Quantum Bits

Physical object? ..... yes, but also...

Mathematical object...

... independent of physical realization.

Qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

α and β are complex numbers,

i.e. $\alpha, \beta \in \mathbb{C}$

such that,

$$|\alpha|^2 + |\beta|^2 = 1$$

Therefore, $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$

Qubit's state is a unit vector in a two-dimensional complex vector space.

Interpretation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{array}{c} 0 \\ 1 \end{array}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Measure $|\psi\rangle$:

0 measured with probability $|\alpha|^2$

1 " " " $|\beta|^2$

Special Cases

Classical: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, probability of measuring zero is 1

" " " one is 0

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, " " " zero is 0

" " " one is 1

non-classical superposition:

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, " " " zero is $\frac{1}{2}$

" " " one is $\frac{1}{2}$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$, " " " zero is $\frac{1}{2}$

" " " one is $\frac{1}{2}$

where $i = \sqrt{-1}$.

Common notation: $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ denoted by "$|+\rangle$"

Possible Physical Realizations of Qubits

Two polarizations of photon

Nuclear spin alignment in uniform magnetic field

Two states of electron orbiting a single atom.

Atomic Qubit

electron can exist in ground state : $|0\rangle$

or

excited state : $|1\rangle$

Move electron to/from $|0\rangle$ to $|1\rangle$ by,

shining light on atom, with energy, e,

and for certain time, t.

$$|0\rangle \xrightarrow[e,\ t/2]{} |+\rangle \xrightarrow[e,\ t/2]{} |1\rangle$$

Re-write Qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Observe:

$$|\alpha|^2 + |\beta|^2 = \cos^2\left(\frac{\theta}{2}\right) + |e^{i\varphi}|^2 \sin^2\frac{\theta}{2} = 1$$

θ and φ define a point on a unit three-dimensional sphere

– Bloch Sphere

Block Sphere

No simple generalization to multiple qubits

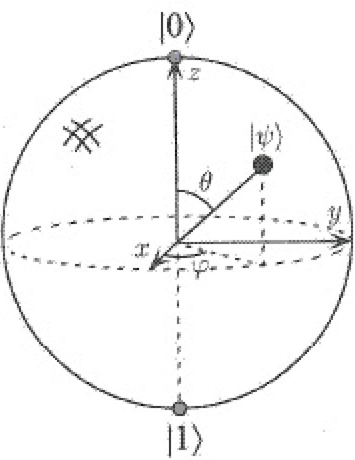


Figure 1.3. Bloch sphere representation of a qubit.

How much information represented by a qubit?

Infinite number of points on a sphere

but ...

qubit measurement only gives 0 or 1.

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \begin{array}{c}\text{measurement} \\ \text{collapse}\end{array} \begin{array}{l} \xrightarrow{0} |0\rangle, \ pr = |\alpha|^2 \\ \xrightarrow{1} |1\rangle, \ pr = |\beta|^2 \end{array}$$

Why collapse? ..... no-one knows

Measure (qubit) $\longrightarrow$ one (classical) bit

Information?

$\alpha|0\rangle + \beta|1\rangle$: (α, β) is quantum information

measure $\left(\alpha|0\rangle + \beta|1\rangle\right) \longrightarrow$ one bit of classical information

. . . . quantum information hidden . . .

Multiple Qubits

Hilbert space is a big place

— Carlton Caves

Bergen is a wet place

— Matthew Parker

Two Qubits

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$= \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix},$$

where $\alpha_{ij} \in \mathbb{C}$, $|\psi\rangle \in (\mathbb{C}^2)^2$,

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

Measurement

$$\text{measure}(|\psi\rangle) \xrightarrow{00} |00\rangle \quad , \quad Pr = |\alpha_{00}|^2$$

$$\text{measure}(|\psi\rangle) \xrightarrow{01} |01\rangle \quad , \quad Pr = |\alpha_{01}|^2$$

$$\text{measure}(|\psi\rangle) \xrightarrow{10} |10\rangle \quad , \quad Pr = |\alpha_{10}|^2$$

$$\text{measure}(|\psi\rangle) \xrightarrow{11} |11\rangle \quad , \quad Pr = |\alpha_{11}|^2$$

