A multi-dimensional approach to the construction and enumeration of Golay complementary sequences

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Abstract

We argue that a Golay complementary sequence is naturally viewed as a projection of a multi-dimensional Golay array. We present a three-stage process for constructing and enumerating Golay array and sequence pairs:

1. construct suitable Golay array pairs from lower-dimensional Golay array pairs;
2. apply transformations to these Golay array pairs to generate a larger set of Golay array pairs; and
3. take projections of the resulting Golay array pairs to lower dimensions.

This process greatly simplifies previous approaches, by separating the construction of Golay arrays from the enumeration of all possible projections of these arrays to lower dimensions.

We use this process to construct and enumerate all 2^h-phase Golay sequences of length 2^m obtainable under any known method, including all 4-phase Golay sequences obtainable from the length 16 examples given in 2005 by Li and Chu [12].

1 Introduction

Golay complementary sequence pairs were introduced by Golay [8] in 1951. They have been applied in diverse areas of digital information processing, including multislit spectrometry [8], optical time domain reflectometry [14], and power control for multicarrier wireless transmission [4].

In 1999 Davis and Jedwab [4] gave an explicit algebraic normal form for a set of 2^h-phase Golay sequence pairs of length 2^m, demonstrating an unexpected connection with Reed-Muller codes. For the next six years this was sufficient to describe all known 2^h-phase Golay sequence pairs of length 2^m. But in 2005 Li and Chu [12] discovered 1024 "non-standard" (not having this algebraic normal form) 4-phase Golay sequences of length 16, by computer search. Although the origin of these sequences was explained shortly afterwards by means of "cross-over" 4-phase Golay sequence pairs of length 8 [6], the question as to which other non-standard Golay sequences of length 2^m could be derived from the cross-over pairs using known constructions proved more difficult. In 2006 the present authors explicitly derived new infinite families of such Golay sequences [7], but were unable to determine explicitly the full generalisation of a key example [7, Example 6]. Furthermore,

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while the framework of constructions for Golay sequences presented in [7] simplifies many of the previous approaches it is still rather cumbersome.

In this paper we demonstrate the power of the recently-introduced view [11] of a Golay sequence pair as the "projection" of a multi-dimensional Golay array pair. While the paper [11] focussed on determining whether at least one Golay array pair of a given size exists, particularly for the binary case, this paper deals with the systematic construction and enumeration of a large set of distinct Golay array pairs of a given size. Although Golay arrays have been previously studied by Lüke [13] and especially Dymond [5], and shown to be of use in coded imaging [15], it appears that for the most part they have been ignored or else regarded as merely another generalisation of a familiar combinatorial object.

We propose on the contrary that a Golay array, constructed in as many dimensions as possible, is in fact the fundamental object of study, and that lower-dimensional Golay arrays (in particular Golay sequences) should be regarded as derived objects! In particular, there is no real distinction to be made between "interleaving" and "concatenation" constructions for Golay sequences (as distinguished in [9, General Properties (9) and (10)] and [7, Lemmas 3 and 4], for example): the two forms are just different projections of the same higher-dimensional construction.

We divide the construction of Golay sequence (and array) pairs into three stages, removing a great deal of the complication of previous approaches. The first stage is to construct suitable Golay array pairs from lower-dimensional Golay array (or sequence) pairs explicitly via repeated use of a generalisation of Dymond's construction [5]. The second stage is to enlarge the constructed set of Golay array pairs by means of a simple transformation. The third stage is to take all possible projections of the resulting Golay array pairs to lower dimensions, including to sequences.

We will illustrate the three-stage process by showing that all "standard" (as constructed in [4]) H-phase Golay sequence pairs of length 2^m can be derived from a single m-dimensional Golay array pair that is constructed from Golay sequence pairs of length 1 in the first stage. We will then determine and count the Golay array pairs with 2^m elements arising from the same m-dimensional array under projection to an intermediate number of dimensions between 1 and m. Finally we will apply the three-stage process once more, using a combination of Golay sequence pairs of length 1 and cross-over 4-phase Golay sequence pairs of length 8 to construct the Golay array pairs of the first stage. The projections to sequences give the desired full generalisation of [7, Example 6], and include as special cases all previously known 4-phase Golay sequence pairs with 2^m elements.

The rest of the paper is organised as follows. Section 2 contains definitions and notation. Section 3 introduces projection mappings for reducing the dimension of an array, and gives a graphical means of tracking the effect of successive projections. Section 4 describes two methods for constructing a Golay array pair from two other Golay array (or sequence) pairs, and gives an explicit form for the result of applying the second method recursively. Section 5 defines a transformation that generates a set of Golay array pairs from a single Golay array pair. Section 6 uses trivial input Golay array pairs in the explicit construction form, followed by the transformation, and finally applies successive projection mappings in order to construct Golay array pairs in all dimensions from 1 to m. Section 7 repeats this process but uses an arbitrary combination of trivial and cross-over 4-phase Golay sequence pairs as inputs to the explicit form. This gives a concise construction and enumeration of all 4-phase Golay array pairs with 2^m elements obtainable under any known method. Section 8 contains a summary of results.

