Schmidt Decomposition

$|\psi\rangle$ is a pure state of a composite system, AB.

Then,

$\exists$ orthonormal states, $|i_A\rangle$, $|i_B\rangle$, s.t.

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

where $\lambda_i \geq 0$, real, and $\sum_i \lambda_i^2 = 1$.

The λ_i are Schmidt coefficients.

Let $\rho = \langle \psi | \psi \rangle$ for system AB.

Then,

$$\rho^A = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^B = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

... so identical eigenvalues for ρ^A and ρ^B.

e.g.

Let $|\psi\rangle = (|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$

$\Rightarrow$

$\mathrm{tr}((\rho^{A})^2) = \mathrm{tr}((\rho^{B})^2) = 7/9$

Proof for A,B of Same Dimension

Let $\{|j\rangle\}$, $\{|k\rangle\}$ be orthonormal bases.

Let

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle |k\rangle.$$

By SVD,

matrix $a = udv$, u, v unitary, d diagonal

$$\Rightarrow \quad |\psi\rangle = \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle |k\rangle.$$

Let $|i_A\rangle = \sum_j u_{ji} |j\rangle$, $|i_B\rangle = \sum_k v_{ik} |k\rangle$, $\lambda_i = d_{ii}$

Then,

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle.$$

Proof of Schmidt Decomposition of $|\psi\rangle$ over AB may be extended to A and B of different dimensions.

Not so simple for 3 component system:

$\exists$ pure states $|\psi\rangle \in ABC$,

that cannot be written as,

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle.$$

Schmidt Bases / Schmidt Number

For $$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle,$$

$\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are called

Schmidt Bases

The number of non-zero λ_i is called the

Schmidt Number.

Schmidt Number Preserved by Unitaries

If

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$$

and

$$U|\psi\rangle = \sum_i \lambda_i \left(U|i_A\rangle\right)|i_B\rangle,$$

then

$$\mathrm{Sch}(|\psi\rangle) = \mathrm{Sch}(U|\psi\rangle).$$

Schmidt Number Roughly Quantifies Entanglement

e.g.

If $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

then

$$\mathrm{Sch}(|\psi\rangle) = 1,$$

as $|\psi\rangle = \sum_i \lambda_i |\psi_A\rangle \otimes |\psi_B\rangle.$

But high entanglement

$\Rightarrow$ high Schmidt Number.

Purification

Consider ρ^A for system A.

Introduce another system R.

Define pure state,

$$|AR\rangle$$

such that

$$\rho^A = \text{tr}_R(|AR\rangle\langle AR|).$$

A is "purified" to AR.

Can purification always be done?

Yes.

Proof:

Let $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$.

Introduce system R with orthonormal bases $\{i^R\}$.

Define
$$|AR\rangle = \sum_i \sqrt{p_i}\, |i^A\rangle |i^R\rangle.$$

Then
$$\begin{aligned}
tr_R(|AR\rangle\langle AR|) &= \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A|\, tr(|i^R\rangle\langle j^R|) \\
&= \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A|\, \delta_{ij} \\
&= \sum_i p_i |i^A\rangle\langle i^A| = \rho^A.
\end{aligned}$$

Find Schmidt Decompositions of

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$\frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}?$$

Let
$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$
Introduce system R with orthonormal basis $\{|i\rangle\}$.
Then,
$$\sum_i \sqrt{p_i}\, |\psi_i\rangle |i\rangle \text{ is a purification of } \rho.$$
Measure R in $\{|i\rangle\}$ basis to obtain i, with $pr = p_i$.
System A collapses to state $|\psi_i\rangle$.

.... there always exists an orthonormal basis $\{|i\rangle\}$ in which R can be measured such that the post-measurement state for system A is $|\psi_i\rangle$ with $pr = p_i$.

EPR and the Bell Inequality

Difference between

classical and quantum?

Conventionally...

we assume an object has existence
independent of observation.

i.e measurement just acts to reveal

.. but..

according to quantum mechanics (QM)..

... an unobserved particle does not possess physical properties that exist independent of observation.

QM:

Properties arise as a consequence of measurements on the system.

... e.g.

QM provides a set of rules specifying, given the state, the probabilities for the measurement outcomes when the observable is measured.

Famous "EPR" paper by,

Einstein, Podolsky and Rosen..

.. aim ...

to demonstrate that QM is not a complete theory of Nature.

EPR's argument

Any element of reality must be represented in any complete physical theory.

... then identify elements of reality not included in QM.