Measure Subset of Qubits

measure first qubit $\left(|\psi\rangle\right)$

$$\xrightarrow{0} \quad \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}, \quad pr = |\alpha_{00}|^2 + |\alpha_{01}|^2$$

re-normalized

$$\xrightarrow{1} \quad \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}, \quad pr = |\alpha_{10}|^2 + |\alpha_{11}|^2$$

Bell State / EPR Pair

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\text{measure first qubit}\left(|\psi\rangle\right) \longrightarrow \begin{cases} |00\rangle, & pr = \frac{1}{2} \\ |11\rangle, & pr = \frac{1}{2} \end{cases}$$

$$\text{measure second qubit}\left(\text{measure first qubit}\left(|\psi\rangle\right)\right) \begin{cases} \xrightarrow{0} |00\rangle \xrightarrow{0} |00\rangle, & pr = \frac{1}{2} \\ \xrightarrow{1} |11\rangle \xrightarrow{1} |11\rangle, & pr = \frac{1}{2} \end{cases}$$

first qubit measurement = second qubit measurement

.... measurement outcomes correlated.

Coding-Theoretic Interpretation

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\text{measure}(|\psi\rangle) \xrightarrow{00} |00\rangle, \quad pr = \tfrac{1}{2}$$

$$\text{measure}(|\psi\rangle) \xrightarrow{11} |11\rangle, \quad pr = \tfrac{1}{2}$$

Binary error-correcting code, $C = \{00, 11\} = [2,1,2]$

$pr(00) = \frac{1}{2}$, $pr(11) = \frac{1}{2}$

Correlated Measurements for $|\psi\rangle$ $\Longleftrightarrow$ High distance between codewords of C.

Einstein, Podolsky, Rosen (EPR) examined

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Bell showed,

measurement correlations in Bell state stronger than possible between classical systems!

n qubits

$$\alpha_{00\ldots0}|00\ldots0\rangle + \alpha_{00\ldots1}|00\ldots1\rangle + \ldots\ldots + \alpha_{11\ldots1}|11\ldots1\rangle$$

$\ldots 2^n$ complex numbers

e.g. $n = 500$ photons

$\Rightarrow$ Nature stores a vector of 2^{500} complex numbers.

Quantum Computation

Classical NOT gate:

$$f : \mathbb{Z}_2 \to \mathbb{Z}_2 \quad : \quad \begin{array}{l} 0 \to 1 \\ 1 \to 0 \end{array}$$

Quantum NOT gate:

$$f : \mathbb{C}^2 \to \mathbb{C}^2 \ : \ \alpha|0\rangle + \beta|1\rangle \to \beta|0\rangle + \alpha|1\rangle$$

Let $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, "Bit-Flip"

then,

$$f : \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \to X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Matrices must be Unitary – that's all

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = U \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

quantum mechanics requires,

$$|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2 = 1$$

$\Rightarrow$ U must be unitary

i.e. $UU^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

"$\dagger$" means transpose matrix then conjugate complex elements

Examples

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = XX^{\dagger} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \checkmark$$

$$\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = I, \qquad \checkmark$$

$$\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = I, \qquad \checkmark$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq I \qquad \times$$

Check

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha + i\beta \\ \alpha - i\beta \end{bmatrix}$$

Verify,

$$|\alpha|^2 + |\beta|^2 = |\alpha + i\beta|^2 + |\alpha - i\beta|^2$$

.... Parseval's Theorem

.... conservation of energy.

