Postulates of Quantum Mechanics

Postulate 1:

Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

Qubit

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1$$

... normalization condition.

Postulate 2

The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U depending only on times t_1 and t_2,

$$|\psi'\rangle = U|\psi\rangle.$$

What unitary operators are naturally realizable?

.... for single qubits ...

... any unitary operator is realizable.

Hadamard gate,

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

What are the eigenvalues and eigenvectors of H?

Closed Systems

Which systems are described by unitary evolution?

Answer: The universe.

... a subsystem may be approximately closed but, in reality, non-closure appears as "noise" or "decoherence".

Evolution in Continuous Time

Postulate 2':

The time evolution of the state of a closed quantum system is described by the Schrödinger equation,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

$\hbar$ is Planck's constant. It is common to absorb $\hbar$ into H, effectively setting $\hbar = 1$. H is a fixed Hermitian operator known as the Hamiltonian of the closed system.

Determining the Hamiltonian for a given physical system is a difficult problem.

We will not consider Hamiltonians.... but sometimes assume a Hamiltonian as a starting point.

Spectral Decomposition

Hamiltonian, H, is a Hermitian operator;

$\Rightarrow$

$$H = \sum_E E|E\rangle\langle E|$$

... with eigenvalues E and eigenvectors $|E\rangle$.

E is the energy of the state $|E\rangle$.

Lowest energy, E_{min}, is the ground state energy.

Corresponding eigenstate, $|E_{min}\rangle$, is the ground state.

The states $|E\rangle$ are known as stationary states because their only change in time is to acquire a phase shift,

$$|E\rangle \rightarrow e^{-iEt/\hbar}|E\rangle.$$

Example

Let $H = \hbar\omega X = \hbar\omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Then H has the same eigenstates as X, namely

$$(|0\rangle + |1\rangle)/\sqrt{2} \quad \text{and} \quad (|0\rangle - |1\rangle)/\sqrt{2},$$

with eigenvalues $\hbar\omega$ and $-\hbar\omega$.

$\Rightarrow$ ground state is $(|0\rangle - |1\rangle)/\sqrt{2}$

and ground state energy is $-\hbar\omega$.

Solution to Schrödinger's equation

$$|\psi(t_2)\rangle = e^{\frac{-iH(t_2-t_1)}{\hbar}}|\psi(t_1)\rangle = U(t_1,t_2)|\psi(t_1)\rangle,$$

where,

$$U(t_1,t_2) = e^{\frac{-iH(t_2-t_1)}{\hbar}} \quad \text{is unitary.}$$

.... in fact, any unitary operator can be written as

$$U = e^{iK}, \quad \text{for } K \text{ a Hermitian operator.}$$

One-to-one correspondence

discrete-time description of dynamics using unitary operators $\Longleftrightarrow$ continuous-time description using Hamiltonians.

we will study

For A, B commuting Hermitian operators, show that,

$$e^A e^B = e^{A+B}.$$

Proof (sketch):

$$e^A = \sum_j e^{a_j} |E_j\rangle\langle E_j|$$

(commuting Hermitian implies orthogonal operators)

$$\Rightarrow e^A e^B = \sum_j e^{a_j} |E_j\rangle\langle E_j| \sum_j e^{b_j} |E_j\rangle\langle E_j|$$

But $\langle E_i | E_j \rangle = \delta_{ij}$

$$\Rightarrow e^A e^B = \sum_j e^{a_j + b_j} |E_j\rangle\langle E_j| = e^{A+B}.$$

Show that $U(t_1,t_2)$ is unitary

Proof (sketch):

$$U(t_1,t_2) = e^{\frac{-iH(t_2-t_1)}{\hbar}} = e^{\frac{it_1}{\hbar}H} e^{\frac{-it_2}{\hbar}H} = e^A e^B.$$

H is Hermitian, so

$$H = \sum_E E |E\rangle\langle E|$$

Let, $V = e^{iH} = \sum_E e^{iE} |E\rangle\langle E|$

$$\Rightarrow V^\dagger = \sum_E e^{-iE} |E\rangle\langle E| \Rightarrow VV^\dagger = I.$$

"Applying" a Unitary operator to a closed quantum system

???

Unitary evolution $\Rightarrow$ closed system.

