POVM Measurements

Postulate 3 $\longrightarrow$ measurement statistics
$\longrightarrow$ post-measurement state

... but ... post-measurement state not always wanted

... in such a case one may use

POVMs (Positive Operator-Value Measurements).

Consider measurement operators $\{M_m\}$ performed on $|\psi\rangle$.

Then,

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

Define,

$$E_m = M_m^\dagger M_m$$

Then,

E_m is a positive operator

$$\sum_m E_m = I$$

$$p(m) = \langle\psi|E_m|\psi\rangle.$$

Operators E_m are POVM elements.

$\{E_m\}$ is a POVM.

Example

Consider projective measurements $\{P_m\}$.

Then,

$$P_m P_{m'} = \delta_{mm'} P_m \quad \text{and} \quad \sum_m P_m = I.$$

For this special case only,

POVM elements = measurement operators,

...because...

$$E_m = P_m^\dagger P_m = P_m.$$

Consider a set $\{E_m\}$ of arbitrary positive operators such that,

$$\sum_m E_m = I.$$

Then,

there always exists a set of measurement operators, $\{M_m\}$, defining a POVM $\{E_m\}$.

This is true if

$$M_m = \sqrt{E_m}.$$

Therefore,

Define POVM to be a set of operators,

$$\{E_m\},$$

such that

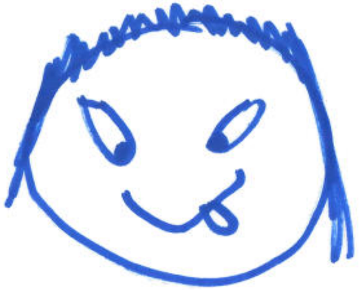
each E_m is positive,

$$\sum_m E_m = I.$$

Then,

$$p(m) = \langle\psi| E_m |\psi\rangle.$$

Example



$$|\psi_1\rangle = |0\rangle$$

or

$$|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$

Bob's problem:

Perform a measurement which distinguishes the two states some of the time, but never mis-identifies.

Remember $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

Solution:

Use POVM with three elements:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1|$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$E_3 = I - E_1 - E_2.$$

Note: $\langle\psi_1|E_1|\psi_1\rangle = 0$, $\langle\psi_2|E_2|\psi_2\rangle = 0.$

Measurement Results:

E_1 : Infer $|\psi_2\rangle$ sent.
E_2 : Infer $|\psi_1\rangle$ sent.
E_3 : Infer nothing.

$\Rightarrow$ Bob never makes a mistake.

Phase

.... different meanings.

Meaning one:

$e^{i\theta}|\psi\rangle$ equals $|\psi\rangle$ up to a global phase

... measurement statistics for two states are identical, i.e.

$$\langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|e^{-i\theta}M_m^\dagger M_m e^{i\theta}|\psi\rangle.$$

... observationally identical.

Meaning two:

relative phase:

Consider

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

amplitude of $|1\rangle$ is $\frac{1}{\sqrt{2}}$

amplitude of $|1\rangle$ is $-\frac{1}{\sqrt{2}}$

same magnitude, different phase

Relative phase is basis-dependent.

Global phase is basis-independent.

Composite Systems

Two or more physical systems?

Postulate 4:

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$.

Example

Show that the average value of the observable $X_1 Z_2$ for a two-qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

Answer:

$$(\langle 00| + \langle 11|)/\sqrt{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} (|00\rangle + |11\rangle)/\sqrt{2}$$

$$= \frac{1}{2} \times (1\,0\,0\,1) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}(1\,0\,0\,1) \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0.$$

Projective measurement + unitary transform

= general measurement.

Proof: Consider state space Q

We wish to perform $\{M_m\}$ on Q.

Introduce ancilla system with state space M, having orthonormal basis $\{|m\rangle\}$ in one-to-one with possible measurement outcomes.

Then we consider,

$$U|\psi\rangle|0\rangle = \sum_m M_m|\psi\rangle|m\rangle$$

... for some matrix U.