Other Gates

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{“phase-flip”}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \text{“Hadamard gate”}$$

Action of Hadamard Gate on Bloch Sphere

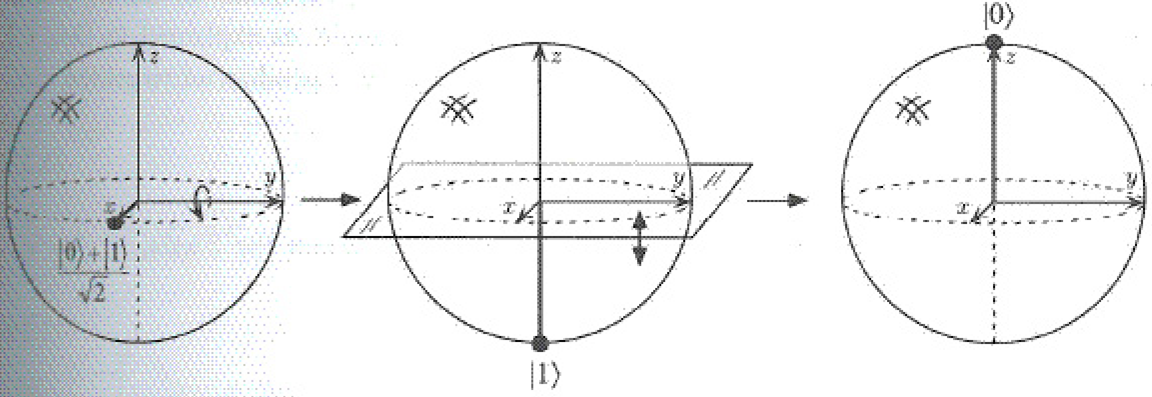


Figure 1.4. Visualization of the Hadamard gate on the Bloch sphere, acting on the input state $(|0\rangle + |1\rangle)/\sqrt{2}$.

Decomposition of Single Qubit Operation

$$U = e^{i\alpha} \overset{\text{rotation}}{\begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix}} \overset{\text{rotation}}{\begin{bmatrix} \cos\gamma/2 & -\sin\gamma/2 \\ \sin\gamma/2 & \cos\gamma/2 \end{bmatrix}} \overset{\text{rotation}}{\begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix}}$$

for arbitrary $\alpha, \beta, \gamma, \delta \in R$

Alternative Decomposition

$$U = \frac{1}{2}\Delta \begin{bmatrix} 1 & a \\ 1 & -a \end{bmatrix} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}$$

where,

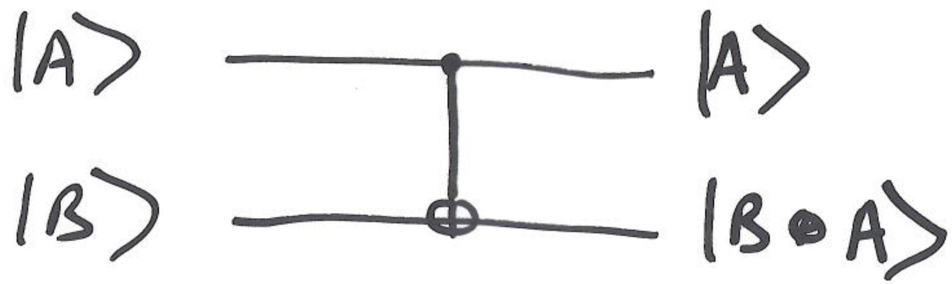
$$a = e^{i\alpha}, \quad b = e^{i\beta},$$

and $\Delta = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} \right\}$

such that $\Delta\Delta^\dagger = I$.

Multi-Qubit Gates

Controlled-NOT



$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Classical Universitality

Universal set of gates:

any Boolean function can be computed using a universal set of gates.

