Saturation up to Redundancy for Tableau and Sequent Calculi

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A FOL Sequent calculus for NNF

$$\alpha \; \frac{\phi,\, \psi,\, \Gamma \vdash}{\phi \wedge \psi,\, \Gamma \vdash} \qquad \beta \; \frac{\phi,\, \Gamma \vdash \quad \psi,\, \Gamma \vdash}{\phi \vee \psi,\, \Gamma \vdash}$$

$$\gamma \; \frac{[x/t]\phi,\, \forall x.\phi,\, \Gamma \vdash}{\forall x.\phi,\, \Gamma \vdash} \qquad \delta \; \frac{[x/c]\phi,\, \Gamma \vdash}{\exists x.\phi,\, \Gamma \vdash}$$

for any ground term t for some new constant c

$$\text{CLOSE} \; \frac{\bot \vdash}{L,\, \neg L,\, \Gamma \vdash}$$

Hintikka Sets

A set of formulae H is a Hintikka Set iff

- $\bot \notin H$
- $\phi, \psi \in H$ for all $\phi \wedge \psi \in H$
- $\phi \in H$ or $\psi \in H$ for all $\phi \vee \psi \in H$

  ...

Completeness beacause:

- Any Hintikka Set is satisfiable.
- Union of all sequents of exhausted open branch is Hintikka set

A simplification rule

Simplification rule of Massacci, 1998:

$$\text{SIMP}\ \frac{L,\ \phi[L],\ \Gamma \vdash}{L,\ \phi,\ \Gamma \vdash}$$

$\phi[L]$:= replace L in ϕ by $\top$ and do Boolean simplification

Example:

$$\text{SIMP}\ \frac{p,\ r \vdash}{p,\ (\neg p \wedge q) \vee (p \wedge r) \vdash}$$

Because: $(\neg\top \wedge q) \vee (\top \wedge r) \ \equiv\ (\bot \wedge q) \vee r \ \equiv\ \bot \vee r \ \equiv\ r$

The problem

$$p,\ (\neg p \wedge q) \vee (p \wedge r) \vdash$$

$$\Downarrow$$

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- In either case, a derivable formula might not be derived
- Formulae on exhausted branch are not a Hintikka Set

Our contibution

[LPAR 2006 article]

Adapt Bachmair/Ganzinger framework of *Saturation up to Redundancy* to Tableaux and Sequent calculi:

- Definitions take splitting rules into account
- Adapted to usual style of describing inferences
- Treatment of rigid free variables

Overview

Input:

- A noetherian order $\succ$ on formulae
- A 'model functor' I like the Defn. of a model from a Hintikka set.

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- rules drop only redundant formulae
- rules reduce counterexamples (like inductive model lemma)

Theorem:

- Any fair proof procedure is complete

Inferences

General form of an inference:

$$\frac{\phi_{11}, \ldots, \phi_{1m_1}, \Gamma \vdash \quad \cdots \quad \phi_{n1}, \ldots, \phi_{nm_n}, \Gamma \vdash}{\phi_{01}, \ldots, \phi_{0m_0}, \Gamma \vdash}$$

Upper semi-sequents are *premises*

Lower semi-sequent is *conclusion*

One of the ϕ_{0i} is identified as *main formula*

Other required formulae are *side formulae*

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Other required formulae are *side formulae*

- ➠ Possibly several premises
- ➠ possibly several introduced formulae
- ➠ possibly several simultaneously removed formulae

Derivations

Derivations are sequences of trees constructed by applying rules.

Define *limit* as union of trees.

$$\mathcal{T}_0 \to \mathcal{T}_1 \to \mathcal{T}_2 \to \cdots \to \mathcal{T}^\infty$$

Branches of $\mathcal{T}^\infty$ are sequences $(\Gamma_i)_{i\in\mathbb{N}}$ of semi-sequents.

Set of *persistent formulae* of a branch:

$$\Gamma^\infty := \bigcup_{i\in\mathbb{N}} \bigcap_{j\geq i} \Gamma_j$$

Redundancy Criteria

A *redundancy criterion* is a pair $(\mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{I}})$ of mappings s.t.

