Interlace Polynomials: Enumeration, Unimodality, and Connections to Codes*

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Abstract

The *interlace polynomial* q was introduced by Arratia, Bollobás, and Sorkin. It encodes many properties of the orbit of a graph under *edge local complementation* (ELC). The interlace polynomial Q, introduced by Aigner and van der Holst, similarly contains information about the orbit of a graph under *local complementation* (LC). We have previously classified LC and ELC orbits, and now give an enumeration of the corresponding interlace polynomials of all graphs of order up to 12. An enumeration of all *circle graphs* of order up to 12 is also given. We show that there exist graphs of all orders greater than 9 with interlace polynomials q whose coefficient sequences are *non-unimodal*, thereby disproving a conjecture by Arratia et al. We have verified that for graphs of order up to 12, all polynomials Q have unimodal coefficients. It has been shown that LC and ELC orbits of graphs correspond to equivalence classes of certain *error-correcting codes* and *quantum states*. We show that the properties of these codes and quantum states are related to properties of the associated interlace polynomials.

1 Introduction

A *graph* is a pair $G = (V, E)$ where V is a set of *vertices*, and $E \subseteq V \times V$ is a set of *edges*. The *order* of G is $n = |V|$. We will only consider *simple undirected* graphs, i.e., graphs where all edges are bidirectional and no vertex can be adjacent to itself. The *neighbourhood* of $v \in V$, denoted $N_v \subset V$, is the set of vertices connected to v by an edge. The number of vertices adjacent to v is called the *degree* of v. An *Eulerian graph* is a graph where all vertices have even degree. The *induced subgraph* of G on $W \subseteq V$ contains vertices W and all edges from E whose endpoints are both in W. The *complement* of G is found by replacing E with $V \times V - E$, i.e., the edges in E are changed to non-edges, and the non-edges to edges. Two graphs $G = (V, E)$ and $G' = (V, E')$ are *isomorphic* if and only if there exists a permutation π on V such that $\{u, v\} \in E$ if and only if $\{\pi(u), \pi(v)\} \in E'$. A *path* is a sequence of vertices, $(v_1, v_2, \ldots, v_i)$, such that $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{i-1}, v_i\} \in E$. A graph is *connected* if there is a path from any vertex to any other vertex in the graph. A graph is *bipartite* if its set

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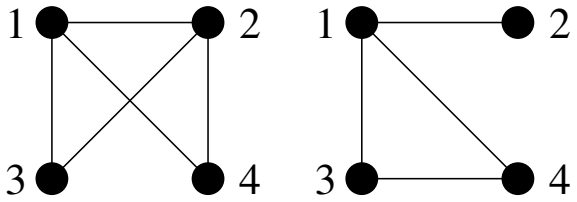


(a) The Graph G **(b)** The Graph $G * 1$

Fig. 1: Example of Local Complementation

of vertices can be decomposed into two disjoint sets such that no two vertices within the same set are adjacent. A *complete graph* is a graph where all pairs of vertices are connected by an edge. A *clique* is a complete subgraph. A k-clique is a clique consisting of k vertices. An *independent set* is the complement of a clique, i.e., an empty subgraph. The *independence number* of G is the size of the largest independent set in G.

Definition 1 ([5, 13, 14])**.** Given a graph $G = (V, E)$ and a vertex $v \in V$, let $N_v \subset V$ be the neighbourhood of v. *Local complementation* (LC) on v transforms G into $G * v$ by replacing the induced subgraph of G on N_v by its complement. (Fig. 1)

Definition 2 ([5])**.** Given a graph $G = (V, E)$ and an edge $\{u, v\} \in E$, *edge local complementation* (ELC) on $\{u, v\}$ transforms G into $G^{(uv)} = G * u * v * u = G * v * u * v$.

Definition 3 ([5])**.** ELC on $\{u, v\}$ can equivalently be defined as follows. Decompose $V \setminus \{u, v\}$ into the following four disjoint sets, as visualized in Fig. 2.

- A Vertices adjacent to u, but not to v.
- B Vertices adjacent to v, but not to u.
- C Vertices adjacent to both u and v.
- D Vertices adjacent to neither u nor v.

To obtain $G^{(uv)}$, perform the following procedure. For any pair of vertices $\{x, y\}$, where x belongs to class A, B, or C, and y belongs to a different class A, B, or C, "toggle" the pair $\{x, y\}$, i.e., if $\{x, y\} \in E$, delete the edge, and if $\{x, y\} \notin E$, add the edge $\{x, y\}$ to E. Finally, swap the labels of vertices u and v.

Definition 4. The *LC orbit* of a graph G is the set of all unlabeled graphs that can be obtained by performing any sequence of LC operations on G. Similarly, the *ELC orbit* of G comprises all unlabeled graphs that can be obtained by performing any sequence of ELC operations on G.

The LC operation was first defined by de Fraysseix [13], and later studied by Fon-der-Flaas [14] and Bouchet [5]. Bouchet defined ELC as "complementation along an edge" [5], but this operation is also known as *pivoting* on a graph [3].

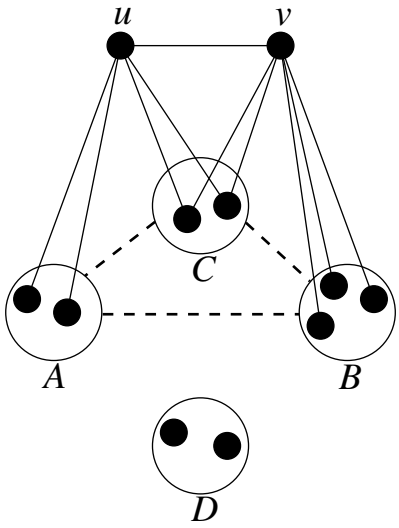


Fig. 2: Visualization of the ELC Operation

The recently defined *interlace polynomials* are based on the LC and ELC operations. Arratia, Bollobás, and Sorkin [3] defined the interlace polynomial $q(G)$ of the graph G. This work was motivated by a problem related to DNA sequencing [2].