NAND - Universal ✓

$\{XOR, NOT\}$ - not universal ✗

CNOT Gate

$$|\psi\rangle' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{01} \end{bmatrix}$$

$|00\rangle \rightarrow |00\rangle$, $|01\rangle \rightarrow |01\rangle$, $|10\rangle \rightarrow |11\rangle$, $|11\rangle \rightarrow |10\rangle$

in general,

$$|A, B\rangle \longrightarrow |A, B \oplus A\rangle$$

Quantum Operations Must be Reversible

(apart from measurement)

$$UU^\dagger = I, \qquad \text{i.e. } U^\dagger = U^{-1}.$$

$$|\psi'\rangle = U|\psi\rangle \iff |\psi\rangle = U^\dagger|\psi'\rangle$$

converting classical logic to a quantum setting requires that all gates are made reversible.

Quantum Universality

Any multi-qubit logic gate may be composed from CNOT and single qubit gates.

Bases for Representation and Measurement of Quantum State

Computational Basis:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

another possible basis uses,

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$$

$$|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

$$\Rightarrow |\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

Observe: $\frac{1}{\sqrt{2}}\begin{pmatrix}\alpha+\beta \\ \alpha-\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

Every unitary defines a basis of representation.

Measurement Bases

measure wrt computational basis $(|\psi\rangle)$

$$(|\psi\rangle) \rightarrow |0\rangle, \quad pr = |\alpha|^2$$

$$(|\psi\rangle) \rightarrow |1\rangle, \quad pr = |\beta|^2$$

measure wrt $(|+\rangle, |-\rangle)$ basis $(|\psi\rangle)$

$$(|\psi\rangle) \rightarrow |+\rangle, \quad pr = \frac{|\alpha+\beta|^2}{2}$$

$$(|\psi\rangle) \rightarrow |-\rangle, \quad pr = \left|\frac{\alpha-\beta}{2}\right|^2$$

Quantum Circuits - SWAP

$$\begin{pmatrix} \alpha'_{00} \\ \alpha'_{01} \\ \alpha'_{10} \\ \alpha'_{11} \end{pmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

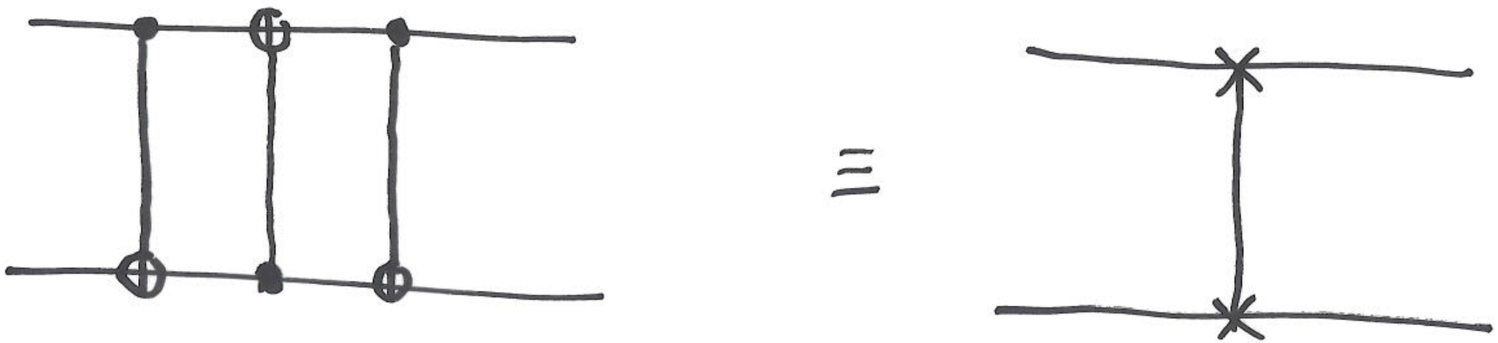
$$= \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \end{bmatrix}$$

$$|a, b\rangle \to |a, a \oplus b\rangle$$

$$\to |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle$$

$$\to |b, (a \oplus b) \oplus b\rangle = |b, a\rangle$$

Circuit Swapping Two Qubits



$$|a, b\rangle \longrightarrow |b, a\rangle$$

(computational basis assumed)

