[i] := x[i] + \theta_{p}b^p[i] 
\]

it follows from (19) that
\[ x[i] > z[i] \text{ if } \theta > 0 \]
i.e. if \( \theta > 0 \) in (21).

Cycling occurs when, following several iterations with \( \theta = 0 \) (in which the basic solution \( x \) does not change), the basic set \( J \) is visited again. Let

\[ Q = \{ \gamma_j \in J : x[\gamma_j] = 0, b^p[\gamma_j] < 0 \} \tag{26} \]

If cycling occurs, and the cycle begins and ends in \( J \), then all intermediate basic sets \( \bar{J} \) satisfy

\[ \bar{J} \setminus J \subset Q \tag{27} \]

since all intermediate pivots \( b^p[q] \) have \( q \in Q \).

We adapt now a standard argument in LP to show that cycling can be prevented by appropriate selection of \( q \) in (22), when \( \theta = 0 \). We denote by

\[ v > 0 \]
\[ \text{lex} \]
the fact that the non-zero vector \( v \) is lexicographically positive, i.e. the first non-zero component is positive. Similarly

\[ u > v \quad \text{ denotes } u - v > 0 \]
\[ \text{lex} \quad \text{lex} \]

The lexicographic minimum of a set of vectors \( S \) is denoted by: \( \text{lexmin } S \).

We assume initially that

\[ c^k > 0 \quad \text{for all } k \in \gamma^{-1}(Q) \]
\[ \text{lex} \]
referred to below as Assumption 1. It can be guaranteed by rearranging the columns of \( A \) so that \( Q \) precedes \( J^c \).
Lemma 1. Under Assumption 1, the basic set \( J \) cannot be repeated if \( q \in Q \) is selected by

\[
q = \gamma_s
\]

where

\[
\frac{c^s}{c^s[\beta_p]} = \operatorname{lexmin}\left\{ \frac{c^j}{c^j[\beta_p]} : j \in \gamma^{-1}(Q) \right\}
\]

Proof: We show first that Assumption 1 holds throughout the iterations. From (3) and (26) it follows that

\[
e^j[\beta_p] > 0 \quad \text{for all} \quad j \in \gamma^{-1}(Q)
\]

and therefore, by (17),

\[
\overline{c}^s > 0 \quad \text{if} \quad c^s > 0 \quad \text{lex}
\]

Similarly, by (18) and (29), for any \( s \neq j \in \gamma^{-1}(Q) \)

\[
\overline{c}^j > 0 \quad \text{if} \quad c^j > 0 \quad \text{lex}
\]

proving that Assumption 1 is invariant.

We show now that the vector \( c^{\gamma^{-1}(i)} \) increases lexicographically at each iteration. Indeed, by (3) and (19),

\[
c^{\gamma^{-1}(i)}[\beta_p] = -b_p[i] < 0
\]

so that, by (18),

\[
\overline{c}^{\gamma^{-1}(i)} := c^{\gamma^{-1}(i)} - c^{\gamma^{-1}(i)}[\beta_p]\overline{c}^s
\]

\[
> c^{\gamma^{-1}(i)} \quad \text{since} \quad \overline{c}^s > 0 \quad \text{lex}
\]
An equivalent anti-cycling rule can be obtained, as in [3], by using the perturbation

\[ b(\epsilon) := b + \sum_{j \in Q} A[j] \epsilon^{r^{-1}(j)}, \quad 0 < \epsilon << 1 \]  \hspace{1cm} (34)

For the perturbed problem

\[ \max \{ \epsilon^T x : Ax = b(\epsilon), x \geq 0 \} \]

the \( J \)-basic solution \( x(\epsilon) \) is

\[ x(\epsilon) := x + \sum_{j \in Q} \epsilon^{r^{-1}(j)} e_j \]

where \( e_j \) is the \( j^{th} \) unit vector. Cycling therefore cannot occur in the perturbed problem, a fact which can be translated into an anti-cycling rule for the original problem.

Another anti-cycling rule can be obtained by adapting Bland's rule [2].