How can we "apply" when we are external to the system?

Example

Laser $\longrightarrow$ atom

$$\text{Hamiltonian}\{\text{Laser} \rightarrow \text{atom}\}$$

$$\simeq \underbrace{\text{Hamiltonian}_{\substack{\text{atom}\\\text{parameters}}}\{\text{laser}\} \otimes \text{Hamiltonian}_{\substack{\text{laser}\\\text{parameters}}}\{\text{atom}\}}_{}$$

model used

.... it is as if the evolution of the atom is described by a Hamiltonian which we can vary at will, despite the atom not being a closed system.

Quantum Measurement

Closed system $\Longleftrightarrow$ unitary evolution

.... but we must model interactions between experimentalist and equipment.

.... measurement implies interaction of closed system with external world.

Postulate 3

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state is $|\psi\rangle$ before measurement then the probability that result m occurs is

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle.$$

Postulate 3 (continued)

State of the system after measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

Completeness equation:

$$\sum_m M_m^\dagger M_m = I.$$

$\Rightarrow$ probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle.$$

Measurement of qubit in computational basis

$$M_0 = |0\rangle\langle 0| , \qquad M_1 = |1\rangle\langle 1|$$

$\because$ Hermitian

$$\Rightarrow M_j^2 = M_j , \qquad M_j^\dagger = M_j .$$

$\Rightarrow$

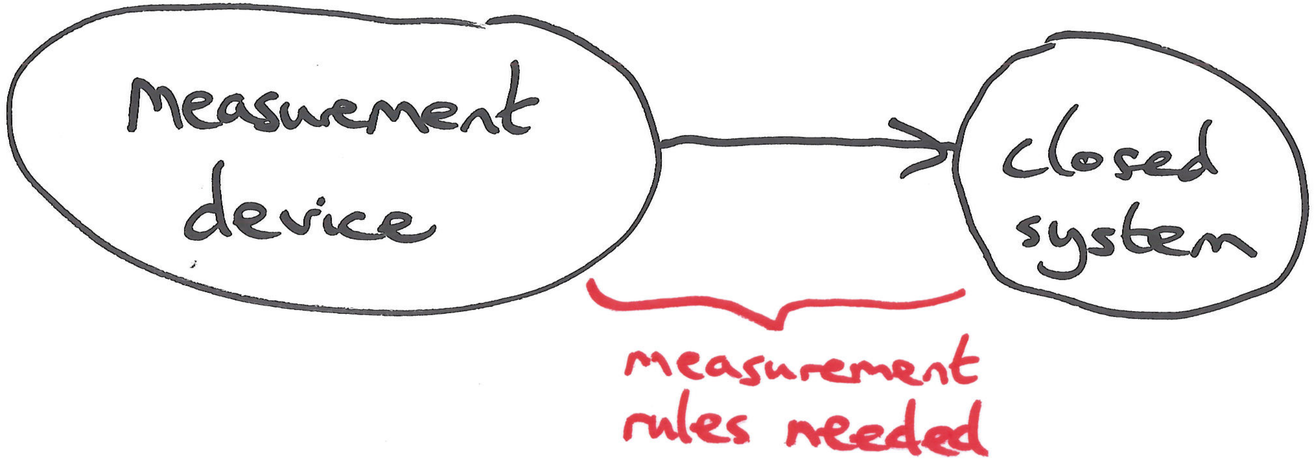
$$I = M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1 \quad \text{is satisfied.}$$