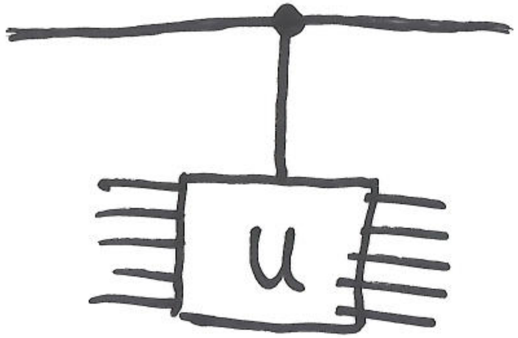
Controlled-U Gate

- generalises controlled-NOT

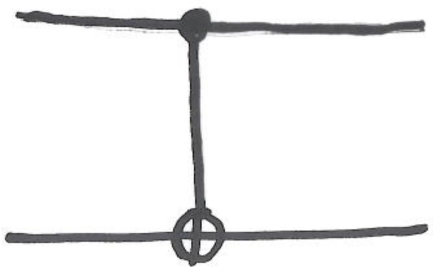
two qubits $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{U}} \\ 0 & 0 & & \end{bmatrix}$, three qubits $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & U & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{bmatrix}$

. . . . etc . . .

Controlled - Gates



Controlled-U gate



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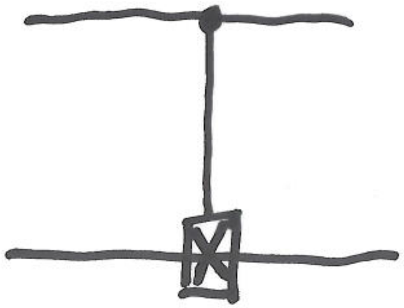
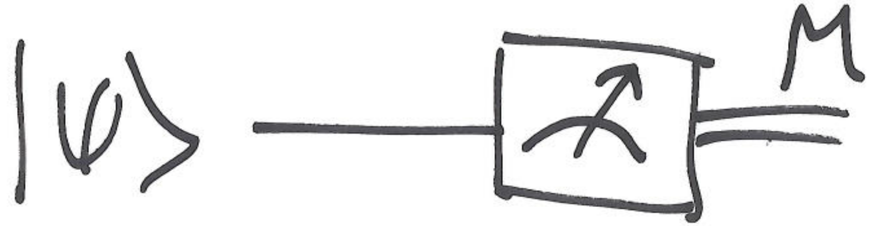


Controlled-NOT

Measurement?



"—" : deterministic

"═" : probabilistic

Qubit Copying? – No-Cloning theorem

Consider,

$$\text{CNOT}\left(\left[a|0\rangle + b|1\rangle\right]|0\rangle\right) = \text{CNOT}\left(a|00\rangle + b|10\rangle\right)$$

$$= a|00\rangle + b|11\rangle$$

i.e.

$$\begin{bmatrix} a \\ 0 \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}$$

The above is "copying" only if a and/or b is 0.

Compare with,

$$|\psi\rangle|\psi\rangle = a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle$$

Bell States (EPR states/pairs)

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

In general,

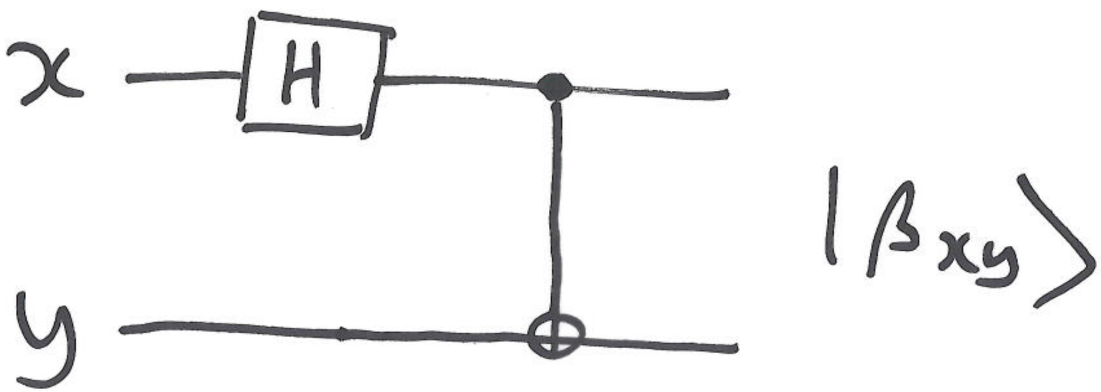
$$|\beta_{xy}\rangle = \frac{|0,y\rangle + (-1)^x|1,\bar{y}\rangle}{2}$$

