Optimal Mappings of the Spectrum of BPSK/QPSK Sequences to Finite Polynomial Fields and Rings

Extended Abstract

M.G.Parker, S.J.Shepherd, S.K.Barton,
Telecommunications Research Group,
Department of Electronic and Electrical Engineering,
University of Bradford,
Bradford, BD7 1DP, UK.
e-mail: mgparker@bradford.ac.uk

Abstract

It is shown how each bin of the Discrete Fourier Transform (DFT) of a Phase Shift Keyed (PSK) sequence can be optimally mapped to a finite polynomial field or ring. This suggests novel solutions to the VLSI implementation of DFTs, and may also help in the search for spectrally-flat PSK sequences.

I Introduction

Consider the N-point DFT of a P-PSK sequence, $d = (d_0, d_1, \ldots, d_{N-1})$, given by,

$$v(n) = \sum_{k=0}^{N-1} d_k e^{j2\pi\frac{kn}{N}} \qquad 0 \le n < N \qquad (1)$$

where $d_k \in \{1, e^{j\frac{2\pi}{P}}, e^{j\frac{4\pi}{P}}, \ldots, e^{j\frac{(P-1)2\pi}{P}}\}$. With $r = \mathrm{lcm}(N, P)$ and $x = e^{j\frac{2\pi}{r}}$, (1) can be expressed as,

$$v_n(x) = \sum_{k=0}^{N-1} d_k(x) x^{\frac{rnk}{N}} \bmod \Phi_r(x) \qquad 0 \le n < N \qquad (2)$$

where $\Phi_r(x)$ is the r^{th}cyclotomic polynomial of degree $\phi(r)$ (ϕ is Euler's Totient Function) [1], and $\deg(v_n(x)) < \phi(r)$. The constellation of polynomials, $\mathbf{V}_n = \{v_n(x)\}$, represent mutually unique points in the complex plane for each bin, n. The 'Pols' column of Tables 1 and 2 shows the constellation size for each bin. Bins, n, which have the same value of $\gcd(N, n)$, generate identical constellations in the complex plane. Therefore only one representative from each class of $\gcd(N, n)$ is tabulated. Note:

- The number of polynomials for bin 0 are,
  - For BPSK : $N+1$. For QPSK : $(N+1)^2$.
- The number of polynomials for bin 1 when N is prime are,
  - For BPSK : $2^N - 1$. For QPSK : $(2^N - 1)^2$.

II Mapping Constellations to Finite Polynomial Fields/Rings

Firstly, the polynomial degree of the constellation representation will be minimised by converting to a polynomial in y. Let $t = \mathrm{lcm}\left(\frac{N}{\gcd(n,N)}, P\right)$. Then substituting $y = x^{\frac{r}{t}}$, $\exists w_n(y)$ such that,

$$w_n(y) \bmod \Phi_t(y) = w_n(x^{\frac{r}{t}}) \bmod \Phi_r(x) = v_n(x) \qquad (3)$$

where $\deg(w_n(y)) < \phi(t)$. One can further map the constellation for bin n from the set $\mathbf{W}_n = \{w_n(y)\}$, (or $\mathbf{V}_n = \{v_n(x)\}$) to the field or ring of finite polynomials, $\mathbf{F}_n = \{f_n(u)\}$, mod $M(u)$, mod m, (i.e. the finite polynomial field/ring, $Z_m[u]/M(u)$). One of the conditions for each element of $\mathbf{W}_n$ to map to a unique element of $\mathbf{F}_n$ is,

$$\exists \alpha(u) \in Z_m[u]/M(u), \alpha(u)^t = 1, \alpha(u)^s \neq 1, 0 < s < t \qquad (4)$$

To find a suitable $Z_m[u]/M(u)$ that, for a given n, gives a unique mapping from $\mathbf{W}_n$ to $\mathbf{F}_n$ and satisfies (4), the following procedure was adopted.