5. EXAMPLES

**Example 1** (taken from [6])

\[
\begin{align*}
\text{max} \quad & -2x_1 + x_2 \\
\text{subject to} \quad & -2x_1 + x_2 + x_3 + x_4 = -3 \\
& -x_1 + x_2 - x_3 + x_5 = -2 \\
& x_j \geq 0, \quad j \in \{4, 5\}
\end{align*}
\]

For the basic set

\[ J := \{4, 5\}, \]
the $J$-basic solution is
\[ x^T = (0, 0, 0, -3, -2), \]
which is not feasible. The $J$-canonical basis of $N(A)$ is
\[ b^1 = (1, 0, 0, 2, 1), \quad \beta_1 = 1 \]
\[ b_2 = (0, 1, 0, -1, -1), \quad \beta_2 = 2 \]
\[ b_3 = (0, 0, 1, -1, 1), \quad \beta_3 = 3 \]

**Iteration 1**

Step 2: Select $i = 4$ (since $x[4] < 0$) and $p = 1$ ($b^1[4] > 0$)

Step 3: $\theta_1 := \frac{3}{2}$, $\theta := \infty$, $q := i = 4$

Step 4: $\bar{x}^T := (0, 0, 0, -3, -2) + \frac{3}{2}(1, 0, 0, 2, 1) = (\frac{3}{2}, 0, 0, 0, -\frac{1}{2})$

Step 5: $\bar{J} := (J \setminus \{4\}) \cup \{1\} = \{5, 1\}$

\[ \bar{b}^1 := \frac{1}{2}(1, 0, 0, 2, 1) = (\frac{1}{2}, 0, 0, 1, \frac{1}{2}), \bar{\beta}_1 := 4 \]

\[ \bar{b}^2 := b^2 - b^2[4]\bar{b}^1 = (\frac{1}{2}, 1, 0, 0, -\frac{1}{2}), \bar{\beta}_2 := 2 \]

\[ \bar{b}_3 := b^3 - b^3[4]\bar{b}^1 = (\frac{1}{2}, 0, 1, 0, \frac{3}{2}), \bar{\beta}_3 := 3 \]

**Iteration 2**

Step 2: Select $i = 5$ and $p = 3$.

Step 3: $\theta_3 := \frac{x[5]}{b^3[5]} = \frac{1}{3}$, $\theta := \infty$, $q := i = 5$

Step 4: $\bar{x}^T := (\frac{3}{2}, 0, 0, 0, -\frac{1}{2}) + \frac{1}{3}(\frac{1}{2}, 0, 1, 0, \frac{3}{2}) = (\frac{5}{3}, 0, \frac{1}{3}, 0, 0)$
Step 5: $\bar{J} := (\{5, 1\} \setminus \{5\}) \cup \{3\} = \{1, 3\}$

$\bar{b}^3 := \frac{2}{3} \left( \frac{1}{2}, 0, 1, 0, \frac{3}{2} \right) = \left( \frac{1}{3}, 0, \frac{2}{3}, 0, 1 \right), \bar{\beta}_3 := 5$

$\bar{b}^1 := b^1 - b^1[5] \bar{b}^3 = \left( \frac{1}{3}, 0, -\frac{1}{3}, 1, 0 \right), \bar{\beta}_1 := 4$

$\bar{b}^2 := b^2 - b^2[5] \bar{b}^3 = \left( \frac{2}{3}, 1, \frac{1}{3}, 0, 0 \right), \bar{\beta}^2 := 2$

Iteration 3

Step 1: $x \geq 0$, exit (to the B-Simplex Algorithm) with $J = \{1, 3\}$.

Example 2

$$\text{max } 3x_1 + x_2 \quad \text{(P)}$$

$$\text{s.t. } x_1 + 2x_2 + x_3 = 3$$

$$-x_1 - x_2 + x_4 = -5$$

$$x_j \geq 0, j \in 1, 4$$

For the basic set

$J = \{3, 4\}$

the basic solution

$x^T = (0, 0, 3, -5)$

is not feasible. The $J$-canonical basis of $N(A)$ is

$b^1 = (1, 0, -1, 1), \beta_1 = 1$

$b^2 = (0, 1, -2, 1), \beta_2 = 2$

Iteration 1
Step 2: Select $i := 4$, $p := 1$

Step 3: $\theta_1 := 5$, $\theta := 3$, $q := 3$

Step 4: $\vec{x}^T := (0, 0, 3, -5) + 3 (1, 0, -1, 1) = (3, 0, 0, -2)$

Step 5: $\tilde{J} := (\{3, 4\}\setminus\{3\}) \cup \{1\} = \{4, 1\}$

$$\vec{b}_1 := \frac{1}{-1} \vec{b}^1 = (-1, 0, 1, -1), \quad \tilde{\beta}_1 := 3$$