(R1) if $\Gamma \subseteq \Gamma'$ then $\mathcal{R}_{\mathcal{F}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma')$, and $\mathcal{R}_{\mathcal{I}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{I}}(\Gamma')$.

(R2) if $\Gamma' \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma)$ then $\mathcal{R}_{\mathcal{F}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma \setminus \Gamma')$, and $\mathcal{R}_{\mathcal{I}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{I}}(\Gamma \setminus \Gamma')$.

(R3) if Γ is unsatisfiable, then so is $\Gamma \setminus \mathcal{R}_{\mathcal{F}}(\Gamma)$.

The criterion is called *effective* if, in addition,

(R4) an inference is in $\mathcal{R}_{\mathcal{I}}(\Gamma)$, whenever it has at least one premise introducing only formulae $P = \{\phi_{k1}, \ldots \phi_{km_k}\}$ with $P \subseteq \Gamma \cup \mathcal{R}_{\mathcal{F}}(\Gamma)$.

Formulae, resp. inferences in $\mathcal{R}_{\mathcal{F}}(\Gamma)$ resp. $\mathcal{R}_{\mathcal{I}}(\Gamma)$ are called *redundant* with respect to Γ.

The Standard Redundancy Criterion

Fix a noetherian ordering $\succ$ on formulae.

For formulae: [just like BG]

A formula ϕ is redundant with respect to a set of formulae Γ, iff

there are formulae $\phi_1, \ldots, \phi_n \in \Gamma$, such that $\phi_1, \ldots, \phi_n \models \phi$

and $\phi \succ \phi_i$ for $i = 1, \ldots, n$.

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For inferences:

An inference with main formula ϕ and side formulae $\phi_1, \ldots \phi_n$ is redundant w.r.t. a set of formulae Γ, iff it has one premise such that for all formulae ξ introduced in that premise, there are formulae $\psi_1, \ldots, \psi_m \in \Gamma$, such that $\psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_n \models \xi$ and $\phi \succ \psi_i$ for $i = 1, \ldots, m$.

Conformance

A calculus *conforms* to a redundancy criterion, if its inferences remove formulae from a branch only if they are redundant with respect to the formulae in the resulting semi-sequent.

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Example:

$$\text{SIMP}\ \frac{L,\ \phi[L],\ \Gamma \vdash}{L,\ \phi,\ \Gamma \vdash}$$

Removes ϕ: Need to show that ϕ redundant w.r.t. $\{L, \phi[L]\}$

In this case: $L \prec \phi$, $\phi[L] \prec \phi$, and $L, \phi[L] \models \phi$.

Reductive Calculi

A calculus is called *reductive* if all new formulae introduced by an inference are smaller than the main formula of the inference w.r.t. $\succ$

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Example:

$$\text{SIMP}\ \frac{L,\ \phi[L],\ \Gamma \vdash}{L,\ \phi,\ \Gamma \vdash}$$

Pick ϕ as main formula

Show that $\phi[L] \prec \phi$.

Counterexamples

Define a *model functor* I that maps

$$\text{a set of formulae } \Gamma \text{ with } \bot \notin \Gamma \qquad \mapsto \qquad \text{a model } I(\Gamma)$$

Let $\Gamma \not\ni \bot$ be a set of formulae

A *counterexample for $I(\Gamma)$ in Γ* is a formula $\phi \in \Gamma$ with $I(\Gamma) \not\models \phi$.

Since $\succ$ is Noetherian, if there is a counterexample for $I(\Gamma)$ in Γ,

then there is also a minimal one.

The Counterexample Reduction Property

A calculus has the *counterexample reduction property*, if:

For any $\Gamma \not\ni \bot$ and minimal counterexample ϕ, the calculus permits an inference

$$\frac{\phi_{11}, \ldots, \phi_{1m_1}, \Gamma_0 \vdash \qquad \cdots \qquad \phi_{n1}, \ldots, \phi_{nm_n}, \Gamma_0 \vdash}{\phi, \phi_{01}, \ldots, \phi_{0m_0}, \Gamma_0 \vdash}$$

with main formula ϕ where $\Gamma = \{\phi, \phi_{01}, \ldots, \phi_{0m_0}\} \cup \Gamma_0$ such that $I(\Gamma)$ satisfies all side formulae, i.e. $I(\Gamma) \models \phi_{01}, \ldots, \phi_{0m_0}$, and each of the premises contains an even smaller counterexample ϕ_{ik_i}, i.e. $I(\Gamma) \not\models \phi_{ik_i}$ and $\phi \succ \phi_{ik_i}$.

