Superdense Coding

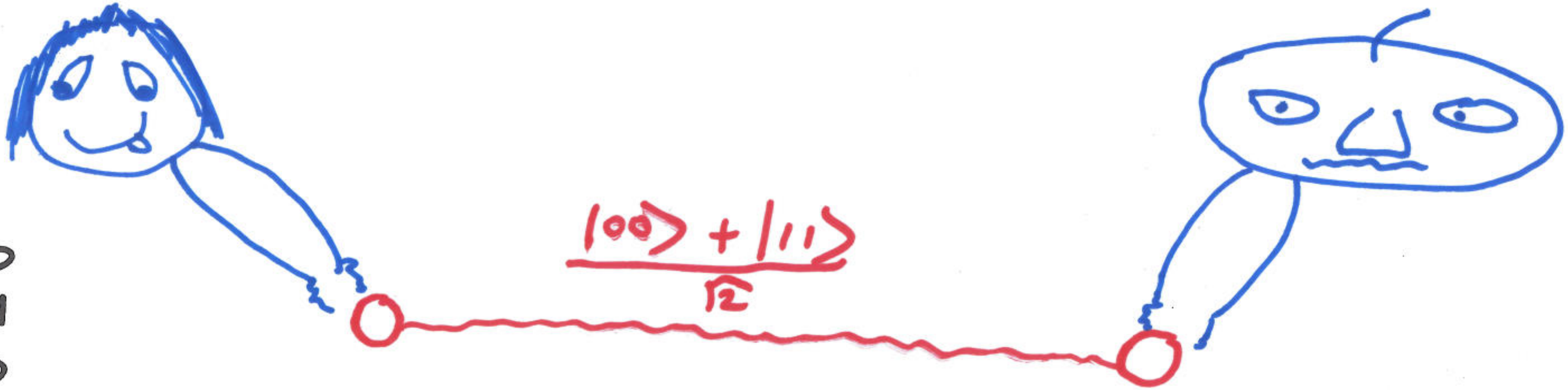
Goal: Transmit classical information from Alice to Bob.

Alice wants to send 2 classical bits to Bob.

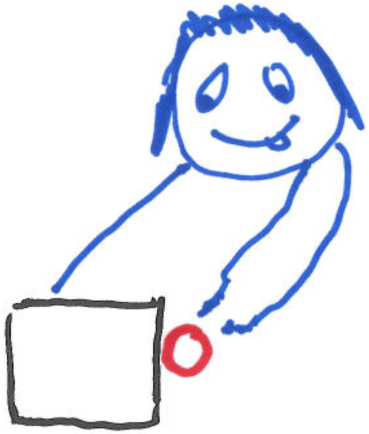
She is allowed to send one qubit.

How?