They introduced a "sufficient condition" for a physical property to be an element of reality - namely:

It is possible to predict with certainty the value that property will have immediately before measurement.

e.g. Let Alice and Bob share

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Alice measures observable $\vec{v}\cdot\vec{\sigma}$
and receives result $+1$.

$\Rightarrow$ she predicts with certainty that Bob will measure -1 if he also measures $\vec{v}\cdot\vec{\sigma}$.

Similarly,
if Alice measures -1 then Bob will measure $+1$.

$\Rightarrow$ physical property is element of reality ..

... but QM as we present it is just a set of rules...

EPR hoped systems could be ascribed properties, independent of measurements.

.... but ... 30 years later ...

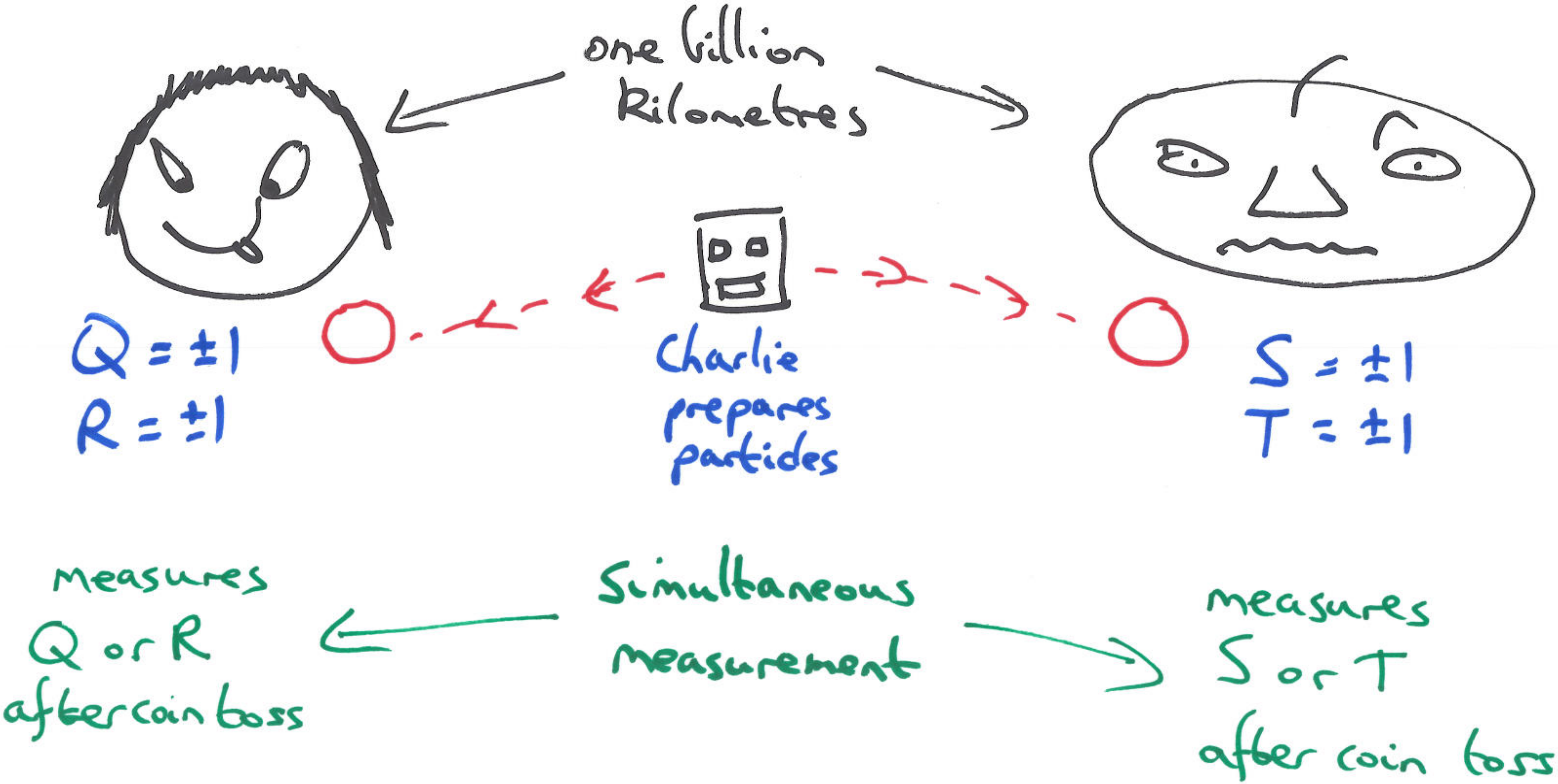
experimental test (i.e. Nature)

invalidated EPR's view.

