

# Saturation up to Redundancy for Tableau and Sequent Calculi

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# Acknowledgment

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# A FOL Sequent calculus for NNF

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$$\alpha \frac{\phi, \psi, \Gamma \vdash}{\phi \wedge \psi, \Gamma \vdash}$$

$$\beta \frac{\phi, \Gamma \vdash \quad \psi, \Gamma \vdash}{\phi \vee \psi, \Gamma \vdash}$$

$$\gamma \frac{[x/t]\phi, \forall x.\phi, \Gamma \vdash}{\forall x.\phi, \Gamma \vdash}$$

$$\delta \frac{[x/c]\phi, \Gamma \vdash}{\exists x.\phi, \Gamma \vdash}$$

for any ground term  $t$

for some new constant  $c$

$$\text{CLOSE} \frac{\perp \vdash}{L, \neg L, \Gamma \vdash}$$

# Hintikka Sets

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A set of formulae  $H$  is a Hintikka Set iff

- $\perp \notin H$
- $\phi, \psi \in H$  for all  $\phi \wedge \psi \in H$
- $\phi \in H$  or  $\psi \in H$  for all  $\phi \vee \psi \in H$
- ...

Completeness because:

- Any Hintikka Set is satisfiable.
- Union of all sequents of exhausted open branch is Hintikka set

# A simplification rule

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Simplification rule of Massacci, 1998:

$$\text{SIMP} \frac{L, \phi[L], \Gamma \vdash}{L, \phi, \Gamma \vdash}$$

$\phi[L]$  := replace  $L$  in  $\phi$  by  $\top$  and do Boolean simplification

Example:

$$\text{SIMP} \frac{p, r \vdash}{p, (\neg p \wedge q) \vee (p \wedge r) \vdash}$$

Because:  $(\neg \top \wedge q) \vee (\top \wedge r) \equiv (\perp \wedge q) \vee r \equiv \perp \vee r \equiv r$

# The problem

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$p, (\neg p \wedge q) \vee (p \wedge r) \vdash$



