Finite volume method for the shallow water equations on a sphere

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Introduction

- SWE simplest model that captures the essential features of fluid flow on the sphere.

  $\rightarrow$ has become the standard test bed for numerical techniques.
- A standard test set of seven problems has been accepted for evaluation of numerical methods for the SWE in spherical geometry, Williamson et. al.

Shallow water equations on a sphere

- spherical coordinates
- regular longitude (ϕ) - latitude (θ) grid
- state variables:
  - $h(\phi, \theta, t)$, depth
  - $u(\phi, \theta, t)$, zonal velocity
  - $v(\phi, \theta, t)$, meridional velocity
- spherical SWE:

$$\mathbf{q_t} + \mathbf{f}_\phi + \mathbf{g}_\theta = \mathbf{s} \tag{1}$$

Finite volume method (FVM)

- Designed to be concervative
- State variables represented by the cell averages in the midle of each cell.
- Cell averages updated through fluxes across the cell interfaces.

CLAWPACK: implements a wave-propagation version of FVM.

Wave-propagation methods

CLAWPACK applies an operator splitting technique on (1):

$$\mathbf{q_t} + \mathbf{f}_\phi + \mathbf{g}_\theta = \mathbf{0} \tag{2}$$

$$\mathbf{q_t} = \mathbf{s}(\mathbf{q}) \tag{3}$$

(2) is solved by Godunovs method:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n) \tag{4}$$

- where Q_i is cell average approximation,
and $F_{i\pm1/2}$ is the approxim. of fluxes across the cell interfaces.

$\rightarrow$ Riemann problem introduced

- allows us to decompose the fluxes into waves determined by the eigenvectors of the Jacobians $\mathbf{A} = \partial\mathbf{f}/\partial\mathbf{q}$ and $\mathbf{B} = \partial\mathbf{g}/\partial\mathbf{q}$ in

$$\mathbf{q}_{\mathrm{t}} + \mathbf{A}\mathbf{q}_{\phi} + \mathbf{B}\mathbf{q}_{\theta} = \mathbf{s}. \tag{5}$$

$\mathbf{A}(\mathbf{q})$ and $\mathbf{B}(\mathbf{q})$ has to be evaluated at the cell interfaces $i \pm 1/2$.
But only Q_i and Q_{i-1} is defined, not $Q_{i\pm1/2}$.
For non-linear problems a Roe-average is used to find $Q_{i\pm1/2}$.

The ”jump” in $\mathbf{q}$ from $\mathbf{q}_i$ to $\mathbf{q}_{i-1}$ is defined as $\Delta\mathbf{q} = \mathbf{q}_i - \mathbf{q}_{i-1}$.

Solving in the ϕ-direction,
the eigendecomposition of $\mathbf{A}$ is given by $\lambda^{1:3}$ and $\mathbf{e^{1:3}}$.

Decomposition of fluxes into waves becomes:

$$\sum_{k=1}^{3} \mathcal{W}^k = \sum_{k=1}^{3} \alpha^k \mathbf{e}^k = \Delta\mathbf{q} \tag{6}$$

Standard wave-decomposition:

At the interfaces $\mathbf{A}\Delta\mathbf{q}$ can be approximated by

$$\mathbf{A}\Delta\mathbf{q} = \sum_{k=1}^{3} \lambda^k \mathcal{W}^k = \sum_{k=1}^{3} \lambda^k \alpha^k \mathbf{e}^k \tag{7}$$

Flux-based wave-decomposition:

$$\Delta\mathbf{f} = \sum_{k=1}^{3} \mathcal{Z}^k = \sum_{k=1}^{3} \beta^k \mathbf{e}^k \tag{8}$$

- where $\mathcal{Z}$ is called *f-waves*.

F-waves is a better method when Roe's crieteria are not met.