1. Assign P and N.
2. Assign bin number, n.
3. Compute $\mathbf{V}_n$ using (2), for all d.
4. Re-express $\mathbf{V}_n$ as $\mathbf{W}_n$ using (3).
5. Choose a $Z_m[u]/M(u)$ that satisfies (4). (Ideally $Z_m[u]/M(u)$ should have as few elements as possible but this must be at least equal to the constellation size).
6. Compute $\mathbf{F}_n = \{f_n(u) = w_n(\alpha(u)) \bmod M(u) \bmod m\}$.
7. If there is a one-to-one mapping from $\mathbf{W}_n$ to $\mathbf{F}_n$, then bin n of the N-point DFT of a length N P-PSK sequence can be

computed using $Z_m[u]/M(u)$. Go to step 2. Otherwise go to step 5.

Tables 1,2 present finite integer or polynomial mappings for BPSK and QPSK, respectively. The mappings are one-to-one (apart from bin 0 when N is odd), and are therefore optimal. Observe that, for bin 1, N prime, $P = 2$, $\alpha(u)$ must be a root of 2, mod m.

III Conclusion

The mappings shown suggest efficient hardware solutions for the DFT inherent to OFDM systems [2, 3]. (For example, to compute bins 1 and 2 of a 3-point QPSK DFT: $f_n(u) = \sum_{k=0}^{2} d'_k(u)(2u)^{4nk}$, mod $(u^2 + 1)$, mod 7, where $d'_k(u) \in \{1, 6u, 6, u\}$ and $2u$ has order 12 over $Z_7[u]/(u^2 + 1)$.) Moreover, the allocation of different mappings for different bin numbers suggests a prime-factor decomposition of the DFT over different finite polynomial fields/rings [1]. Finally, it is hoped these mappings will help to categorise PSK sequences by spectral shape [4].

P	N	n	Pols	m	$M(u)$	t	$\alpha(u)$
2	3	0	4	5	–	2	4
		1	7	7	–	6	3
	4	0	5	5	–	2	4
		1	9	3	u^2+1	4	u
		2	5	5	–	2	4
	5	0	6	7	–	2	6
		1	31	31	–	10	27
	6	0	7	7	–	2	6
		1	19	19	–	6	8
		2	19	19	–	6	8
		3	7	7	–	2	6
	7	0	8	9	–	2	8
		1	127	127	–	14	63
	8	0	9	9	–	2	8
		1	81	3	u^4+1	8	u
		2	25	5	u^2+1	4	u
		4	9	9	–	2	8
	9	0	10	11	–	2	10
		1	$343 = 7^3$	7	u^3+2	18	u
		3	37	37	–	6	11
	10	0	11	11	–	2	10
		1	211	211	–	10	23
		2	211	211	–	10	23
		5	11	11	–	2	10
	11	0	12	13	–	2	12
		1	$2047 = 23.89$	2047	–	22	1983

Table 1: Finite Polynomial Mappings for BPSK DFT Output Bins

P	N	n	Pols	m	$M(u)$	t	$\alpha(u)$
4	2	0	9	3	u^2+1	4	u
		1	9	3	u^2+1	4	u
	3	0	16	17	–	4	4
		1	49	7	u^2+1	12	$2u$
	4	0	25	5	u^2+1	4	u
		1	25	5	u^2+1	4	u
		2	25	5	u^2+1	4	u
	5	0	36	37	–	4	6
		1	$961 = 31^2$	31	u^2+1	20	$2u$
	6	0	49	7	u^2+1	4	u
		1	$361 = 19^2$	19	u^2+1	12	$7u$
		2	361	19	u^2+1	12	$7u$
		3	49	7	u^2+1	4	u
	7	0	64	$65 = 5.13$	–	4	8
		1	$16129 = 127^2$	127	u^2+1	28	$2u$
	8	0	81	$9 = 3^2$	u^2+1	4	u
		1	$625 = 5^4$	5	u^4+1	8	u
		2	81	$9 = 3^2$	u^2+1	4	u
		4	81	$9 = 3^2$	u^2+1	4	u
	9	0	100	101	–	4	10
		1	$117649 = 7^6$	7	u^6+2	36	u
		3	$1369 = 37^2$	37	u^2+1	12	$10u$
	10	0	$121 = 11^2$	11	u^2+1	4	u
		1	$44521 = 211^2$	211	u^2+1	20	$55u$
		2	$44521 = 211^2$	211	u^2+1	20	$55u$
		5	$121 = 11^2$	11	u^2+1	4	u

Table 2: Finite Polynomial Mappings for QPSK DFT Output Bins

References

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