Definition 5 ([3])**.** For every graph G, there is an associated interlace polynomial $q(G, x)$, which we will usually denote $q(G)$ for brevity. For the edgeless graph of order n, $E_n = (V, \emptyset)$, $q(E_n) = x^n$. For any other graph $G = (V, E)$, choose an arbitrary edge $\{u, v\} \in E$, and let

$$q(G) = q(G \setminus u) + q(G^{(uv)} \setminus u),$$

where $G \setminus u$ is the graph G with vertex u and all edges incident on u removed.

It was proven by Arratia et al. [3] that the polynomial is independent of the order of removal of edges, and that the polynomial is invariant under ELC, i.e., that $q(G) = q(G^{(uv)})$ for any edge $\{u, v\}$.

Aigner and van der Holst [1] later defined the interlace polynomial $Q(G)$ which similarly encodes properties of the LC orbit of G.

Definition 6 ([1])**.** For every graph G, there is an associated interlace polynomial $Q(G, x)$, which we will usually denote $Q(G)$ for brevity. For the edgeless graph of order n, $E_n = (V, \emptyset)$, $Q(E_n) = x^n$. For any other graph $G = (V, E)$, choose an arbitrary edge $\{u, v\} \in E$, and let

$$Q(G) = Q(G \setminus u) + Q(G^{(uv)} \setminus u) + Q(G * u \setminus u).$$

Again, the order of removal of edges is irrelevant, and the polynomial is invariant under LC and ELC. It was shown by Aigner and van der Holst [1] that both $q(G)$ and $Q(G)$ can also be derived from the ranks of matrices obtained by certain modifications of the adjacency matrix of G. A similar approach, but expressed in terms of certain sets of *local unitary transforms*, was shown by Riera

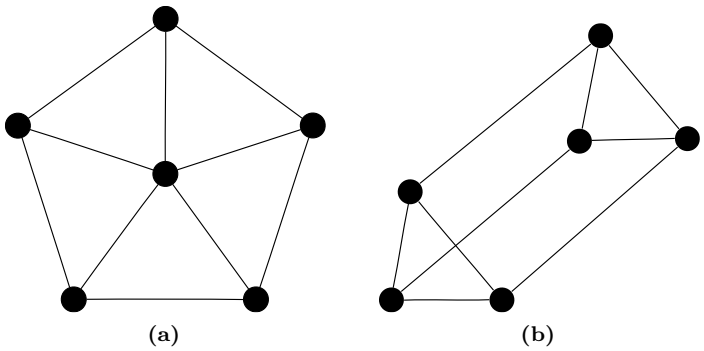


Fig. 3: Example of an LC Orbit

and Parker [20]. If G is an unconnected graph with components G_1 and G_2, then $q(G) = q(G_1)q(G_2)$ and $Q(G) = Q(G_1)Q(G_2)$.

The interlace polynomials $q(G)$ and $Q(G)$ summarize several properties of the ELC and LC orbits of the graph G. The degree of the lowest-degree term of $q(G)$ equals the number of connected components of G, and is therefore one for a connected graph [3]. The degree of $q(G)$ equals the maximum independence number over all graphs in the ELC orbit of G [1]. It follows that the degree of $q(G)$ is also an upper bound on the independence number of G. Likewise, the degree of $Q(G)$ gives the size of the largest independent set in the LC orbit of G [10]. The degree of $Q(G)$ will always be greater than or equal to the degree of $q(G)$. Evaluating interlace polynomials for certain values of x can also gives us some information about the associated graphs. For a graph G of order n, it always holds that $q(G, 2) = 2^n$ and $Q(G, 3) = 3^n$. $q(G, 1)$ equals the number of induced subgraphs of G with an odd number of *perfect matchings* [1]. $Q(G, 2)$ equals the number of general induced subgraphs of G (with possible loops attached to the vertices) with an odd number of general perfect matchings [1]. $Q(G, 4)$ equals 2^n times the number of induced Eulerian subgraphs of G [1]. It has been shown that $q(G, -1) = (-1)^n 2^{n-r}$, where n is the order of G and r is the rank over $\mathbb{F}_2$ of $\Gamma + I$, where Γ is the adjacency matrix of G [1, 4]. $q(G, 3)$ is always divisible by $q(G, 1)$, and the quotient is an odd integer [1].

Example 7. The two graphs in Fig. 3 comprise an LC orbit, and an ELC orbit. (Note that in general, an LC orbit can be decomposed into one or more ELC orbits.) Both graphs have interlace polynomials $q(G) = 12x + 10x^2$ and $Q(G) = 108x + 45x^2$. The fact that $\deg(Q) = 2$ matches the observation that none of the two graphs have an independent set of size greater than two. That $\frac{Q(G,4)}{2^6} = 18$ means that each graph has 18 Eulerian subgraphs.

In their list of open problems [3], Arratia et al. pose the question of how many different interlace polynomials there are for graphs of order n. In Section 2, we answer this question for $n \leq 12$, for both interlace polynomials q and Q.

In the DNA sequencing setting [2], interlace polynomials of *circle graphs* are of particular interest. Arratia et al. [2] enumerated the circle graphs of order up to 9. In Section 3, we extend this enumeration to order 12.

Let $q(G) = a_1x + a_2x^2 + \cdots + a_dx^d$. Then the sequence of coefficients of q is $\{a_i\} = (a_1, a_2, \ldots, a_d)$. Arratia et al. [3] conjecture that this sequence is

unimodal for all q. The sequence $\{a_i\}$ is unimodal if, for some $1 \leq k \leq d$, $a_i \leq a_j$ for all $i < j \leq k$, and $a_i \geq a_j$ for all $i > j \geq k$. In other words, the sequence is non-decreasing up to some coefficient k, and the rest of the sequence is non-increasing. In Section 4, we show that there exist interlace polynomials q whose coefficient sequences are non-unimodal, and thereby disprove the conjecture by Arratia et al. Our enumeration shows that all interlace polynomials of graphs of order up to 9 are unimodal, but that there are two graphs of order 10 with the same non-unimodal interlace polynomial. From these graphs of order 10 it is possible to construct graphs of any order greater than 10 with non-unimodal interlace polynomials. We verify that all interlace polynomials $Q(G)$ and all polynomials $x \cdot q(G, x+1)$ of graphs of order up to 12 have unimodal coefficients.