i.e.

$$|\beta_{xy}\rangle = U(I\otimes H)|x,y\rangle$$

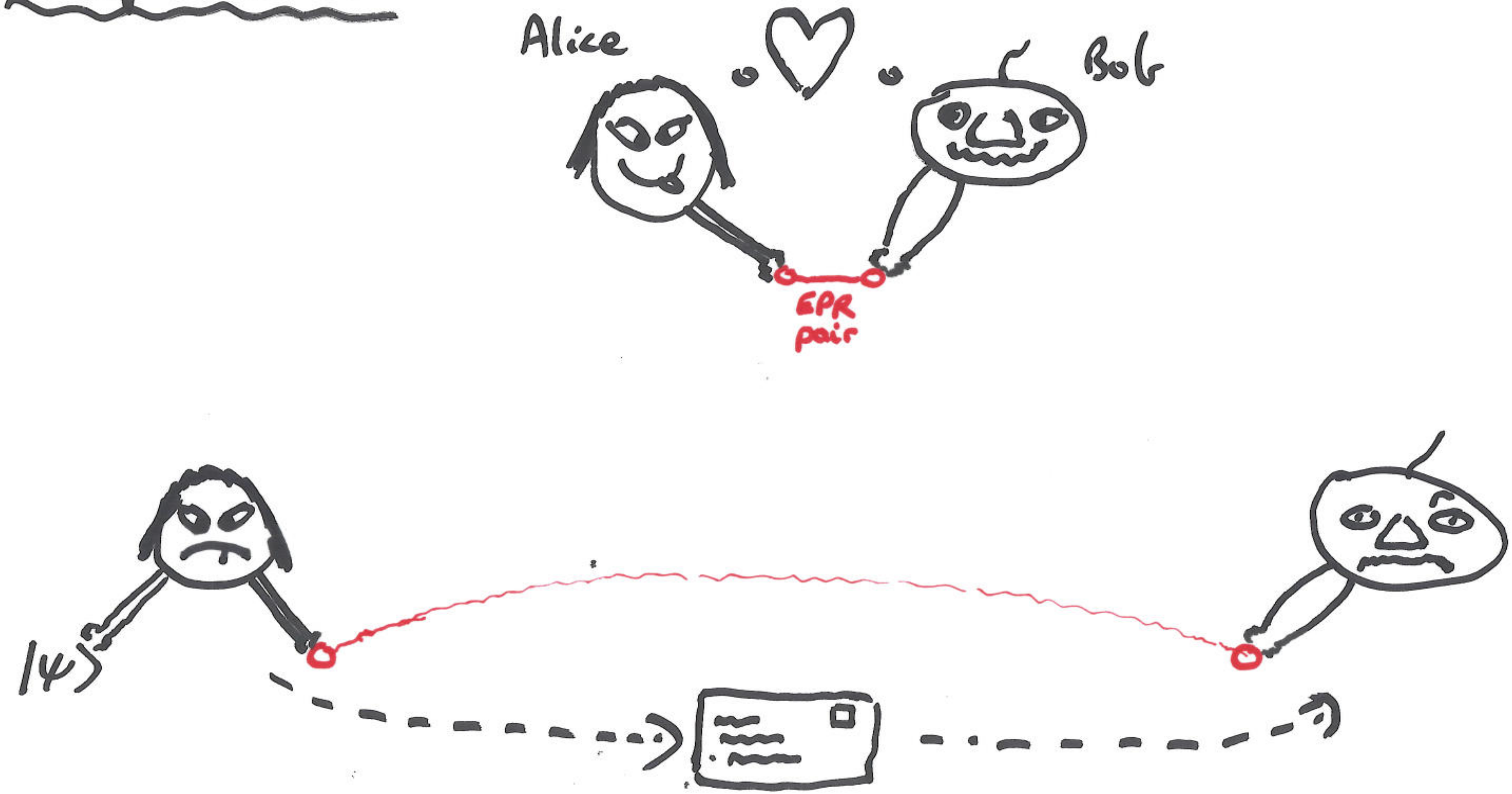
where $U = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$, $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $H = \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