$$\vec{b}_2 := \vec{b}^2 - \vec{b}^2[3] \vec{b}_1 = (-2, 1, 0, -1), \quad \tilde{\beta}_2 := 2$$

Iteration 2

Step 2: Select $i := 4$, but $\vec{b}^1[4] < 0$, $\vec{b}^2[4] < 0$. Therefore $(P)$ is infeasible, exit.

6. A DUAL ALGORITHM

The primal algorithm of § 4 uses adjacent updates of canonical bases of $N(A)$ to improve the $(P)$-feasibility of the current basic solutions. Analogously, a dual algorithm can be given which uses canonical bases of $R(A^T)$ to improve the $(\tilde{D})$-feasibility.

Dual Algorithm.

**Input:** $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ (* problem data *)

$J \subset \{1, n\}$ (* a basic set *)

$y$ (* the $J$-basic solution of $(\tilde{D})$ *)

$C = \{c^1, c^2, \ldots, c^m\}$, $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ (* the $J$-canonical basis of $R(A^T)$ *)
Basic Solutions

\[ \hat{x} \text{ any solution of } Ax = b \]

Step 1: if \( y[J^c] \geq c[J^c] \) then exit (to the C-Dual Algorithm);

if \( \hat{x}^T c^k \geq 0 \) for all \( k \in 1, m \) then exit (to the B-Simplex Algorithm); \(^8\)

Step 2: let \( j \) be the smallest integer such that \( y[\hat{j}] < c[\hat{j}] \); \(^9\)

if \( c^k[\hat{j}] \leq 0 \) for all \( k \in 1, m \) then exit ((\( D \)) is infeasible)

else select \( s \) such that \( c'[j] > 0; \)

Step 3: (* calculation of step size \( \theta_\ast \ast \))

\[
\theta_\ast := -\frac{y[j] - c[j]}{c'[j]}
\]

\[
\theta := \begin{cases} 
\min \{ -\frac{y[k] - c[k]}{c'[k]} : k \in J^c, c'[k] < 0, y[k] \geq c[k] \} \\
\infty \text{ if } c'[J^c] \geq 0
\end{cases}
\]

if \( \theta_\ast \leq \theta \) then  \( t := j \), go to Step 4;

else select \( t \) such that \(^{10}\)

\[ \theta = -\frac{y[t] - c[t]}{c'[t]}, \quad y[t] \geq c[t], \quad c'[t] < 0 \]

let \( \theta_\ast := \theta \)

Step 4:

\(^8\) If \( \hat{x}^T c^k \geq 0 \) for all \( k \in 1, m \) then the \( J \)-basic solution \( z \) of \((P)\) is feasible, by (6) and (11).

\(^9\) In analogy with the primal algorithm, \( j \) can be chosen arbitrarily. This particular choice guarantees that the algorithm stays with \( j \) until \( y[j] = c[j] \), or \((\tilde{D})\) is declared infeasible.

\(^{10}\) As in the primal algorithm, an anti-cycling rule (such as given in Lemma 1) should be used in this selection.
Basic Solutions

\[
\bar{y} := y + \theta^* e^*; \quad (** \text{update } y \ **) \\
\text{Step 5:} \\
\bar{J} := (J \setminus \{\gamma_i\}) \cup \{i\}; \quad (** \text{update } J \ **) \\
\text{execute (17);} \quad (** \text{update } C \ **) \\
\text{execute (18);} \\
\text{Step 6: remove all bars} \quad (** \bar{y} \text{ becomes } y, \text{etc. } **) \\
\text{go to Step 1.} \\
\]

Finiteness can be proved here analogously to the finiteness proof in Lemma 1.

REFERENCES