2 bits :
00
or 01
or 10
or 11



$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



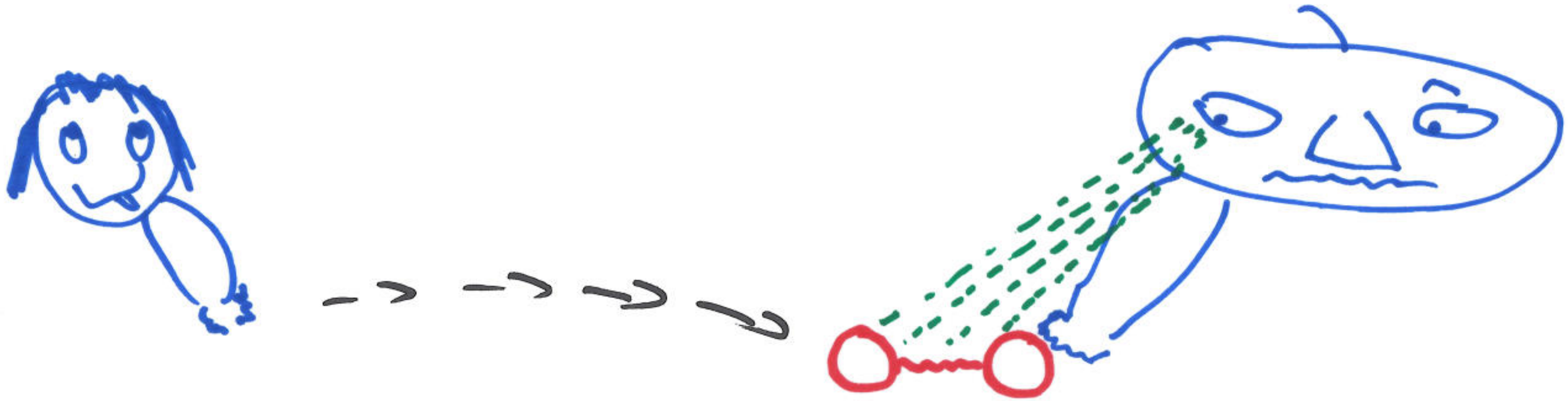
00 : I
01 : Z
10 : X
11 : iY

$$\left.\begin{aligned}
I|\psi\rangle &= |\psi\rangle \\
Z|\psi\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
X|\psi\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
iY|\psi\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned}\right\}$$

Bell basis,
Bell states,
EPR pairs.

Bell States form an orthonormal basis

Alice then sends her qubit to Bob:



Bob chooses to measure
$\{P_{00}, P_{01}, P_{10}, P_{11}\}$,

where $P_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

Similarly, $P_{01} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, $P_{10} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $P_{11} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

e.g.

$$\text{for} \quad |\psi'\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

$$\langle\psi'|P_i|\psi'\rangle = 0, \qquad i = 00, 01, 10$$

$$\langle\psi'|P_i|\psi'\rangle = 1, \qquad i = 11, \qquad \text{where } P_{11} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In general, Bob measures $\{P_{00}, P_{01}, P_{10}, P_{11}\}$

$$\downarrow$$

$$j \Rightarrow \langle\psi'|P_j|\psi'\rangle = 1.$$

therefore Alice sends j.

Averages?

Choose, arbitrarily, eigenvalues $1, -1, i, -i$

for P_{00}, P_{01}, P_{10}, P_{11}, respectively.

Then,

Observable,

$$M = \sum_j m_j P_j = 1.P_{00} + (-1).P_{01} + i P_{10} + (-i).P_{11}$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

e.g. let $|\psi'\rangle$ always be $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$,

then, average $= \langle \psi' | M | \psi' \rangle = -i$.

Super-Dense Coding Summary

1 pre-shared EPR pair
(in known basis?)

2 classical bits with Alice

1 qubit communicated

$\Longleftrightarrow$

1 unentangled pair of qubits with Bob

2 classical bits shared by Alice and Bob.

i.e.

2 bits communicated : cost = 1 EPR pair
1 qubit communicated

... partial verification in lab.

Density Operator

... alternative to state formulation

... but equivalent.

Ensembles of quantum states

Which state do I have, $|\psi_0\rangle$, $|\psi_1\rangle$, or $|\psi_2\rangle$ ... ?

... and with what probability? ...

density matrix $= \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

..... realistic situation ...

Evolution

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger$$

$$= U\rho U^\dagger .$$

Measurement

Probability of getting m given i:

$$p(m|i) = \langle \psi_i | M_m^\dagger M_m | \psi_i \rangle$$
$$= tr(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|)$$

$\Rightarrow$

$$p(m) = \sum_i p(m|i) p_i$$
$$= \sum_i p_i \, tr(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|)$$
$$= tr(M_m^\dagger M_m \rho).$$

Density Operator after measurement result m?

Assume initial state is $|\psi_i\rangle$.