Creating Bell States



	In		Out
	x	y	$\lvert\beta_{xy}\rangle$
$\lvert 00\rangle$:	0	0	$(\lvert 00\rangle + \lvert 11\rangle)/\sqrt{2}$
$\lvert 01\rangle$:	0	1	$(\lvert 01\rangle + \lvert 10\rangle)/\sqrt{2}$
$\lvert 10\rangle$:	1	0	$(\lvert 00\rangle - \lvert 11\rangle)/\sqrt{2}$
$\lvert 11\rangle$:	1	1	$(\lvert 01\rangle - \lvert 10\rangle)/\sqrt{2}$

Teleportation



Challenge for Alice:

Send qubit, $|\psi\rangle$, to Bob.

Initially,

Alice has one qubit
Bob has one qubit $\Bigg\}$ EPR pair, $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

also Alice has one unknown qubit state,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Therefore system state is,

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle\left(|00\rangle + |11\rangle\right) + \beta|1\rangle\left(|00\rangle + |11\rangle\right)\right]$$

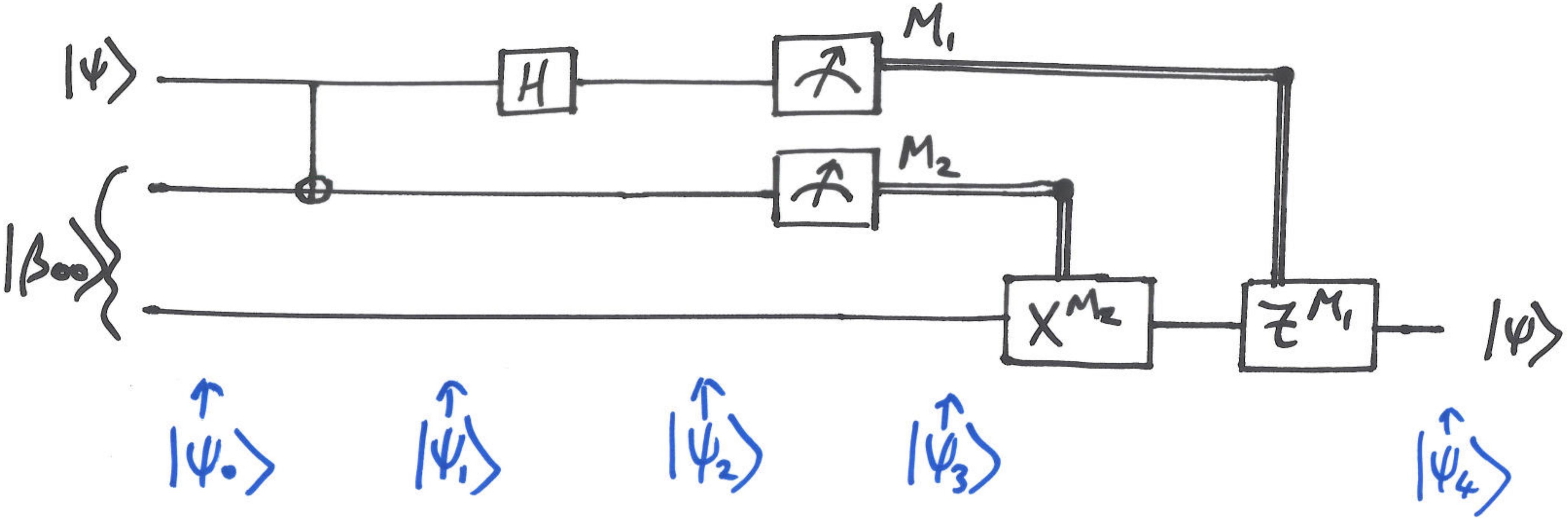
$$
|\psi_2\rangle = \frac{1}{2}\begin{bmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix} \leftarrow |\psi_0\rangle
$$