In Section 5 we highlight an interesting relationship between interlace polynomials, error-correcting codes, and quantum states. The LC orbit of a graph corresponds to the equivalence class of a *self-dual quantum code* [26], and ELC orbits of bipartite graphs correspond to equivalence classes of *binary linear codes* [12]. In both cases, the *minimum distance* of a code is given by $\delta + 1$, where δ is the minimum vertex degree over all graphs in the corresponding orbit. We have previously shown [10] that a self-dual quantum code with high minimum distance often corresponds to a graph G where $\deg(Q)$, the degree of $Q(G)$, is small. A self-dual quantum code can also be interpreted as a *quantum graph state* [17]. A code with high minimum distance will correspond to a quantum state with a high degree of *entanglement*. The degree of $Q(G)$ gives an indicator of the entanglement in the graph state represented by G known as the *peak-to-average power ratio* [10] with respect to certain transforms. Another indicator of the entanglement in a graph state is the *Clifford merit factor* (CMF) [18], which can be derived from the evaluation of $Q(G)$ at $x = 4$ [21]. In Section 5 we give the range of possible values of δ, $\deg(Q)$, and $Q(G, 4)$ for graphs of order up to 12, and bounds on these parameters for graphs of order up to 25, derived from the best known self-dual quantum codes.

2 Enumeration of Interlace Polynomials

In the context of error-correcting codes, we have previously classified the LC orbits [11] and ELC orbits [12, 19] of all graphs on up to 12 vertices. In Table 1, the sequence $\{c_{L,n}\}$ gives the number of LC orbits of connected graphs on n vertices, while $\{t_{L,n}\}$ gives the total number of LC orbits of graphs on n vertices. Similarly, the sequence $\{c_{E,n}\}$ gives the number of ELC orbits of connected graphs on n vertices, while $\{t_{E,n}\}$ gives the total number of ELC orbits of graphs on n vertices. A database containing one representative from each LC orbit is available at `http://www.ii.uib.no/~larsed/vncorbits/`. A similar database of ELC orbits can be found at `http://www.ii.uib.no/~larsed/pivot/`.

Note that the value of t_n (for either ELC or LC orbits) can be derived easily once the sequence $\{c_m\}$ is known for $1 \leq m \leq n$, using the *Euler transform* [24],

$$\begin{aligned} a_n &= \sum_{d|n} d c_d, \\ t_1 &= a_1, \\ t_n &= \frac{1}{n}\left(a_n + \sum_{k=1}^{n-1} a_k t_{n-k}\right). \end{aligned}$$

Table 1: Number of LC and ELC Orbits

n	$c_{L,n}$	$t_{L,n}$	$c_{E,n}$	$t_{E,n}$
1	1	1	1	1
2	1	2	1	2
3	1	3	2	4
4	2	6	4	9
5	4	11	10	21
6	11	26	35	64
7	26	59	134	218
8	101	182	777	1068
9	440	675	6702	8038
10	3132	3990	104,825	114,188
11	40,457	45,144	3,370,317	3,493,965
12	1,274,068	1,323,363	231,557,290	235,176,097

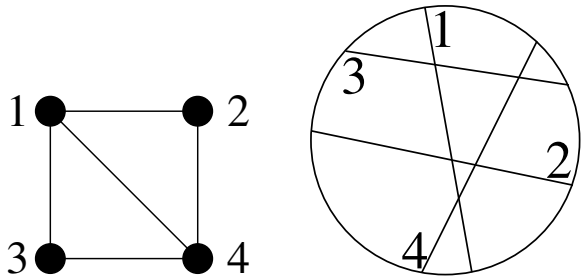
Table 2: Number of Distinct Interlace Polynomials

n	$c_{Q,n}$	$t_{Q,n}$	$c_{q,n}$	$t_{q,n}$
1	1	1	1	1
2	1	2	1	2
3	1	3	2	4
4	2	6	4	8
5	4	11	9	17
6	10	24	24	41
7	23	52	71	112
8	84	152	257	369
9	337	521	1186	1555
10	2154	2793	7070	8625
11	22,956	26,178	56,698	65,323
12	486,488	515,131	614,952	680,275

The question of how many distinct interlace polynomials there are for graphs of order n was posed by Arratia et al. [3]. For a representative from each LC and ELC orbit, we have calculated the interlace polynomials Q and q, respectively. We then counted the number of distinct interlace polynomials. In Table 2, the sequence $\{c_{Q,n}\}$ gives the number of interlace polynomials Q of connected graphs of order n, while $\{t_{Q,n}\}$ gives the total number of interlace polynomials Q of graphs of order n. Similarly, $\{c_{q,n}\}$ and $\{t_{q,n}\}$ give the numbers of interlace polynomials q. We observe that in Table 2, the relationship $t_{q,n} = c_{q,n} + t_{q,n-1}$ holds.

3 Enumeration of Circle Graphs

A graph G is a *circle graph* if each vertex in G can be represented as a chord on a circle, such that two chords intersect if and only if there is an edge between



(a) The Circle Graph G **(b)** The Circle Representation of G

Fig. 4: Example of a Circle Graph

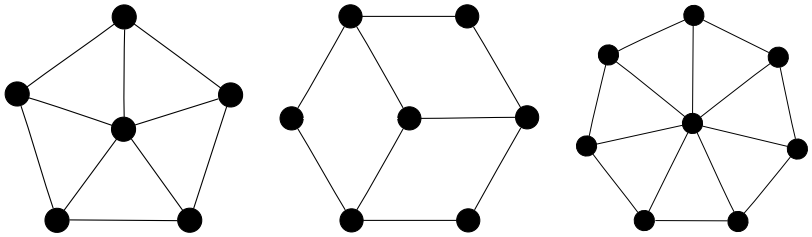


Fig. 5: Circle Graph Obstructions

the two corresponding vertices in G. An example of a circle graph and its corresponding circle diagram is given in Fig. 4.