Source terms

After the homogenous equation

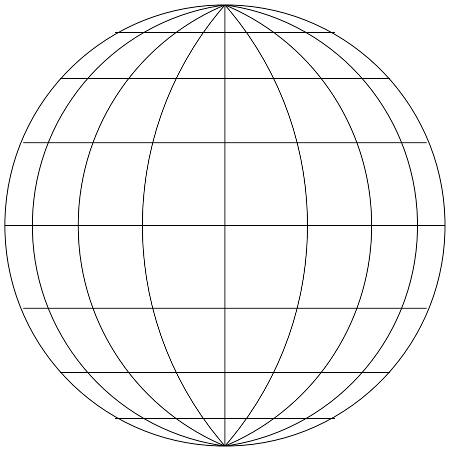
$$\mathbf{q}_\mathrm{t}^* + \mathbf{f}_\phi + \mathbf{g}_\theta = \mathbf{0} \tag{9}$$

has been solved, the ODE including the source term is solved

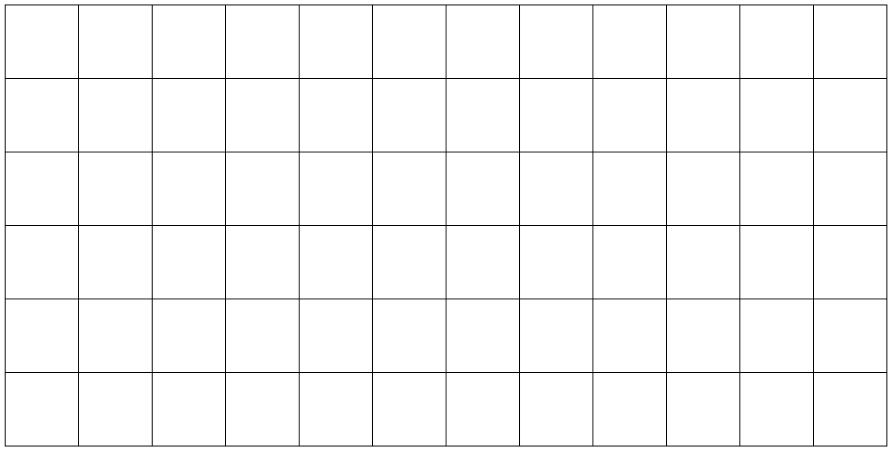
$$\mathbf{q}_\mathrm{t} = \mathbf{s}(\mathbf{q}^*). \tag{10}$$

Both equations will introduce errors, and these errors may not cancel. This can introduce problems for stationary flow.