Counterexample Reduction, Example

Example:

$$\Gamma = \{\phi \vee \psi\} \cup \Gamma_0$$

and $I(\Gamma) \not\models \phi \vee \psi$ is minimal counterexample

Apply

$$\beta \; \frac{\phi,\, \Gamma_0 \vdash \qquad \psi,\, \Gamma_0 \vdash}{\phi \vee \psi,\, \Gamma_0 \vdash}$$

$\phi,\, \psi \prec \phi \vee \psi$ and $I(\Gamma) \not\models \phi,\, \psi$ ➠ smaller counterexamples

Fairness

A derivation $(\mathcal{T}_i)_{i \in \mathbb{N}}$ in a calculus that conforms to an effective redundancy criterion is called *fair* if for every limit branch $(\Gamma_i)_{i \in \mathbb{N}}$ of $\mathcal{T}^\infty$, and any inference

$$\frac{\phi_{11}, \ldots, \phi_{1m_1}, \Gamma_0 \vdash \quad \cdots \quad \phi_{n1}, \ldots, \phi_{nm_n}, \Gamma_0 \vdash}{\phi_{01}, \ldots, \phi_{0m_0}, \Gamma_0 \vdash}$$

possible on formulae in Γ^∞,

- the inference is redundant in Γ^∞, or
- some of the ϕ_{0i} is redundant in Γ^∞, or
- There is a $j \in \{1, \ldots, n\}$ such that for all $k \in \{1, \ldots, m_j\}$
  - ϕ_{jk} is redundant in $\bigcup_i \Gamma_i$ or
  - $\phi_{jk} \in \bigcup_i \Gamma_i$

Completeness

Theorem: If a calculus

- conforms to the standard redundancy criterion, and
- is reductive, and
- has the counterexample reduction property, then

any fair derivation for an unsatisfiable formula ϕ contains a closed tableau.

Case study in paper: NNF variant of hyper-tableaux calculus

Free Variables

Treatment of free variables using *constraints*.

$$\textsc{Simp}\ \frac{p(a),\ r(X) \ll X \equiv a,\ \neg p(X) \vee r(X) \ll X \not\equiv a \vdash}{p(a),\ \neg p(X) \vee r(X) \vdash}$$

- Correspondence between ‘constrained formula’ tableaux and ‘ground’ tableaux
- Completeness theorem for free variable tableaux
- Fairness in some cases not easy to achieve

Syntactic (Dis-)unification Constraints

A *constraint* is a formula built from

- equality $\equiv$ between terms with (free) variables X, Y, Z,
- negation !, and
- conjunction $\&$

and interpreted over the term universe.

$\mathrm{Sat}(C)$ is the set of ground substitutions satisfying C:

$$\begin{aligned} \mathrm{Sat}(s \equiv t) &= \{\sigma \in \mathcal{G} \mid \sigma s = \sigma t\} \\ \mathrm{Sat}(C \,\&\, D) &= \mathrm{Sat}(C) \cap \mathrm{Sat}(D) \\ \mathrm{Sat}(!\, C) &= \mathcal{G} \setminus \mathrm{Sat}(C) \end{aligned}$$

Constrained Formula Tableaux

A *constrained formula* is a pair

$$\phi \ll C$$

of a constraint and a formula.

A *constrained formula semi-sequent* is a set of constrained formulae.

A *(constrained formula) tableau* is a tree where each node is labeled with a constrained formula semi-sequent.

It is *closed* under $\sigma \in \mathcal{G}$ if every branch contains a semi-sequent Γ containing a constrained formula $\perp \ll C$ with $\sigma \in \mathrm{Sat}(C)$

It is *closable* if there is a $\sigma \in \mathcal{G}$ under which it is closed.