$p, r \vdash$

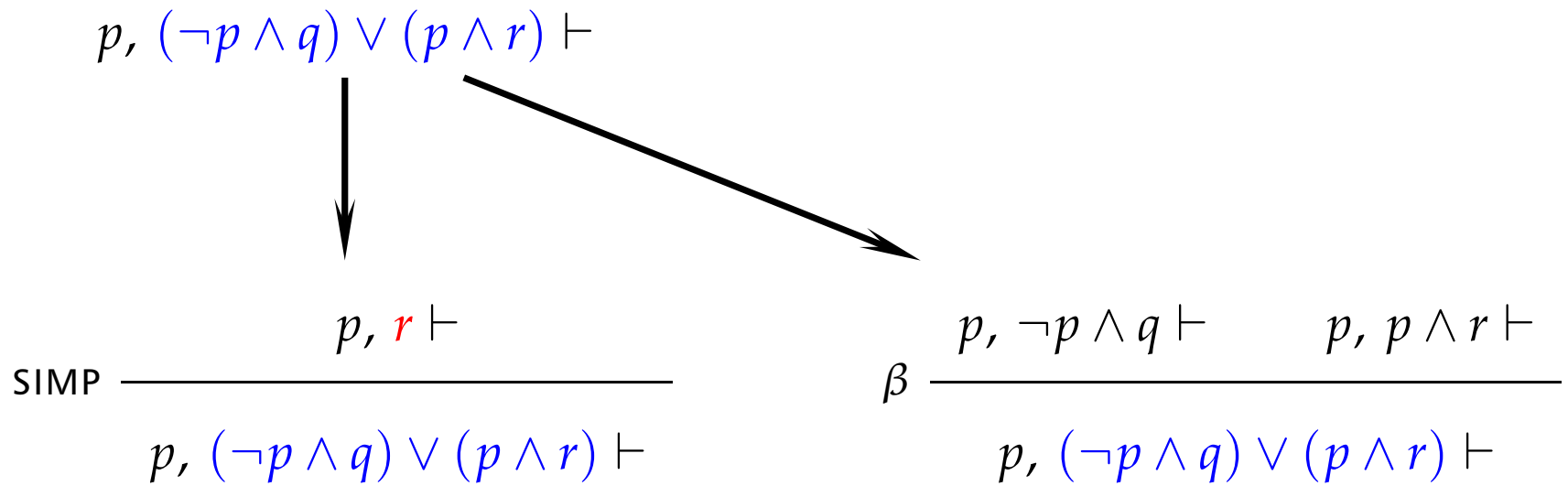
SIMP



$p, (\neg p \wedge q) \vee (p \wedge r) \vdash$

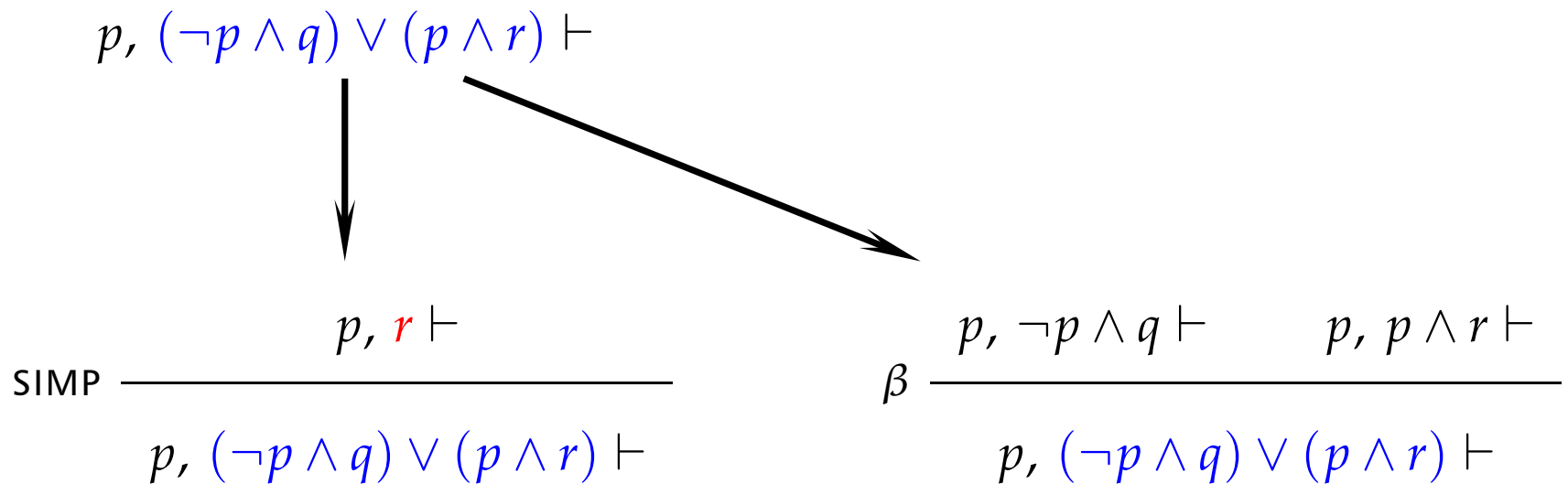
# The problem

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# The problem

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- ▶ In either case, a derivable formula might not be derived
- ▶ Formulae on exhausted branch are not a Hintikka Set



# Our contribution

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[LPAR 2006 article]

Adapt Bachmair/Ganzinger framework of *Saturation up to Redundancy* to Tableaux and Sequent calculi:

- Definitions take splitting rules into account
- Adapted to usual style of describing inferences
- Treatment of rigid free variables

# Overview

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Input:

- A noetherian order  $\succ$  on formulae
- A 'model functor'  $I$  like the Defn. of a model from a Hintikka set.

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- rules drop only redundant formulae
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- rules reduce counterexamples (like inductive model lemma)

Theorem:

- Any fair proof procedure is complete

# Inferences

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General form of an inference:

$$\frac{\phi_{11}, \dots, \phi_{1m_1}, \Gamma \vdash \quad \dots \quad \phi_{n1}, \dots, \phi_{nm_n}, \Gamma \vdash}{\phi_{01}, \dots, \phi_{0m_0}, \Gamma \vdash}$$

Upper semi-sequents are *premises*

Lower semi-sequent is *conclusion*

One of the  $\phi_{0i}$  is identified as *main formula*

Other required formulae are *side formulae*

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- ▣➡ Possibly several premises
- ▣➡ possibly several introduced formulae
- ▣➡ possibly several simultaneously removed formulae

# Derivations

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*Derivations* are sequences of trees constructed by applying rules.

Define *limit* as union of trees.

$$\mathcal{T}_0 \rightarrow \mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \dots \rightarrow \mathcal{T}^\infty$$

*Branches* of  $\mathcal{T}^\infty$  are sequences  $(\Gamma_i)_{i \in \mathbb{N}}$  of semi-sequents.