Bell's Inequality

"Common-sense" analysis inconsistent with QM analysis.

... so ask Nature (via experiment) to decide.


one billion kilometres
Q = ±1
R = ±1
Charlie prepares particles
S = ±1
T = ±1
measures Q or R after coin toss
Simultaneous measurement
measures S or T after coin toss

Assume a Classical Experiment

Measurements are of physical properties P_Q and P_R, with values Q and R, respectively,
(similarly for Bob).

Assume Alice and Bob measure

at the same time

Observe,

$$QS + RS + RT - QT = (Q+R)S + (R-Q)T$$
$$= \pm 2.$$

Let $p(q,r,s,t)$ be the probability that, before measurement,

$$Q=q,\ R=r,\ S=s,\ T=t.$$

Then the expected value,

$$E(QS+RS+RT-QT) = \sum_{q,r,s,t} p(q,r,s,t)(qs+rs+rt-qt)$$
$$\leq \sum_{q,r,s,t} p(q,r,s,t) \times 2$$
$$= 2.$$

Also,

$$E(QS + RS + RT - QT)$$

$$= \sum_{q,r,s,t} p(q,r,s,t)\,qs + \sum_{q,r,s,t} p(q,r,s,t)\,rs$$

$$+ \sum_{q,r,s,t} p(q,r,s,t)\,rt + \sum_{q,r,s,t} p(q,r,s,t)\,qt$$

$$= E(QS) + E(RS) + E(RT) - E(QT)$$

$\Rightarrow$

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2.$$

Bell's Inequality

$$E(QS)+E(RS)+E(RT)-E(QT)\leq 2$$

Bell's Inequality
(CHSH inequality).

Repeated experiments determine validity or otherwise of Bell's inequality

... but ...

now assume a QM experiment.

Charlie prepares

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$Q = Z_1$

$R = X_1$

$$S = \frac{-Z_2 - X_2}{\sqrt{2}} \qquad T = \frac{Z_2 - X_2}{\sqrt{2}}$$

$\Rightarrow \langle QS\rangle = \frac{1}{\sqrt{2}}$; $\langle RS\rangle = \frac{1}{\sqrt{2}}$; $\langle RT\rangle = \frac{1}{\sqrt{2}}$; $\langle QT\rangle = \frac{-1}{\sqrt{2}}$

$\Rightarrow \langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}$.

??? Inconsistency ???

Bell's Inequality

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2.$$ Classical.

... but...

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$ Quantum.

Which is correct? Classical or quantum?

.... experiments say quantum.

.... So classical assumptions must be wrong...

(1) Assumption that physical properties have definite values Q, R, S, T, independent of observation.

— Realism.

(2) Assumption that Alice's measurement doesn't influence Bob's measurement.

— Locality.

.. So experiments based on Bell's Inequality invalidate Local Realism.

The world is not locally realistic.

... substantial experimental evidence.

.... what makes the difference ...?

... entanglement ...?

Observe

$$f(\theta\vec{n}\cdot\vec{\sigma}) = \frac{f(\theta)+f(-\theta)}{2} I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\cdot\vec{\sigma}$$

Properties of Schmidt Number

$$\mathrm{Sch}(|\psi\rangle) = \mathrm{rank}\left(\rho_A \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|)\right).$$

Let $|\psi\rangle = \sum_j |\alpha_j\rangle|\beta_j\rangle$, where $|\alpha_j\rangle$ and $|\beta_j\rangle$ are unnormalised. Then,

$$\mathrm{Sch}(|\psi\rangle) \leq \#\text{ terms in decomposition}.$$

Let $|\psi\rangle = \alpha|\varphi\rangle + \beta|\gamma\rangle$. Then,

$$\mathrm{Sch}(|\psi\rangle) \geq |\mathrm{Sch}(|\varphi\rangle) - \mathrm{Sch}(|\gamma\rangle)|.$$

Tsirelson's Inequality

Let $Q = \vec{q}\cdot\vec{\sigma}$, $R = \vec{r}\cdot\vec{\sigma}$, $S = \vec{s}\cdot\vec{\sigma}$, $T = \vec{t}\cdot\vec{\sigma}$:

Then,

$$(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 = 4I + [Q,R]\otimes[S,T]$$

$$\Rightarrow$$

$$\langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle \leq 2\sqrt{2}$$

... So Bell's Inequality is maximum possible.