Moreover,

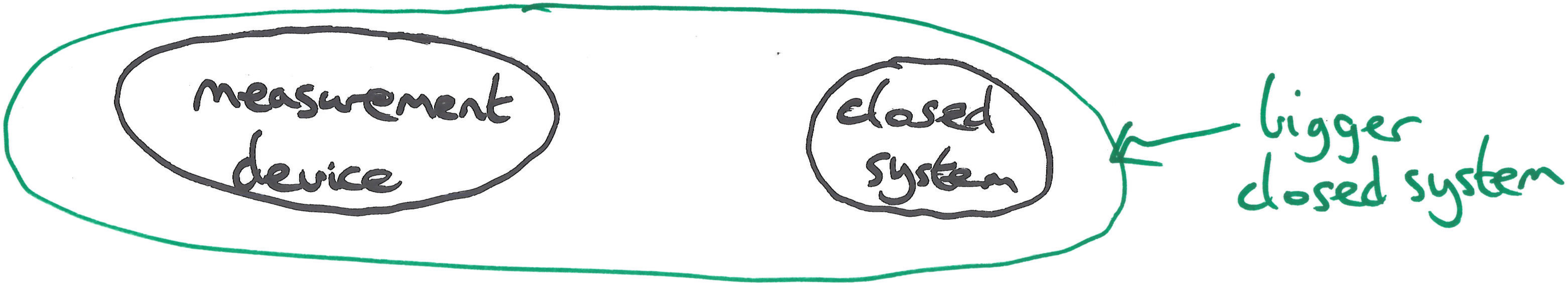
$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = \langle \psi | M_0 | \psi \rangle = |a|^2$$

and, state after measurement is

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|} |0\rangle .$$



... but... why not...



... just using unitaries?

... may be possible.... physicists disagree.

Cascaded Measurements are Single Measurements

$$\text{measure}_{M_m}\left(\text{measure}_{L_l}(|\psi\rangle)\right) = \text{measure}_{N_{lm}}(|\psi\rangle),$$

where

$$N_{lm} \equiv M_m L_l$$

... proof..?

Distinguishing Quantum States

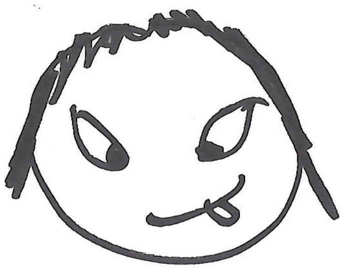
"Ideal" classical objects are distinguishable,

e.g. coin toss "heads" or "tails"

(... but consider two or more CDMA sequences..)

Quantum states are not, in general, deterministically distinguishable.

Distinguishable if orthonormal



$$\{|\psi_i\rangle\}_{1\leq i\leq n} \xrightarrow{\quad |\psi_j\rangle \quad} \{M_i\} \cup M_0$$

where $\langle\psi_i|\psi_j\rangle = \delta_{ij}$,

and $M_0 = \sqrt{I - \sum_{i\neq 0} |\psi_i\rangle\langle\psi_i|}^{\,+ve}$

(to ensure completeness)

$$\Rightarrow \quad p(i) = \langle\psi_i|M_i|\psi_i\rangle = 1.$$

Non-distinguishable if not orthonormal

Assume

$$\langle \psi_i | \psi_j \rangle \neq \delta_{ij} \quad \text{in general.}$$

Let

$$M_k = p(i)|i\rangle\langle i| + p(j)|j\rangle\langle j|,$$

$$p_i, p_j < 1.$$

Then,

$$p(i) = \langle \psi_i | M_k | \psi_i \rangle < 1.$$

Projective Measurements

Postulate 3 $\equiv$ Projective measurements
+ unitary transforms

A projective measurement is described by an observable, M, a Hermitian operator on the system state space being observed. Spectral decomposition is:

$$M = \sum_m m P_m ,$$

where P_m is the projector onto the eigenspace of M with eigenvalue m.

Projective measurement -

observable, $M = \sum_m m P_m$.

Possible measurement outcomes correspond to eigenvalues, m, of the observable.

Upon measuring $|\psi\rangle$, the probability of getting result m is

$$p(m) = \langle\psi|P_m|\psi\rangle.$$

Given that outcome m occured, the resultant state is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}}.$$

Projective measurements as special case of Postulate 3

Let $\{M_m\}$ satisfy completeness
and be orthogonal projectors,
(i.e. the M_m are Hermitian and
$M_m M_{m'} = \delta_{mm'} M_m$)

..... then Postulate 3 defines
projective measurements.