Set of *persistent formulae* of a branch:

$$\Gamma^\infty := \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} \Gamma_j$$

# Redundancy Criteria

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A *redundancy criterion* is a pair  $(\mathcal{R}_{\mathcal{F}}, \mathcal{R}_{\mathcal{I}})$  of mappings s.t.

**(R1)** if  $\Gamma \subseteq \Gamma'$  then  $\mathcal{R}_{\mathcal{F}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma')$ , and  $\mathcal{R}_{\mathcal{I}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{I}}(\Gamma')$ .

**(R2)** if  $\Gamma' \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma)$  then  $\mathcal{R}_{\mathcal{F}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{F}}(\Gamma \setminus \Gamma')$ , and  $\mathcal{R}_{\mathcal{I}}(\Gamma) \subseteq \mathcal{R}_{\mathcal{I}}(\Gamma \setminus \Gamma')$ .

**(R3)** if  $\Gamma$  is unsatisfiable, then so is  $\Gamma \setminus \mathcal{R}_{\mathcal{F}}(\Gamma)$ .

The criterion is called *effective* if, in addition,

**(R4)** an inference is in  $\mathcal{R}_{\mathcal{I}}(\Gamma)$ , whenever it has **at least one premise**

introducing **only formulae**  $P = \{\phi_{k1}, \dots, \phi_{km_k}\}$  with  $P \subseteq \Gamma \cup \mathcal{R}_{\mathcal{F}}(\Gamma)$ .

Formulae, resp. inferences in  $\mathcal{R}_{\mathcal{F}}(\Gamma)$  resp.  $\mathcal{R}_{\mathcal{I}}(\Gamma)$  are called

*redundant* with respect to  $\Gamma$ .



# The Standard Redundancy Criterion

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Fix a noetherian ordering  $\succ$  on formulae.

For formulae: [just like BG]

A formula  $\phi$  is redundant with respect to a set of formulae  $\Gamma$ , iff

there are formulae  $\phi_1, \dots, \phi_n \in \Gamma$ , such that  $\phi_1, \dots, \phi_n \models \phi$

and  $\phi \succ \phi_i$  for  $i = 1, \dots, n$ .

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For inferences:

An inference with main formula  $\phi$  and side formulae  $\phi_1, \dots, \phi_n$  is redundant w.r.t. a set of formulae  $\Gamma$ , iff it has **one premise** such that

**for all formulae**  $\xi$  introduced in that premise, there are formulae

$\psi_1, \dots, \psi_m \in \Gamma$ , such that  $\psi_1, \dots, \psi_m, \phi_1, \dots, \phi_n \models \xi$  and

$\phi \succ \psi_i$  for  $i = 1, \dots, m$ .

# Conformance

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A calculus *conforms* to a redundancy criterion, if its inferences remove formulae from a branch only if they are redundant with respect to the formulae in the resulting semi-sequent.

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Example:

$$\text{SIMP} \frac{L, \phi[L], \Gamma \vdash}{L, \phi, \Gamma \vdash}$$

Removes  $\phi$ : Need to show that  $\phi$  redundant w.r.t.  $\{L, \phi[L]\}$

In this case:  $L \prec \phi$ ,  $\phi[L] \prec \phi$ , and  $L, \phi[L] \models \phi$ .

# Reductive Calculi

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A calculus is called *reductive* if all new formulae introduced by an inference are smaller than the main formula of the inference w.r.t.  $\succ$

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Example:

$$\text{SIMP} \frac{L, \phi[L], \Gamma \vdash}{L, \phi, \Gamma \vdash}$$

Pick  $\phi$  as main formula

Show that  $\phi[L] \prec \phi$ .

# Counterexamples

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Define a *model functor*  $I$  that maps

a set of formulae  $\Gamma$  with  $\perp \notin \Gamma$   $\mapsto$  a model  $I(\Gamma)$

Let  $\Gamma \not\ni \perp$  be a set of formulae

A *counterexample for  $I(\Gamma)$  in  $\Gamma$*  is a formula  $\phi \in \Gamma$  with  $I(\Gamma) \not\models \phi$ .

Since  $\succ$  is Noetherian, if there is a counterexample for  $I(\Gamma)$  in  $\Gamma$ , then there is also a minimal one.