Whether a given graph is a circle graph can be recognized in polynomial time [25]. It is also known that LC operations will map a circle graph to a circle graph, and a non-circle graph to a non-circle graph [6]. Bouchet [6] proved that a graph G is a circle graph if and only if certain *obstructions*, shown in Fig. 5, do not appear as subgraphs anywhere in the LC orbit of G.

Arratia et al. [2] pointed out that an enumeration of circle graphs did not seem to have appeared in the literature before, and then gave an enumeration of circle graphs of order up to 9. Using our previous classification of LC orbits, and the fact that the circle graph property is preserved by LC operations, we are able to generate all circle graphs of order up to 12. In Table 3, the sequence $\{c_{c,n}\}$ gives the number of connected circle graphs of order n, while $\{t_{c,n}\}$ gives the total number of circle graphs of order n. The sequences $\{c'_{c,n}\}$ and $\{t'_{c,n}\}$ give the number of LC orbits containing circle graphs. A database with one representative from each LC orbit of connected circle graphs is available at `http://www.ii.uib.no/~larsed/circle/`.

4 Unimodality

Having calculated the interlace polynomials q of all graphs of order up to 12, it was possible to check whether their coefficient sequences were unimodal, as conjectured by Arratia et al. [3]. Note that a similar conjecture has been disproved for the related *Tutte polynomial* [23].

Table 3: Number of Circle Graphs

n	$c_{c,n}$	$t_{c,n}$	$c'_{c,n}$	$t'_{c,n}$
1	1	1	1	1
2	1	2	1	2
3	2	4	1	3
4	6	11	2	6
5	21	34	4	11
6	110	154	10	25
7	789	978	23	55
8	8336	9497	81	157
9	117,283	127,954	293	499
10	2,026,331	2,165,291	1403	2059
11	40,302,425	42,609,994	7968	10,543
12	892,278,076	937,233,306	55,553	68,281

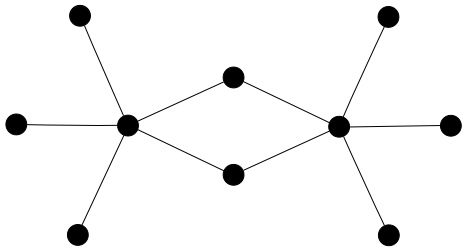


Fig. 6: The Smallest Graph with Non-Unimodal Interlace Polynomial q

Our results show that all interlace polynomials q of graphs of order $n \leq 9$ are unimodal, but that for $n = 10$ there exists a single non-unimodal interlace polynomial with coefficient sequence $\{a_i\} = (2, 7, 6, 7, 4, 3, 2, 1, 0, 0)$. Only two graphs on 10 vertices, comprising a single ELC orbit, correspond to this polynomial. One of these graphs is shown in Fig. 6.

We have further found that, up to ELC equivalence, there are 4 graphs on 11 vertices with non-unimodal interlace polynomials, 3 of which are connected graphs, and 20 graphs on 12 vertices with non-unimodal polynomials, 15 of which are connected.

Given the single non-unimodal interlace polynomial of a graph of order $n = 10$, it is easy to show that there must exist non-unimodal interlace polynomials for all $n > 10$, since the following methods of extending a graph will preserve the non-unimodality of the associated interlace polynomial. Given a graph G on n vertices with non-unimodal interlace polynomial, we can add an isolated vertex to obtaining an unconnected graph G' on $n + 1$ vertices, where $q(G') = x \cdot q(G)$ is clearly also non-unimodal. Non-unimodality is also preserved by *substituting* a vertex v of G by a clique of size m, i.e., we obtain G' where v is replaced by m vertices, all connected to each other and all connected to w whenever $\{v, w\}$ is an edge in G. It can then be shown that $q(G') = 2^m q(G)$ [3, Prop. 38].

Proposition 8. *Given a graph G, let G' be the graph obtained by* duplicating *a vertex v of G, i.e., by adding a vertex v' such that v' is connected to w*

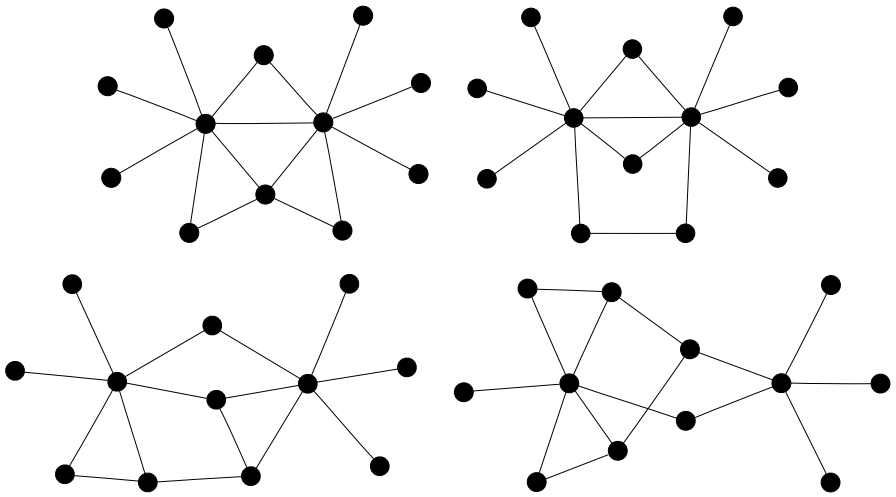


Fig. 7: Non-trivial Graphs of Order 12 with Non-Unimodal Interlace Polynomial q

whenever $\{v, w\}$ *is an edge in* G*. The interlace polynomial of* G *can be written* $q(G) = a(x) + cx^j + x^{j+1}b(x)$*, where* a *and* b *are arbitrary polynomials,* c *is a constant, and* $j = \deg(a) + 1$*. The unimodality or lack thereof of* G *will be preserved in* G' *if* $q(G \setminus v) = a(x) + x^j b(x)$*.*