Then, after measuring m,

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

$$\Rightarrow$$

$$\rho_m = \sum_i p(i|m)\,|\psi_i^m\rangle\langle\psi_i^m|$$

$$= \sum_i p(i/m)\,\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}.$$

Remember:
$$\rho_m = \sum_i \frac{p(i|m) M_m |\psi_i\rangle\langle\psi_i| M_m^\dagger}{\langle\psi_i| M_m^\dagger M_m |\psi_i\rangle}.$$
$$p(m|i) = \langle\psi_i| M_m^\dagger M_m |\psi_i\rangle.$$
$$p(m) = tr(M_m^\dagger M_m \rho).$$

But,
$$p(i|m) = \frac{p(m,i)}{p(m)} = \frac{p(m|i)\, p_i}{p(m)}$$

$$\Rightarrow \rho_m = \sum_i p_i \frac{M_m |\psi_i\rangle\langle\psi_i| M_m^\dagger}{tr(M_m^\dagger M_m \rho)}$$
$$= \frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}.$$

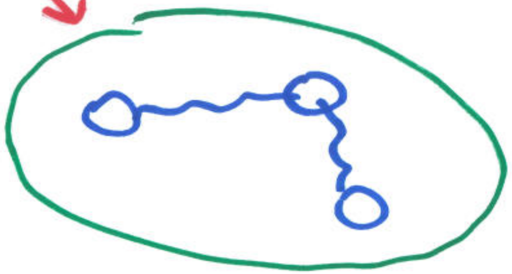
Language

If state $|\psi\rangle$ is known then it is a pure state,

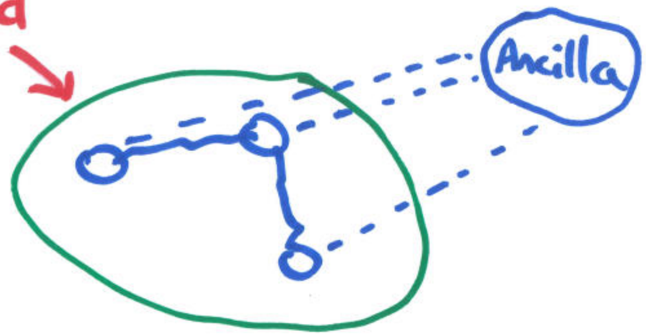
$$\rho = |\psi\rangle\langle\psi|.$$

Otherwise, ρ is a mixed state, and is a mixture of pure states:

Pure



Mixed



Properties

Pure state : $\mathrm{tr}(\rho^2) = 1$

Mixed state : $\mathrm{tr}(\rho^2) < 1$.

.... Sometimes people use "mixed" to include "pure".

Mixture of Density Matrices

Imagine ρ_i is prepared with probability p_i.

Then,

$$\text{density matrix} = \sum_i p_i \rho_i \quad .$$

Ignorance

... noise introduces ignorance as to our knowledge of the state,

e.g. measure then forget result

$$\Rightarrow \quad \rho = \sum_m p(m)\rho_m$$

$$= \sum_m \mathrm{tr}(M_m^\dagger M_m \rho) \frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$$

$$= \sum_m M_m \rho M_m^\dagger .$$

Density Operator - intrinsic characterization

ρ is a density operator associated to $\{p_i, |\psi_i\rangle\}$

iff

1. ρ has trace one. (trace condition).
2. ρ is positive. (positivity condition).

i.e. $\langle\psi|\rho|\psi\rangle \geq 0$.