i.e.

$$
\begin{aligned}
|\psi_2\rangle &= H^{(0)}\, CNOT^{(01)} |\psi_0\rangle \\
&= (H \otimes I \otimes I)(U \otimes I)|\psi\rangle,
\end{aligned}
$$

where $U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Quantum Circuit for Teleporting a Qubit



top two lines are Alice's system

bottom line is Bob's system

Alice now measures $|\psi\rangle$ and her EPR qubit:

$$\text{measure}_{\text{ALICE}}\left(|\psi_2\rangle\right) : \frac{1}{2}\begin{bmatrix} \alpha \\ \alpha \\ \beta \\ -\beta \\ \beta \\ -\beta \\ \alpha \\ \alpha \end{bmatrix}$$

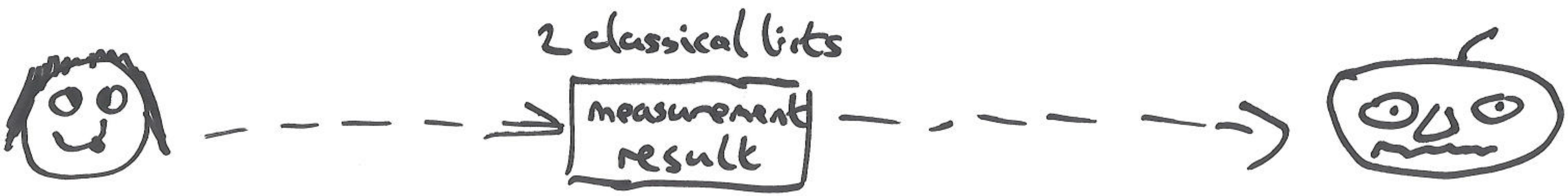
Bob

$\alpha|0\rangle + \beta|1\rangle$

$\alpha|0\rangle - \beta|1\rangle$

$\beta|0\rangle + \alpha|1\rangle$

$-\beta|0\rangle + \alpha|1\rangle$



Communicating result is limited by speed of light.

Bob Corrects His State According to Alice's Measurement

Alice measures		Bob performs
00	$\longrightarrow$	$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$
10	$\longrightarrow$	$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}\alpha\\ -\beta\end{pmatrix} = Z\left(\alpha\lvert 0\rangle - \beta\lvert 1\rangle\right)$
01	$\longrightarrow$	$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = X\left(\beta\lvert 0\rangle + \alpha\lvert 1\rangle\right)$
11	$\longrightarrow$	$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}-\beta\\ \alpha\end{pmatrix} = ZX\left(-\beta\lvert 0\rangle + \alpha\lvert 1\rangle\right)$

Teleportation Summary

1. Alice and Bob share EPR pair
2. Alice also has unknown qubit state, $|\psi\rangle$
3. Alice performs controlled-NOT on her EPR qubit and $|\psi\rangle$
4. Alice performs H on ($|\psi\rangle$ after CNOT).
5. Alice measures her unknown qubit and her EPR qubit.
6. Alice sends her measurement results to Bob by classical means.
7. Bob corrects his EPR qubit according to the message sent from Alice to obtain $|\psi\rangle$.

Important:

Without classical communication, no information is transferred at all.

one shared EPR pair + two classical bits of communication $\geq$ one qubit of communication.