Proof. By duplicating the vertex v, we obtain a graph G' with interlace polynomial $q(G') = (1 + x)q(G) - x \cdot q(G \setminus v)$ [3, Prop. 40]. If the condition above is satisfied, $q(G') = x^{j+2}a(x) + c(x^{j+1} + x^j) + b(x)$. The only difference between the coefficient sequences of $q(G)$ and $q(G')$ is that the coefficient c is repeated in $q(G')$, and unimodality or non-unimodality must therefore be preserved. □

Let G be the graph depicted in Fig. 6, and let v be one of the six vertices of degree one in this graph. If we duplicate v we obtain a graph whose interlace polynomial has the non-unimodal coefficient sequence $(2, 7, 6, 7, 6, 4, 3, 2, 1, 0, 0)$. According to Prop. 8, we can repeat the duplication of a vertex with degree one and the coefficient sequence will remain $(2, 7, 6, 7, 6, \ldots, 6, 4, 3, 2, 1, 0, 0)$, i.e., non-unimodal.

By the described extension methods we can obtain, from the single graph on 10 vertices shown in Fig. 6, all the 4 inequivalent graphs on 11 vertices and 16 of the 20 inequivalent graphs on 12 vertices with non-unimodal interlace polynomials. Representatives from the ELC orbits of the 4 non-trivial graphs on 12 vertices with non-unimodal interlace polynomials are shown in Fig. 7.

The two following conjectures have been checked for all graphs on up to 12 vertices, and no counterexamples have been found.

Conjecture 1 ([3]). For any interlace polynomial $q(G, x)$, the associated polynomial $x \cdot q(G, x + 1)$ has a unimodal coefficient sequence.

Conjecture 2. For any graph G, the interlace polynomial $Q(G)$ has a unimodal coefficient sequence.

Table 4: Range of $\deg(Q)$ For Given δ and n

$\delta \backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	2	2,3	3,4	3–5	3–6	3–7	4–8	4–9	4–10	4–11
2				2	3	3,4	3,4	3–5	4–6	4–7	4–8
3					2		3,4	3,4	3–5	4–6	4–7
4										4	4
5											4

5 Connections to Codes and Quantum States

An important question is what the interlace polynomials $q(G)$ and $Q(G)$ actually compute about the graph G itself. When G is a circle graph, $q(G)$ can be used to solve counting problems relevant to DNA sequencing [2]. We will show that the interlace polynomials also give clues about the error-correction capability of codes and the entanglement of quantum states.

It is known that *self-dual quantum codes*, so called because they correspond to self-dual additive codes over $\mathbb{F}_4$ [7], can be represented as graphs [16, 22]. The LC orbit of a graph corresponds to the equivalence class of a self-dual quantum code [26]. Similarly, the ELC orbits of bipartite graphs correspond to equivalence classes of *binary linear codes* [12]. In both cases the *minimum distance*, an important parameter that determines the error-correcting capability of a code, is given by $\delta + 1$, where δ is the minimum vertex degree over all graphs in the corresponding LC or ELC orbit. A self-dual quantum code can also be interpreted as a *quantum graph state* [17], and the δ-value of the associated LC orbit is then an indicator of the degree of *entanglement* in the state.

Although the value δ can not be obtained from an interlace polynomial, several values that are correlated with δ are encoded in the interlace polynomial. The size of the largest independent set over all members of the LC orbit of G equals $\deg(Q)$, the degree of $Q(G)$ [1, 10]. We have previously shown that optimal self-dual quantum codes correspond to LC orbits where $\deg(Q)$ is small [10]. These codes have largest possible minimum distance for a given length n, and thus the associated LC orbits of graphs on n vertices have maximum possible values of δ. The data in Table 4 implies that the LC orbits with the highest δ-values also have the lowest values of $\deg(Q)$, but that the converse is not always true. In the context of quantum graph states, the value $2^{\deg(Q)}$ is equal to the *peak-to-average power ratio* [10] with respect to certain transforms, which is another indicator of the degree of entanglement in the state.

Another measure of the entanglement in a quantum graph state is the *Clifford merit factor* (CMF) [18]. A quantum graph state can be represented as a graph G, and the CMF of the state can be derived from the value obtained by evaluating the associated interlace polynomial $Q(G)$ at $x = 4$ [21]. The CMF value can be obtained with the formula $\frac{6^n}{2^n Q(G,4) - 6^n}$. Interestingly, $\frac{Q(G,4)}{2^n}$ is also the number of induced Eulerian subgraphs of a graph on n vertices [1], which is invariant over the LC orbit. As can be seen in Table 5, the LC orbits with the highest δ-values also have the lowest values of $Q(G, 4)$. Other evaluations of the interlace polynomials are also of interest in the context of quantum graph states, for

Table 5: Range of $\frac{Q(G,4)}{2^n}$ For Given n and δ

$n\backslash\delta$	1	2	3	4	5
2	3				
3	5				
4	8–9				
5	13–17	12			
6	20–33	19	18		
7	30–65	29–30			
8	47–129	45–48	44–45		
9	73–257	69–80	68–69		
10	112–513	106–128	104–109		
11	172–1025	160–183	157–180	156	
12	260–2049	244–362	237–288	238–239	234

instance $q(G,1)$ and $Q(G,2)$ give the number of *flat spectra* with respect to certain sets of transforms of the state [21].