Reformulated Postulates using Density Operator

Postulate 1:

Associated to any isolated system is a complex vector with inner product known as the state space of the system. The system is completely described by its density operator - a positive operator ρ with trace one, acting on the system,

where,

$$\rho = \sum_i p_i \rho_i \, .$$

Postulate 2

The evolution of a closed quantum system is described by a unitary transformation. The state ρ at time t_1 is related to ρ' at time t_2 by a unitary U, depending only on t_1 and t_2,

$$\rho' = U \rho U^{\dagger}.$$

Postulate 3

Measurements are described by a collection $\{M_m\}$ of measurement operators. Index m refers to measurement outcomes. The probability that result m occurs is,

$$p(m) = tr(M_m^\dagger M_m \rho),$$

the state after measurement is,

$$M_m \rho M_m^\dagger \,/\, tr(M_m^\dagger M_m \rho),$$

The completeness equation is satisfied,

$$\sum_m M_m^\dagger M_m = I.$$

Postulate 4

The state space of a composite system is the tensor product of the component spaces, where,

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \rho_n .$$

Is a state mixed or pure?

Given ρ,

Test $\mathrm{tr}(\rho^2)$ $\rightarrow$ 1 : pure

$\rightarrow$ <1 : mixed.

Two Different Ensembles can give rise to the same density matrix

e.g. Let $|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$

$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle.$

Let states $|a\rangle$ and $|b\rangle$ be prepared with $pr = \frac{1}{2}$ each.

Then,

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|.$$

... but also..

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|.$$

.... so which ensemble is it ??

What Class of Ensembles Gives a Particular Density Matrix

Let $\{|\tilde{\psi}_i\rangle\}$ be an ensemble of unnormalised vectors,

such that,

$$|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$$

Let,

$$\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

$\{|\tilde{\psi}_i\rangle\}$ and $\{|\tilde{\varphi}_i\rangle\}$ generate the same density matrix iff,

$$|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle,$$

where u_{ij} is a unitary matrix.

The smallest vector of $|\tilde{\psi}_i\rangle$ or $|\tilde{\varphi}_j\rangle$ is padded with zeroes.

It follows that ....

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\varphi_j\rangle\langle\varphi_j|$$

iff

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\varphi_j\rangle .$$

Bloch Sphere for Mixed States

For $\vec{r}$ a real vector $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$, and $\vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$

be a vector of Pauli matrices,

Then a mixed-state qubit can be written as,

$$\rho = \frac{I + \vec{r}.\vec{\sigma}}{2}, \quad \text{where } \|\vec{r}\| \leq 1.$$

If ρ is pure then $\|\vec{r}\| = 1$

$\Rightarrow$ Mixed-state is interior of sphere.

Minimal Ensemble

$\{p_i, |\psi_i\rangle\}$ is minimal for ρ if

$$|\{p_i, |\psi_i\rangle\}| = \text{rank}(\rho).$$

The support of ρ is the vector space spanned by the eigenvectors of ρ with non-zero eigenvalues.

Let $|\psi\rangle \in \text{support}(\rho)$.

Then, $\exists$ minimal ensemble for ρ containing $|\psi\rangle$.

In such an ensemble, $|\psi\rangle$ appears with probability

$$p = 1/\langle\psi|\rho^{-1}|\psi\rangle,$$

where ρ^{-1} is the inverse of ρ' - the operator acting on support(ρ).

Reduced Density Operator

Describes subsystems of composite system.

Consider systems A and B, described by

$$\rho^{AB}.$$

The reduced density operator for system A is,

$$\rho^{A} = tr_B(\rho^{AB}),$$

where tr_B is the partial trace over system B,

$$tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \, tr(|b_1\rangle\langle b_2|)$$

and,

$$tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle.$$

$\rho^A = tr_B(\rho^{AB})$ provides correct measurement statistics for measurements on A,

e.g. let $\rho^{AB} = \rho_A \otimes \rho_B$.

Then,
$$\rho^A = tr_B(\rho_A \otimes \rho_B) = \rho_A \, tr(\rho_B) = \rho_A,$$
as expected.