Expected Values for Projective Measurements

$$E(M) = \sum_m m\,p(m)$$
$$= \sum_m m \langle\psi|P_m|\psi\rangle$$
$$= \langle\psi|\left(\sum_m m P_m\right)|\psi\rangle$$
$$= \langle\psi|M|\psi\rangle.$$

Average value of observable is,

$$\langle M\rangle \equiv \langle\psi|M|\psi\rangle.$$

Standard deviation^2 $= [\Delta(M)]^2 = \langle(M - \langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2$,

so, after many experiments,

$$\Delta(M) = \sqrt{\langle M^2\rangle - \langle M\rangle^2}.$$

Alternative Nomenclature

List a complete set of orthogonal projectors, P_m, such that,

$$\sum_m P_m = I \quad \text{and} \quad P_m P_{m'} = \delta_{mm'} P_m .$$

Then corresponding observable is,

$$M = \sum_m m P_m .$$

"Measure in a basis $|m\rangle$"

... where $\{|m\rangle\}$ form an orthonormal basis.

... means,

Perform projective measurements with projectors,

$$P_m = |m\rangle\langle m|.$$

Example

Measure observable $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

eigenvectors $|0\rangle$ and $|1\rangle$ with eigenvalues $+1$ and -1.

e.g. Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$

Then measurement of Z on $|\psi\rangle \xrightarrow{+1} |0\rangle$, $pr = \langle\psi|0\rangle\langle 0|\psi\rangle = \frac{1}{2}$

$\xrightarrow{-1} |1\rangle$, $pr = \langle\psi|1\rangle\langle 1|\psi\rangle = \frac{1}{2}$

Note:

Observable, $Z = 1.|0\rangle\langle 0| + -1.|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\left(Z = \sum_m m P_m.\right)$

and $|0\rangle\langle 0| + |1\rangle\langle 1| = I$ $\left(\sum_m P_m = I\right)$

Measurement of spin "along $\vec{v}$ axis"

Let $\vec{v} = (v_1, v_2, v_3)$, $\quad v_i \in R$

Let $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\quad \sigma_i$ Pauli matrices.

Then the observable,

$$\vec{v} \cdot \vec{\sigma} = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3$$

is (historically) referred to as,

"measurement of spin along $\vec{v}$ axis."

$\vec{\sigma}$??

Heisenberg Uncertainty Principle

Let A, B be two Hermitian operators.

Let

$$\langle\psi|AB|\psi\rangle = x + iy, \qquad x, y \in \mathbb{R}$$

Note:

$$\langle\psi|[A,B]|\psi\rangle = 2iy$$

$$\langle\psi|\{A,B\}|\psi\rangle = 2x$$

$$\Rightarrow |\langle\psi|[A,B]|\psi\rangle|^2 + |\langle\psi|\{A,B\}|\psi\rangle|^2 = 4|\langle\psi|AB|\psi\rangle|^2.$$

By Cauchy-Schwarz

$$|\langle\psi|AB|\psi\rangle|^2 \leq \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle$$

$$\Rightarrow |\langle\psi|[A,B]|\psi\rangle|^2 \leq 4\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle$$

Let C, D be observables, where $A = C - \langle C\rangle$, $B = D - \langle D\rangle$

$$\Rightarrow \Delta(C)\Delta(D) \geq |\langle\psi|[C,D]|\psi\rangle|/2 \qquad \leftarrow \text{HUP}$$

Heisenberg Uncertainty Principle

$$\Delta(C)\,\Delta(D) \geq \frac{|\langle\psi|[C,D]|\psi\rangle|}{2}.$$

Prepare large number of $|\psi\rangle$ states.
Perform measurement of C on some and D on the rest.

Then standard deviation of C results times standard deviation of D results satisfies above inequality.

Example

Consider observables X and Y, measured on $|0\rangle$.

We know that
$$[X,Y] = 2iZ$$
$$\Rightarrow \Delta(X)\Delta(Y) \geq \langle 0|Z|0\rangle = 1$$

.... "uncertainty" is lower bounded.

Note: eigenvectors/eigenvalues of X are $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, with $1, -1$, respectively.

$$\left.\begin{aligned} \Rightarrow P_1 &= \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\tfrac{1}{\sqrt{2}}(\langle 0| + \langle 1|) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ P_{-1} &= \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\tfrac{1}{\sqrt{2}}(\langle 0| - \langle 1|) = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned}\right\} \Rightarrow X = 1.P_1 + (-1).P_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly, for Y,
$$\left.\begin{aligned} P_1 &= \tfrac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\ P_{-1} &= \tfrac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \end{aligned}\right\} \Rightarrow Y = 1.P_1 + (-1).P_{-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$