Although no algorithm is known for computing the interlace polynomial of a graph efficiently, it is in general faster to generate interlace polynomials, by simply using the recursive algorithm given in Definitions 5 and 6, than it is to generate the entire ELC or LC orbits of a graph. Note that calculating δ can also be done faster than by generating the complete LC orbit, by using methods for calculating the minimum distance of a self-dual quantum code represented as a graph [11]. We have calculated the interlace polynomials Q of graphs corresponding to the best known self-dual quantum codes, obtained from `http://www.codetables.de/` and from a search we have previously performed of *circulant graph codes* [8]. An adjacency matrix is called *circulant* if the i-th row is equal to the first row, cyclically shifted $i-1$ times to the right. The results, for graphs of order n up to 25, are given in Table 6. Values printed in bold font are the best values we have found, and are thus upper bounds on the minimum possible values of $\deg(Q)$ and $Q(G,4)$ for the given n. The values of δ printed in bold font are known to be optimal, except for $n=23$ and $n=25$, where a graph with $\delta=8$ could exist, and $n=24$, $n=26$, and $n=27$, where $\delta=9$ is possible. In general, the following bounds hold [7].

$$\delta \leq 2\left\lfloor \frac{n}{6} \right\rfloor + 1,$$

if the corresponding self-dual quantum code is of *Type II*, which means that its graph representation is *anti-Eulerian* [11], i.e., a graph where all vertices have odd degree. Such graphs must have an even number of vertices, and it is interesting to note that the anti-Eulerian property is preserved by LC operations.

$$\delta \leq \begin{cases} 2\left\lfloor \frac{n}{6} \right\rfloor, & \text{if } n \equiv 0 \pmod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 2, & \text{if } n \equiv 5 \pmod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 1, & \text{otherwise,} \end{cases}$$

if the corresponding self-dual quantum code is of *Type I*, i.e., corresponds to a graph where at least one vertex has even degree. Table 6 also lists the first

row of those adjacency matrices that are circulant. The remaining adjacency matrices are as follows.

$$\Gamma_{13,1} = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&1&1&1&1\\
1&0&1&0&0&0&0&1&0&0&0&1&1\\
0&1&0&1&0&1&0&0&0&0&0&1&1\\
0&0&1&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&1&0&1&0&0&1&1&1\\
0&0&1&0&1&0&0&0&0&1&1&0&1\\
0&0&0&1&0&0&0&0&1&1&1&0&1\\
0&1&0&0&1&0&0&0&1&0&1&1&0\\
0&0&0&0&0&0&1&1&0&1&1&1&0\\
1&0&0&0&0&1&1&0&1&0&0&1&0\\
1&0&0&1&1&1&1&1&1&0&0&0&0\\
1&1&1&1&1&0&0&1&1&1&0&0&0\\
1&1&1&1&1&1&1&0&0&0&0&0&0
\end{pmatrix}$$

$$\Gamma_{13,2} = \begin{pmatrix}
0&1&1&0&0&0&0&0&1&0&0&0&1\\
1&0&0&0&1&0&1&0&0&0&0&1&1\\
1&0&0&1&0&0&0&1&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&1&1&1&0\\
0&1&0&0&0&1&0&0&0&0&1&1&1\\
0&0&0&0&1&0&1&1&0&1&0&1&0\\
0&1&0&0&0&1&0&0&1&1&0&1&0\\
0&0&1&0&0&1&0&0&1&1&1&0&1\\
1&0&0&0&0&0&1&1&0&1&1&1&1\\
0&0&0&1&0&1&1&1&1&0&1&0&0\\
0&0&1&1&1&0&0&1&1&1&0&1&1\\
0&1&0&1&1&1&1&0&1&0&1&0&1\\
1&1&0&0&1&0&0&1&1&0&1&1&0
\end{pmatrix}$$

$$\Gamma_{18} = \begin{pmatrix}
0&0&1&0&0&0&0&0&1&1&0&0&0&1&0&1&1&1\\
0&0&0&1&1&0&0&0&0&1&0&1&0&0&0&1&1&1\\
1&0&0&0&0&1&0&0&1&1&1&1&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&1&0&0&0&1&1&1&1\\
0&1&0&0&0&1&1&1&0&0&0&1&1&0&1&0&0&0\\
0&0&1&0&1&0&0&1&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&1&0&0&1&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&1&1&1&0&0&1&1&0&0&1&1&0&0&0\\
1&0&1&1&0&0&1&0&0&0&1&0&1&0&0&0&0&1\\
1&1&1&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1\\
0&0&1&1&0&0&1&1&1&0&0&0&0&1&0&1&0&0\\
0&1&1&0&1&1&0&0&0&1&0&0&1&0&1&1&1&0\\
0&0&0&0&0&1&0&1&0&1&1&0&1&0&1&0&1&0&0\\
1&0&1&0&0&1&0&1&0&0&1&0&1&0&1&0&0&0\\
0&0&0&0&1&1&0&1&1&0&0&0&1&0&1&0&0&1&0\\
1&1&0&1&0&1&0&0&0&0&1&1&1&0&0&0&0&0\\
1&1&0&1&0&0&1&0&0&0&0&1&0&0&1&0&0&1\\
1&1&0&1&0&1&0&0&1&1&0&0&0&0&0&0&1&0
\end{pmatrix}$$

$$\Gamma_{21} = \begin{pmatrix}
0&0&0&0&0&1&0&0&0&1&1&0&1&0&1&0&1&1&1&0&1\\
0&0&1&1&0&0&1&1&0&1&1&0&0&0&1&1&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&1&0&1&0&0&0&0&0&1&1&1&0\\
0&1&0&0&1&0&1&1&0&0&0&1&1&0&0&0&1&0&1&0&0\\
0&0&1&1&0&1&0&0&0&1&0&0&0&1&1&0&0&0&1&1&0\\
1&0&1&0&1&0&1&1&1&0&0&0&0&1&0&0&0&1&0&1&0\\
0&1&0&1&0&1&0&1&0&1&1&0&1&0&0&0&0&0&1&0&1\\
0&1&0&1&0&1&1&0&0&0&0&0&0&1&0&0&0&1&0&1&1\\
0&0&0&0&0&1&0&0&0&1&1&1&1&1&0&1&0&0&1&0&1\\
1&1&1&0&1&0&1&0&1&0&0&0&0&0&1&1&0&0&0&1&0\\
1&1&0&0&0&0&1&0&1&0&0&0&1&1&0&0&1&0&0&0&1\\
0&0&1&1&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&0&1\\
1&0&0&1&0&0&1&0&1&0&1&0&0&0&0&1&0&1&0&0&1\\
0&0&0&0&1&1&0&1&1&0&1&1&0&0&0&1&0&1&0&0&0\\
1&1&0&0&1&0&0&0&0&1&0&0&0&0&0&1&1&1&0&0&1\\
0&1&0&0&0&0&0&0&1&1&0&1&1&1&1&0&0&0&0&1&0\\
1&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&1&1&1&0\\
1&0&1&0&0&1&0&1&0&0&0&0&1&1&1&0&1&0&0&0&0\\
1&0&1&1&1&0&1&0&1&0&0&1&0&0&0&0&1&0&0&1&0\\
0&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&1&0&1&0&0\\
1&0&0&0&0&0&1&1&1&0&1&1&1&0&1&0&0&0&0&0&0
\end{pmatrix}$$