Similarly,
$$\rho^B = \rho_B.$$

Example:

Let $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ with density operator,

$$\rho = \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\rho' = tr_2(\rho) = \left[tr_2\left(|00\rangle\langle 00|\right) + tr_2\left(|11\rangle\langle 00|\right) + tr\left(|00\rangle\langle 11|\right) + tr_2\left(|11\rangle\langle 11|\right)\right]/2$$

$$= \frac{|0\rangle\langle 0|\langle 0|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}.$$

Intuitively,

$$\begin{array}{cc|l} x_2 & x_1 & \\ 0 & 0 & \\ 0 & 1 & : \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ 1 & 0 & \\ 1 & 1 & \end{array}$$

"tracing over qubit x_2",

when $x_2 = 0$, we have $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for x_1

when $x_2 = 1$, we have $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for x_1.

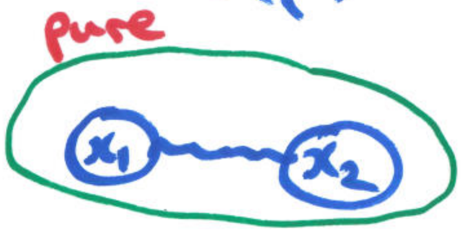
In density matrix terms,

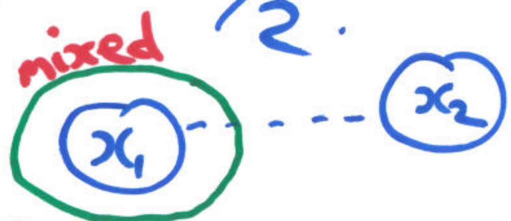
$$\rho_{x_2=0} = \frac{1}{2}|0\rangle\langle 0| = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho_{x_2=1} = \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \rho' = tr_2(\rho) = \rho_{x_2=0} + \rho_{x_2=1} = I/2.$$

Visually,

$\rho =$ 

, $\rho' = tr_2(\rho) =$ 

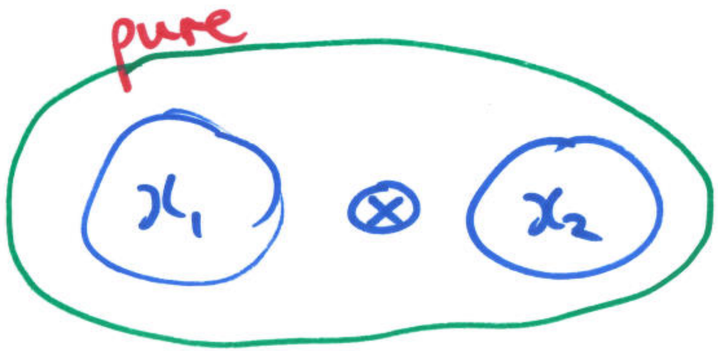
tracing out system 2 externalises system 2 $\rightarrow$

Special Case:

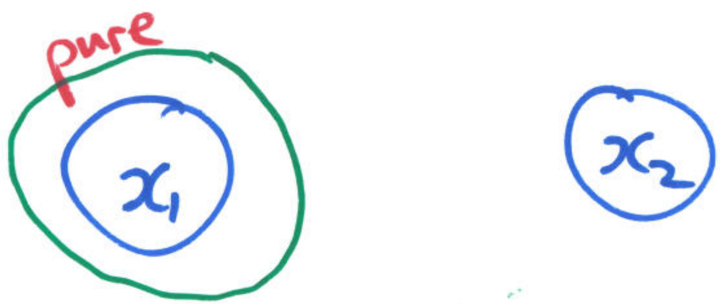
Let $\rho = \rho_1 \otimes \rho_2$.

Then $\rho' = \mathrm{tr}_2(\rho) = \rho_1$: pure state

$\rho =$



$\rho' = \mathrm{tr}_2(\rho) =$



Teleportation

Before Alice makes her measurement.