# The Counterexample Reduction Property

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A calculus has the *counterexample reduction property*, if:

For any  $\Gamma \not\perp$  and minimal counterexample  $\phi$ , the calculus permits an inference

$$\frac{\phi_{11}, \dots, \phi_{1m_1}, \Gamma_0 \vdash \quad \dots \quad \phi_{n1}, \dots, \phi_{nm_n}, \Gamma_0 \vdash}{\phi, \phi_{01}, \dots, \phi_{0m_0}, \Gamma_0 \vdash}$$

with main formula  $\phi$  where  $\Gamma = \{\phi, \phi_{01}, \dots, \phi_{0m_0}\} \cup \Gamma_0$  such that

$I(\Gamma)$  satisfies all side formulae, i.e.  $I(\Gamma) \models \phi_{01}, \dots, \phi_{0m_0}$ , and

each of the premises contains an even smaller counterexample  $\phi_{ik_i}$ ,

i.e.  $I(\Gamma) \not\models \phi_{ik_i}$  and  $\phi \succ \phi_{ik_i}$ .



# Counterexample Reduction, Example

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Example:

$$\Gamma = \{\phi \vee \psi\} \cup \Gamma_0$$

and  $I(\Gamma) \not\models \phi \vee \psi$  is minimal counterexample

Apply

$$\beta \frac{\phi, \Gamma_0 \vdash \quad \psi, \Gamma_0 \vdash}{\phi \vee \psi, \Gamma_0 \vdash}$$

$\phi, \psi \prec \phi \vee \psi$  and  $I(\Gamma) \not\models \phi, \psi \implies$  smaller counterexamples

# Fairness

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A derivation  $(\mathcal{T}_i)_{i \in \mathbb{N}}$  in a calculus that conforms to an effective redundancy criterion is called *fair* if for every limit branch  $(\Gamma_i)_{i \in \mathbb{N}}$  of  $\mathcal{T}^\infty$ , and any inference

$$\frac{\phi_{11}, \dots, \phi_{1m_1}, \Gamma_0 \vdash \quad \dots \quad \phi_{n1}, \dots, \phi_{nm_n}, \Gamma_0 \vdash}{\phi_{01}, \dots, \phi_{0m_0}, \Gamma_0 \vdash}$$

possible on formulae in  $\Gamma^\infty$ ,

- the inference is redundant in  $\Gamma^\infty$ , or
- some of the  $\phi_{0i}$  is redundant in  $\Gamma^\infty$ , or
- There is a  $j \in \{1, \dots, n\}$  such that for all  $k \in \{1, \dots, m_j\}$ 
  - $\phi_{jk}$  is redundant in  $\bigcup_i \Gamma_i$  or
  - $\phi_{jk} \in \bigcup_i \Gamma_i$

# Completeness

---

**Theorem:** If a calculus

- conforms to the standard redundancy criterion, and
- is reductive, and
- has the counterexample reduction property, then

any fair derivation for an unsatisfiable formula  $\phi$  contains a closed tableau.

Case study in paper: NNF variant of hyper-tableaux calculus

# Free Variables

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Treatment of free variables using *constraints*.

$$\text{SIMP} \frac{p(a), r(X) \ll X \equiv a, \neg p(X) \vee r(X) \ll X \neq a \vdash}{p(a), \neg p(X) \vee r(X) \vdash}$$

- Correspondence between 'constrained formula' tableaux and 'ground' tableaux
- Completeness theorem for free variable tableaux
- Fairness in some cases not easy to achieve

# Syntactic (Dis-)unification Constraints

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A *constraint* is a formula built from

- equality  $\equiv$  between terms with (free) variables  $X, Y, Z,$
- negation  $!$ , and
- conjunction  $\&$

and interpreted over the term universe.

$\text{Sat}(C)$  is the set of ground substitutions satisfying  $C$ :

$$\text{Sat}(s \equiv t) = \{\sigma \in \mathcal{G} \mid \sigma s = \sigma t\}$$

$$\text{Sat}(C \& D) = \text{Sat}(C) \cap \text{Sat}(D)$$

$$\text{Sat}(!C) = \mathcal{G} \setminus \text{Sat}(C)$$

# Constrained Formula Tableaux

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A *constrained formula* is a pair

$$\phi \ll C$$

of a constraint and a formula.

A *constrained formula semi-sequent* is a set of constrained formulae.

A *(constrained formula) tableau* is a tree where each node is labeled with a constrained formula semi-sequent.