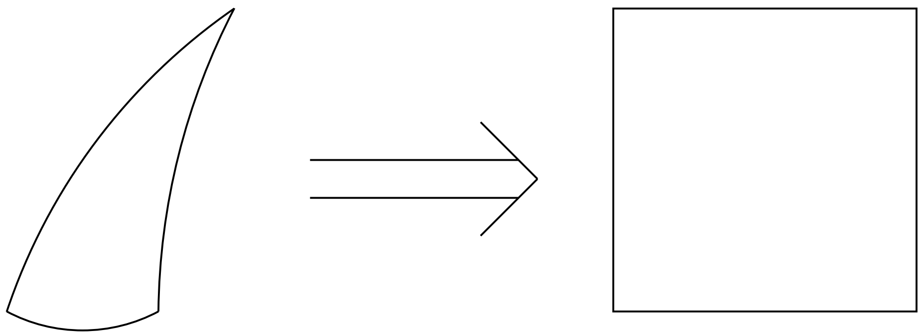
Grids



Physical Grid



Computational Grid



A cell at the North pole

The corresponding computational cell

Numerical results

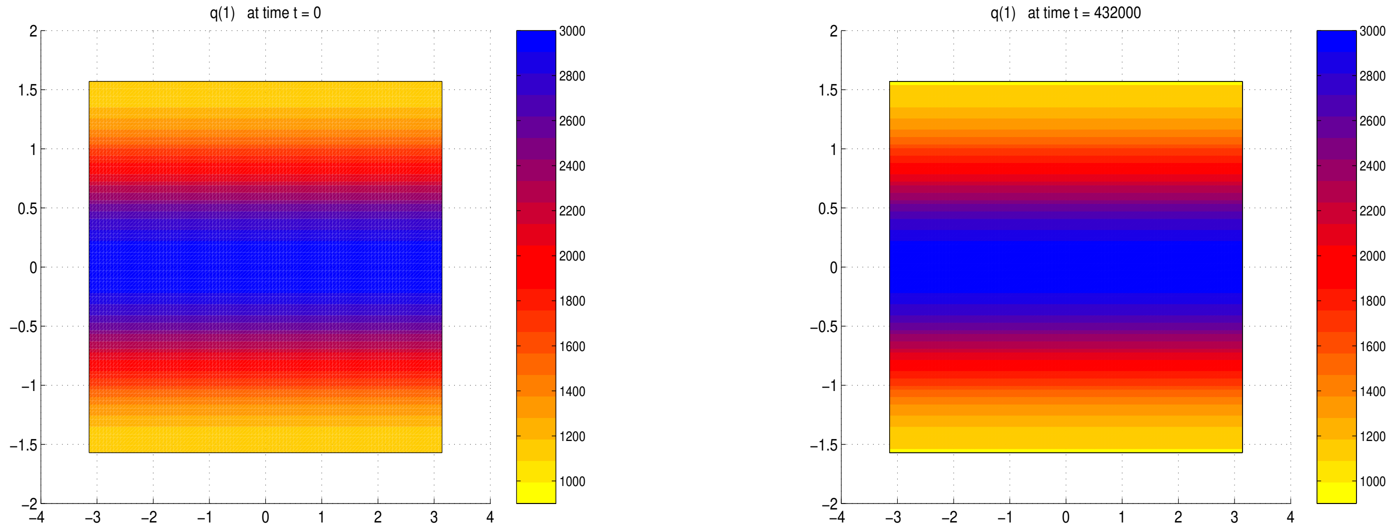


Figure 1. Test case 2, Steady state zonal flow, 2nd order f-waves.
a) Initial state of h, b) State of h after 5 days.

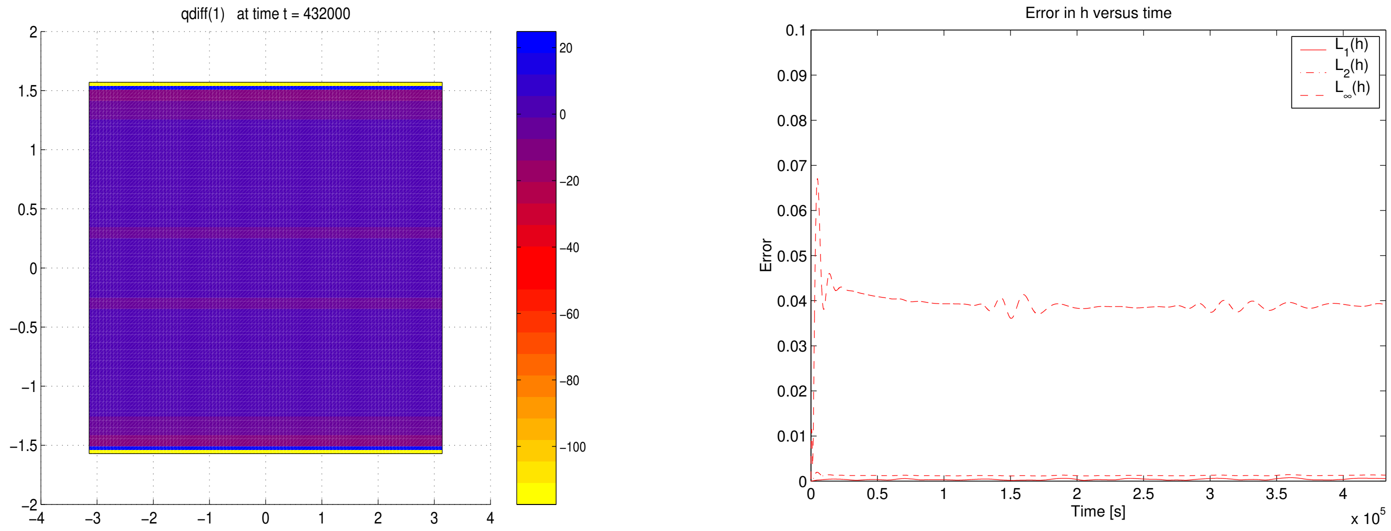


Figure 2. Test case 2, Steady state zonal flow, 2nd order f-waves.
a) Error-difference plot of h at T = 5 days,
b) Error-norms for h as function of time.

Convergence rates

Resolution	$L_1(h)$	CR	$L_2(h)$	CR	$L_\infty(h)$	CR
50×50	0.0031		0.0044		0.0412	
100×100	0.0005	5.6	0.0013	3.3	0.0392	1.1
200×200	0.0001	5.2	0.0006	2.1	0.0439	0.9

Table 1. Test case 2. Error norms after 5 days for different resolutions and corresponding convergence rates (CR).

Conclusions

+ 100% conservation
- Flux-based waves better than standard waves.
- 2nd order of accuracy necessary

+ Stable solution, no increase in error over time.

- Pole problem