For $n = 13$ and $n = 14$ we were able to compute the interlace polynomial Q of all graphs with optimal δ, since the corresponding codes have been classified [11, 27]. For other n, codes with the same δ but with lower $\deg(Q)$ or $Q(G, 4)$ may exist. The best self-dual quantum codes correspond to LC orbits where δ is maximized, and our results for graphs on up to 12 vertices suggested that these LC orbits also minimize $\deg(Q)$ and $Q(G, 4)$. However, in Table 6 we find several examples where the graph we have found with lowest $\deg(Q)$ does not have maximum δ. We have not found a single example where the lowest $Q(G, 4)$ is found in a graph with suboptimal δ, which indicates that $Q(G, 4)$ may be a better indicator of the minimum distance of a code that $\deg(Q)$, and leads to the following conjecture.

Conjecture 3. Let G be a graph on n vertices, and let δ be the minimum vertex degree over all graphs in the LC orbit of G. If there exists no other graph G' on n vertices such that $Q(G', 4) < Q(G, 4)$, then there exists no other graph on n vertices where the minimum vertex degree over all graphs in the LC orbit is greater than δ.

Note that once we have found a graph G on n vertices with a certain $\deg(Q(G))$, we can obtain a graph G' on $n - 1$ vertices with $\deg(Q(G')) = \deg(Q(G))$ or $\deg(Q(G')) = \deg(Q(G)) - 1$ by simply deleting any vertex of G. This process is equivalent to *shortening* a quantum code [15], and it is known that if the minimum vertex degree in the LC orbit of G is δ, then the minimum vertex degree in the LC orbit of G' is δ or $\delta - 1$.

The following theorem gives an upper bound on $Q(G, 4)$ for a graph G with a given value of δ. Note that the proof relies on certain properties of the *aperiodic propagation criteria* [9] for Boolean functions, which will not be defined here.

Theorem 9.

$$Q(G, 4) \leq \frac{\gamma(\delta + 1) + 6^n}{2^n},$$

Table 6: Best Found Values of δ, $\deg(Q)$, and $\frac{Q(G,4)}{2^n}$

n	δ	$\deg(Q)$	$\frac{Q(G,4)}{2^n}$	Adjacency matrix
13	**4**	**4**	361	$\Gamma_{13,1}$
13	**4**	5	**360**	$\Gamma_{13,2}$
14	**5**	**4**	**549**	(00001011101000)
15	**5**	6	**830**	(001110011001110)
15	4	**5**	833	(001111011011110)
16	**5**	**5**	**1264**	(0010101101101010)
17	**6**	6	**1872**	(00100011111100010)
17	5	**5**	1906	(00001111001110000)
18	**7**	6	**2808**	Γ_{18}
18	5	**5**	2835	(001001111111110010)
19	**6**	**6**	**4296**	(0000101001100101000)
20	**7**	**6**	**6444**	(00000100111110010000)
21	**7**	9	**9672**	Γ_{21}
21	6	**6**	9756	(000001100100100110000)
22	**7**	**6**	**14,688**	(0000010011111100100000)
23	**7**	7	**22,013**	(00000011101111011100000)
23	6	**5**	22,036	(00000111110110111110000)
24	**7**	**6**	**33,156**	(00100111010010010110010)
25	**7**	7	**49,812**	(00011000011111111100001100)
25	**7**	**6**	49,862	(00000111110011001111110000)

where

$$\gamma(d) = \sum_{t=0}^{n} \binom{n}{t} 2^t \left(\sum_{k=\max(1,d+t-n)}^{t} \binom{t}{k} 2^{n-k} \right).$$

Proof. The graph G corresponds to a Boolean function with APC distance $d = \delta + 1$, which means that all fixed aperiodic autocorrelation coefficients [9] up to and including weight $d-1$ are set to zero. As the Clifford merit factor (CMF) can be computed with the out-of-phase sum-of-squares of these autocorrelation coefficients in the denominator, then we immediately have a lower bound on CMF dependent on d. For a Boolean function f of n variables with APC distance d, it can thus be shown that the sum-of-squares is upper-bounded by $\gamma(d)$. The CMF is then lower-bounded by

$$\mathrm{CMF}(f) \geq \frac{6^n}{\gamma(d)}.$$

When f is a quadratic Boolean function representing a graph G, the upper bound on $Q(G,4)$ follows. □

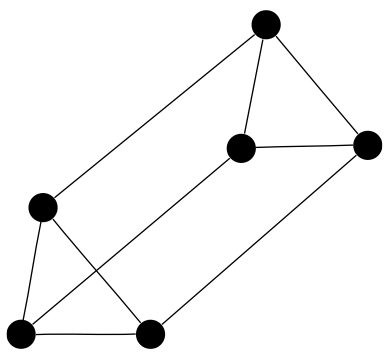
A class of self-dual quantum codes known to have high minimum distance are the *quadratic residue codes*. The graphs corresponding to these codes are *Paley graphs*. To construct a Paley graph on n vertices, where n must be a prime power and $n \equiv 1 \pmod 4$, let the elements of the finite field $\mathbb{F}_n$ be the set of vertices, and let two vertices, i and j, be joined by an edge if and only if their difference is a quadratic residue in $\mathbb{F}_n \backslash \{0\}$, i.e., there exists an $x \in \mathbb{F}_n \backslash \{0\}$ such

that $x^2 \equiv i - j$. This construction will result in a circulant adjacency matrix, where the first row is called a *Legendre sequence*. Paley graphs are known to have low independence numbers, and, since they correspond to strong quantum codes, the degrees of their interlace polynomials Q are also low, i.e., the size of the largest independent set in the LC orbit of a Paley graph is small, compared to other graphs on the same number of vertices. This suggests that Paley graphs, due to their high degree of symmetry, have the property that their independence numbers remain largely invariant with respect to LC. Another code construction is the *bordered quadratic residue code*, equivalent to extending a Paley graph by adding one vertex and connecting it to all existing vertices. For example, optimal quantum codes of length 5, 6, 29, and 30 can be constructed using Paley graphs or extended Paley graphs.