$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |01\rangle(\alpha|1\rangle+\beta|0\rangle)$$
$$+ |10\rangle(\alpha|0\rangle-\beta|1\rangle)\Big]$$

After Alice measures in computational basis,

$|00\rangle[\alpha|0\rangle+\beta|1\rangle]$ with pr $= \frac{1}{4}$

$|01\rangle[\alpha|1\rangle+\beta|0\rangle]$ " "

$|10\rangle[\alpha|0\rangle-\beta|1\rangle]$ " "

$|11\rangle[\alpha|1\rangle-\beta|0\rangle]$ " " .

... after Alice measures in computational basis...

Density operator is

$$\rho = \frac{1}{4}\Big[|00\rangle\langle 00| \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)$$

$$+ |01\rangle\langle 01| \left(\alpha|1\rangle + \beta|0\rangle\right)\left(\alpha^*\langle 1| + \beta^*\langle 0|\right)$$

$$+ |10\rangle\langle 10| \left(\alpha|0\rangle - \beta|1\rangle\right)\left(\alpha^*\langle 0| - \beta^*\langle 1|\right)$$

$$+ |11\rangle\langle 11| \left(\alpha|1\rangle - \beta|0\rangle\right)\left(\alpha^*\langle 1| - \beta^*\langle 0|\right)\Big]$$

.... so what does Bob see?

... what does Bob see?

.... trace out Alice's system ...

$$\rho^B = \frac{1}{4}\Big[\big(\alpha|0\rangle+\beta|1\rangle\big)\big(\alpha^*\langle 0|+\beta^*\langle 1|\big)$$
$$+\big(\alpha|1\rangle+\beta|0\rangle\big)\big(\alpha^*\langle 1|+\beta^*\langle 0|\big)$$
$$+\big(\alpha|0\rangle-\beta|1\rangle\big)\big(\alpha^*\langle 0|-\beta^*\langle 1|\big)$$
$$+\big(\alpha|1\rangle-\beta|0\rangle\big)\big(\alpha^*\langle 1|-\beta^*\langle 0|\big)\Big]$$
$$= \frac{2}{4}\big(|\alpha|^2+|\beta|^2\big)|0\rangle\langle 0| + \frac{2}{4}\big(|\alpha|^2+|\beta|^2\big)|1\rangle\langle 1|$$
$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = I/2.$$

Thus,

Bob's state after Alice has measured but before Bob has learned the measurement result is,

$$I/2$$

.... no dependence on state $|\psi\rangle$ being teleported

$\Rightarrow$

no information sent faster than speed of light.

More General Example

Consider

$$\begin{array}{cc|c} B & A & \\ 0 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & c \\ 1 & 1 & d \end{array} \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Tr_B : $x_B = 0$, $pr = |a|^2 + |b|^2$

$\Rightarrow$ normalised state $= \frac{1}{|a|^2+|b|^2}\begin{pmatrix} a \\ b \end{pmatrix}$

$$= \begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix}.$$

$x_B = 1$, $pr = |c|^2 + |d|^2$

$\Rightarrow$ norm. state $= \frac{1}{|c|^2+|d|^2}\begin{pmatrix} c \\ d \end{pmatrix}$

$$= \begin{bmatrix} |c|^2 & cd^* \\ dc^* & |d|^2 \end{bmatrix}.$$

$$\Rightarrow Tr_B = \begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix} + \begin{bmatrix} |c|^2 & cd^* \\ dc^* & |d|^2 \end{bmatrix},$$

e.g. $a = d = \frac{1}{\sqrt{2}}$, $b = c = 0$

$$\Rightarrow \rho_B = Tr_B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = I/2.$$

e.g.

Measure observable $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on ρ_B.

Therefore,

$$p(0) = |0\rangle\langle 0| \rho_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} |a|^2 + |c|^2 & ab^* + cd^* \\ ba^* + dc^* & |b|^2 + |d|^2 \end{bmatrix}$$

$$= |a|^2 + |c|^2.$$

Similarly,

$$p(1) = |1\rangle\langle 1| \rho_B = |b|^2 + |d|^2$$

.... equivalent to "summarising over B."