We have,

$$U|\psi\rangle|0\rangle = \sum_m M_m|\psi\rangle|m\rangle$$

Using orthonormality of $\{|m\rangle\}$ and completeness of $\{M_m\}$, we see that U preserves inner product of $|\psi\rangle|0\rangle$ with $|\psi\rangle|0\rangle$,

$$\langle\psi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_{m,m'}\langle\psi|M_m^\dagger M_{m'}|\psi\rangle\langle m|m'\rangle$$

$$= \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

$$= \langle\psi|\psi\rangle$$

$\Rightarrow U$ can be made a unitary operator acting on $Q \otimes M$.

Let V be a Hilbert space with subspace W.

Suppose $U : W \to W$ is a linear operator preserving inner products, i.e.

$$\langle w_1 | U^\dagger U | w_2 \rangle = \langle w_1 | w_2 \rangle, \quad |w_1\rangle, |w_2\rangle \in W.$$

Then there exists a unitary operator $U' : V \to V$, which extends U,

i.e.

$$U' |w\rangle = U |w\rangle, \quad \forall |w\rangle \in W \ldots$$

but U' is defined on V.

Consider a projector,

$$P_m = I_Q \otimes |m\rangle\langle m|.$$

Then,

$$p(m) = \langle\psi|\langle 0| U^\dagger P_m U |\psi\rangle|0\rangle$$

$$= \sum_{m',m''} \langle\psi| M_{m'}^\dagger \langle m'| \left(I_Q \otimes |m\rangle\langle m|\right) M_{m''} |\psi\rangle |m''\rangle$$

$$= \langle\psi| M_m^\dagger M_m |\psi\rangle.$$

Joint state of system $Q \otimes M$ after measurement of m is,

$$\frac{P_m U|\psi\rangle|0\rangle}{\sqrt{\langle\psi|U^\dagger P_m U|\psi\rangle}} = \frac{M_m|\psi\rangle|m\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

$\Rightarrow$ post-measurement state of system M is $|m\rangle$, and the state of system Q is,

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}.$$

Conclusion

Unitary dynamics, projective measurements, and added ancilla, allow any form of measurement.

$$U \; |\psi\rangle \otimes |0\rangle \longrightarrow M_m \, |\psi\rangle \; \{|m\rangle\}$$

$$\Rightarrow \quad P_m \left\{ M_m \, |\psi\rangle \; \{|m\rangle\} \right\} \begin{array}{l} \longrightarrow |m\rangle \\ \longrightarrow M_m |\psi\rangle \end{array}$$

Why general measurement (Postulate 3)?

.... why not just use projective measurements + unitary operations?

1. General measurements are simpler
   – fewer restrictions, e.g. $P_i P_j = \delta_{ij} P_i$ not required.
2. Optimal distinguishing of quantum states involve general measurement, not projective measurement.
3. Projective measurements are "repeatable"
   i.e. first measurement gives m, subsequent measurements give m with prob. = 1.

But (to re-iterate),

I prefer,

ancilla.

$$U\left(|\psi\rangle \otimes |0\rangle \right)$$

$$\Rightarrow \left\{ M_m \left(|\psi\rangle \otimes |m\rangle \right) \right\}$$

such that,

throw away

$$P_m\left(\left\{ M_m \left(|\psi\rangle \otimes |m\rangle \right) \right\} \right) \rightarrow |m\rangle, \quad \rightarrow M_m|\psi\rangle.$$

Entanglement

Consider,

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$\nexists\ |a\rangle, |b\rangle$ such that $|\psi\rangle = |a\rangle|b\rangle$

i.e. tensor factorisation of $|\psi\rangle$ is impossible.

POVMs?

.... Special case of general measurement

... don't need/wish to know post-measurement state.

To Emphasise

This matrix notation is <u>not</u> necessary.

To measure state $|\psi\rangle$ generally:

1. Extend $|\psi\rangle$ by adding ancilla to give,

   $$|\psi\rangle \otimes |a\rangle.$$

2. Apply some unitary, U, to $|\psi\rangle \otimes |a\rangle$ to give $|\psi'\rangle$.
3. Measure $|\psi'\rangle$ in computational basis.
4. Apply $U^{\dagger}$ to $|\psi'\rangle$
5. Throw away ancilla.