Example: SIMP with constraints

$$\text{SIMP}\ \frac{L \ll B,\ \mu\phi[\mu L] \ll L \equiv M \,\&\, A \,\&\, B,\ \phi \ll A \,\&\, !(L \equiv M \,\&\, B),\ \Gamma \vdash}{L \ll B,\ \phi \ll A,\ \Gamma \vdash}$$

where μ is a mgu of L and M, and M occurs in ϕ

e.g.:

$$\text{SIMP}\ \frac{p(a),\ r(X) \ll X \equiv a,\ \neg p(X) \vee r(X) \ll X \not\equiv a \vdash}{p(a),\ \neg p(X) \vee r(X) \vdash}$$

Substitutions and Constraints

Let Γ be a set of constrained formulae. We define

$$\sigma\Gamma := \{\sigma\phi \mid \phi \ll C \in \Gamma \text{ with } \sigma \in \mathrm{Sat}(C)\} \quad .$$

Let $\mathcal{T}$ be a tableau.

We construct $\sigma\mathcal{T}$ by replacing the semi-sequent Γ in each node of $\mathcal{T}$ by $\sigma\Gamma$.

Correspondence

Let

$$\frac{\Gamma_1 \vdash \quad \cdots \quad \Gamma_n \vdash}{\Gamma_0}$$

be an inference of a constrained formula tableau calculus. The *corresponding ground inference under σ* for some $\sigma \in \mathcal{G}$ is

$$\frac{\sigma\Gamma_1 \vdash \quad \cdots \quad \sigma\Gamma_n \vdash}{\sigma\Gamma_0} \quad .$$

The *corresponding ground calculus* is the calculus consisting of all corresponding ground inferences under any σ of any inferences in the constrained formula calculus.

Corresponding inferences for SIMP

$$\text{SIMP}\ \frac{L \ll B,\ \mu\phi[\mu L] \ll L \equiv M \,\&\, A \,\&\, B,\ \phi \ll A \,\&\, !(L \equiv M \,\&\, B),\ \Gamma \vdash}{L \ll B,\ \phi \ll A,\ \Gamma \vdash}$$

Corresponding ground inference under $\sigma \in \mathrm{Sat}(L \equiv M \,\&\, A \,\&\, B)$:

$$\text{SIMP}\ \frac{\sigma L,\ \sigma\phi[\sigma L],\ \Gamma \vdash}{\sigma L,\ \sigma\phi,\ \Gamma \vdash}$$

For all $\sigma \notin \mathrm{Sat}(L \equiv M \,\&\, A \,\&\, B)$: ground semi-sequent unchanged

Lifting of notions

A constrained formula calculus *conforms* to a given redundancy criterion, has the *counterexample reduction property*, or is *reductive* iff the corresponding ground calculus has that property.

A constrained formula tableau derivation $(\mathcal{T}_i)_{i \in \mathbb{N}}$ in a calculus that conforms to an effective redundancy criterion is called *fair* if there is a $\sigma \in \mathcal{G}$, such that $(\sigma\mathcal{T}_i)_{i \in \mathbb{N}}$ is a fair derivation of the corresponding ground calculus. We call such a σ a *fair instantiation* for the constrained formula tableau derivation.

Completeness

Theorem: If a constrained formula calculus

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- is reductive, and
- has the counterexample reduction property, then

any fair derivation for an unsatisfiable formula ϕ contains a closed tableau.

Case study in paper: NNF variant of hyper-tableaux calculus with rigid variables.

The Problem with Fairness

Consider rules deriving

$$\phi \ll C_0 \quad \to \quad \phi \ll C_1 \quad \to \quad \phi \ll C_2 \quad \to \quad \cdots$$

such that for some $\sigma \in \mathcal{G}$:

$$\sigma \in \mathrm{Sat}(C_0) \cap \mathrm{Sat}(C_1) \cap \mathrm{Sat}(C_2) \cdots$$

- None of the $\phi \ll C_i$ is persistent
- But $\sigma\phi$ is in the corresp. ground derivation

➠ fairness in general requires rule application on some $\phi \ll C_i$

How can this be implemented?

Conclusion

- Generalized Bachmair/Ganzinger saturation framework to Tableaux/Sequent calculi
- Permits semantic completeness proofs for destructive calculi
- Free-variable tableaux considered, but results preliminary

Future work:

- more uniform treatment of free variables
- alternatives to constraints for lifting