It is *closed* under  $\sigma \in \mathcal{G}$  if every branch contains a semi-sequent  $\Gamma$  containing a constrained formula  $\perp \ll C$  with  $\sigma \in \text{Sat}(C)$

It is *closable* if there is a  $\sigma \in \mathcal{G}$  under which it is closed.

# Example: SIMP with constraints

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$$\text{SIMP} \frac{L \ll B, \mu\phi[\mu L] \ll L \equiv M \& A \& B, \phi \ll A \& !(L \equiv M \& B), \Gamma \vdash}{L \ll B, \phi \ll A, \Gamma \vdash}$$

where  $\mu$  is a mgu of  $L$  and  $M$ , and  $M$  occurs in  $\phi$

e.g.:

$$\text{SIMP} \frac{p(a), r(X) \ll X \equiv a, \neg p(X) \vee r(X) \ll X \neq a \vdash}{p(a), \neg p(X) \vee r(X) \vdash}$$

# Substitutions and Constraints

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Let  $\Gamma$  be a set of constrained formulae. We define

$$\sigma\Gamma := \{ \sigma\phi \mid \phi \ll C \in \Gamma \text{ with } \sigma \in \text{Sat}(C) \} \quad .$$

Let  $\mathcal{T}$  be a tableau.

We construct  $\sigma\mathcal{T}$  by replacing the semi-sequent  $\Gamma$  in each node of  $\mathcal{T}$  by  $\sigma\Gamma$ .



# Correspondence

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Let

$$\frac{\Gamma_1 \vdash \quad \dots \quad \Gamma_n \vdash}{\Gamma_0}$$

be an inference of a constrained formula tableau calculus. The

*corresponding ground inference under  $\sigma$*  for some  $\sigma \in \mathcal{G}$  is

$$\frac{\sigma\Gamma_1 \vdash \quad \dots \quad \sigma\Gamma_n \vdash}{\sigma\Gamma_0} .$$

The *corresponding ground calculus* is the calculus consisting of all corresponding ground inferences under any  $\sigma$  of any inferences in the constrained formula calculus.

# Corresponding inferences for SIMP

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$$\text{SIMP} \frac{L \ll B, \mu\phi[\mu L] \ll L \equiv M \& A \& B, \phi \ll A \& !(L \equiv M \& B), \Gamma \vdash}{L \ll B, \phi \ll A, \Gamma \vdash}$$

Corresponding ground inference under  $\sigma \in \text{Sat}(L \equiv M \& A \& B)$ :

$$\text{SIMP} \frac{\sigma L, \sigma\phi[\sigma L], \Gamma \vdash}{\sigma L, \sigma\phi, \Gamma \vdash}$$

For all  $\sigma \notin \text{Sat}(L \equiv M \& A \& B)$ : ground semi-sequent unchanged

# Lifting of notions

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A constrained formula calculus *conforms* to a given redundancy criterion, has the *counterexample reduction property*, or is *reductive* iff the corresponding ground calculus has that property.

A constrained formula tableau derivation  $(\mathcal{T}_i)_{i \in \mathbb{N}}$  in a calculus that conforms to an effective redundancy criterion is called *fair* if there is a  $\sigma \in \mathcal{G}$ , such that  $(\sigma\mathcal{T}_i)_{i \in \mathbb{N}}$  is a fair derivation of the corresponding ground calculus. We call such a  $\sigma$  a *fair instantiation* for the constrained formula tableau derivation.

# Completeness

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**Theorem:** If a constrained formula calculus

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- is reductive, and
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Case study in paper: NNF variant of hyper-tableaux calculus with rigid variables.

# The Problem with Fairness

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Consider rules deriving

$$\phi \ll C_0 \rightarrow \phi \ll C_1 \rightarrow \phi \ll C_2 \rightarrow \dots$$

such that for some  $\sigma \in \mathcal{G}$ :

$$\sigma \in \text{Sat}(C_0) \cap \text{Sat}(C_1) \cap \text{Sat}(C_2) \dots$$

- None of the  $\phi \ll C_i$  is persistent
- But  $\sigma\phi$  is in the corresp. ground derivation

▮▮▮▮► fairness in general requires rule application on some  $\phi \ll C_i$

How can this be implemented?

# Conclusion

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- Generalized Bachmair/Ganzinger saturation framework to Tableaux/Sequent calculi
- Permits semantic completeness proofs for destructive calculi
- Free-variable tableaux considered, but results preliminary

## Future work:

- more uniform treatment of free variables
- alternatives to constraints for lifting