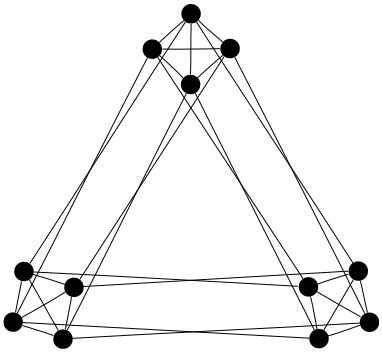
We have previously discovered [10] that many strong self-dual quantum codes can be represented as highly structured *nested clique graphs*. Some of these graphs are shown in Fig. 8. For instance, Fig. 8b shows a graph consisting of three 4-cliques. The remaining edges form a *Hamiltonian cycle*, i.e., a cycle that visits every vertex of the graph exactly once. Fig. 8c shows five 4-cliques interconnected by one Hamiltonian cycle and two cycles of length 10. Ignoring edges in the cliques, there are no cycles of length shorter than 5 in the graph. The graph in Fig. 8a can be viewed as two interconnected 3-cliques. Note that the graphs in Fig. 8 have values of δ, $\deg(Q)$, and $Q(G, 4)$ that match the optimal or best known values in Tables 4, 5, and 6. Also note that they are all *regular* graphs, with all vertices having degree δ, which means that the number of edges is minimal for the given δ.

It is interesting to observe that the problem of finding good quantum codes, or highly entangled quantum states, can be reformulated as the problem of finding LC orbits of graphs with certain properties, and that these properties are related to the interlace polynomials of the graphs. Even though certain construction techniques are known, as shown above, many open problems remain, such as providing better bounds on δ, $\deg(Q)$, and $Q(G, 4)$, and finding new methods for constructing graphs with optimal or good values for these parameters. It would also be interesting to study possible connections between the observation that the best self-dual quantum codes have a minimal number of Eulerian subgraphs, and the fact that many optimal self-dual quantum codes are of Type II, i.e., correspond to anti-Eulerian graphs. Note that all the graphs in Fig. 8 are anti-Eulerian. The graphs in Fig. 8 also give other clues as to the types of graphs that may optimise $\deg(Q)$ and $Q(G, 4)$. If a graph contains a k-clique, performing LC on any vertex in the clique will produce a graph with an independent set of size at least $k - 1$. Thus the interlace polynomial Q of a complete graph will have the highest possible degree of any connected graph. This explains why our graphs contain several relatively small cliques. That the graphs contain a few long cycles reduces the number of cycles in the graph, which makes sense when we consider that a cycle is an Eulerian subgraph.

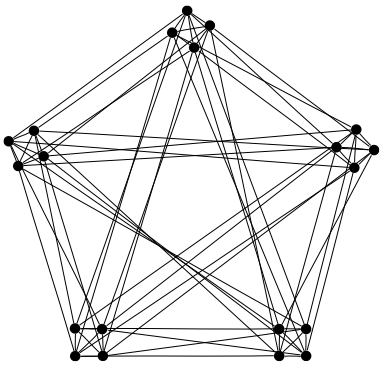
It is also possible to say something about which properties should not be present in a graph with optimal δ, $\deg(Q)$, or $Q(G, 4)$. A bipartite graph on n vertices will have an independence number of at least $\lceil \frac{n}{2} \rceil$. Thus the interlace polynomial Q associated with an LC orbit that contains a bipartite graph will have degree at least $\lceil \frac{n}{2} \rceil$. Note that bipartiteness is preserved by ELC, but not by LC. In Table 7, we give the number of LC orbits containing connected bipartite graphs on n vertices with a given value of δ. Compare this to Table 9, which



(a) $n = 6$, $\delta = 3$, $\deg(Q) = 2$, $\frac{Q(G,4)}{2^n} = 18$



(b) $n = 12$, $\delta = 5$, $\deg(Q) = 4$, $\frac{Q(G,4)}{2^n} = 234$



(c) $n = 20$, $\delta = 7$, $\deg(Q) = 6$, $\frac{Q(G,4)}{2^n} = 6444$

Fig. 8: Examples of Nested Clique Graphs

Table 7: Number of LC Orbits Containing Connected Bipartite Graphs by δ and n

$\delta\backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	1	2	3	7	14	40	106	352	1218	5140
2					1	1	2	4	16	41	215
3							1		2	1	11
All	1	1	2	3	8	15	43	110	370	1260	5366

Table 8: Number of LC Orbits of Connected Circle Graphs by δ and n

$\delta\backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	1	2	3	9	21	75	277	1346	7712	54,067
2				1	1	2	5	16	55	254	1474
3							1		2	2	12
All	1	1	2	4	10	23	81	293	1403	7968	55,553

Table 9: Number of LC Orbits of Connected Graphs by δ and n

$\delta\backslash n$	2	3	4	5	6	7	8	9	10	11	12
1	1	1	2	3	9	22	85	363	2436	26,750	611,036
2				1	1	4	11	69	576	11,200	467,513
3					1		5	8	120	2506	195,455
4										1	63
5											1
All	1	1	2	4	11	26	101	440	3132	40,457	1,274,068

includes LC orbits of all connected graphs. It also turns out that circle graphs are bad. This is not surprising, given that the circle graph obstructions shown in Fig. 5 all have optimal values of δ. The obstruction on 6 vertices also has optimal value of $Q(G, 4)$, and the two other obstructions have $Q(G, 4)$ only one greater than optimal. In Table 8, we give the number of LC orbits of connected circle graphs on n vertices with a given value of δ.

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