2 Definitions and notation

We define an *array* of size $s_1 \times \cdots \times s_r$ to be an r-dimensional matrix $\mathcal{A} = (A[i_1, \ldots, i_r])$ of complex-valued entries, where $i_1, \ldots, i_r$ are integer, for which

$$A[i_1, \ldots, i_r] = 0 \text{ if, for any } k \in \{1, 2, \ldots, r\},\ i_k < 0 \text{ or } i_k \geq s_k.$$

In the case $r = 1$, $\mathcal{A} = (A[i_1])$ is a *length s_1 sequence.* Call the set of array elements

$$\{A[i_1, \ldots, i_r] \mid 0 \leq i_k < s_k \text{ for all } k\}$$

the *in-range entries* of $\mathcal{A}$. The *energy* of $\mathcal{A}$ is

$$\epsilon(\mathcal{A}) := \sum_{i_1} \cdots \sum_{i_r} |A[i_1, \ldots, i_r]|^2, \tag{1}$$

which equals the *volume* $s_1 \cdots s_r$ of $\mathcal{A}$ if $|A[i_1, \ldots, i_r]| = 1$ for all in-range entries of $\mathcal{A}$.

Usually the in-range entries of $\mathcal{A}$ are constrained to lie in a small finite set S called the array *alphabet.* Let ξ denote $\exp(2\pi\sqrt{-1}/H)$ (a primitive H-th root of unity) for some H, where H represents an even integer throughout the paper. If $S = \{1, \xi, \xi^2, \ldots, \xi^{H-1}\}$ then $\mathcal{A}$ is an *H-phase* array. Particular cases of interest are the *binary* case $H = 2$, for which $S = \{1, -1\}$, and the *quaternary* case $H = 4$, for which $S = \{1, \sqrt{-1}, -1, -\sqrt{-1}\}$. If $S = \mathbb{Z}_H$ then $\mathcal{A}$ is an *array over* $\mathbb{Z}_H$. The in-range entries of an $s_1 \times \cdots \times s_r$ H-phase array $\mathcal{A} = (A[i_1, \ldots, i_r])$ can be represented in the form

$$\xi^{a[i_1,\ldots,i_r]} := A[i_1, \ldots, i_r], \text{ where each } a[i_1, \ldots, i_r] \in \mathbb{Z}_H. \tag{2}$$

We say that the $s_1 \times \cdots \times s_r$ array $(a[i_1, \ldots, i_r])$ given by (2) is the *array over $\mathbb{Z}_H$ corresponding to $\mathcal{A}$.* (Here and elsewhere, in defining the elements of an array of a stated size, the definition implicitly applies only to the in-range entries.) We will consistently use lower-case letters for arrays over $\mathbb{Z}_H$ and upper-case letters for complex-valued arrays; the same letter (for example a and A) will indicate that the arrays correspond.

The *aperiodic autocorrelation function* of an $s_1 \times \cdots \times s_r$ complex-valued array $\mathcal{A} = (A[i_1, \ldots, i_r])$ is given by

$$C_{\mathcal{A}}(u_1, \ldots, u_r) := \sum_{i_1} \cdots \sum_{i_r} A[i_1, \ldots, i_r]\overline{A[i_1 + u_1, \ldots, i_r + u_r]} \text{ for integer } u_1, \ldots, u_r, \tag{3}$$

where bar represents complex conjugation. The aperiodic autocorrelation function of an array over $\mathbb{Z}_H$ is that of the corresponding H-phase array. An $s_1 \times \cdots \times s_r$ *Golay array pair* is a pair of $s_1 \times \cdots \times s_r$ arrays $\mathcal{A}$ and $\mathcal{B}$ for which

$$C_{\mathcal{A}}(u_1, \ldots, u_r) + C_{\mathcal{B}}(u_1, \ldots, u_r) = 0 \text{ for all } (u_1, \ldots, u_r) \neq (0, \ldots, 0).$$

An array $\mathcal{A}$ is called a *Golay array* if it forms a Golay array pair with some array $\mathcal{B}$. The *standard* Golay sequence pairs of length 2^m over $\mathbb{Z}_H$ are those given in Corollary 10.

The *generating function* corresponding to the complex-valued array $(A[i_1, \ldots, i_r])$ is the polynomial

$$A(y_1, \ldots, y_r) := \sum_{i_1} \cdots \sum_{i_r} A[i_1, \ldots, i_r] y_1^{i_1} \cdots y_r^{i_r} \text{ in indeterminates } (y_1, \ldots, y_r) \neq (0, \ldots, 0)$$

(where we distinguish the generating function from the complex-valued array by means of round brackets instead of square brackets). The generating function and energy of an array over $\mathbb{Z}_H$ are those of the corresponding H-phase array. Straightforward manipulation shows that

$$A(y_1, \ldots, y_r)\overline{A(y_1^{-1}, \ldots, y_r^{-1})} = \sum_{u_1} \cdots \sum_{u_r} \overline{C_{\mathcal{A}}(u_1, \ldots, u_r)} y_1^{u_1} \cdots y_r^{u_r},$$

from which it follows that $\mathcal{A} = (A[i_1, \dots, i_r])$ and $\mathcal{B} = (B[i_1, \dots, i_r])$ form an $s_1 \times \cdots \times s_r$ Golay array pair if and only if

$$A(y_1, \dots, y_r)\overline{A(y_1^{-1}, \dots, y_r^{-1})} + B(y_1, \dots, y_r)\overline{B(y_1^{-1}, \dots, y_r^{-1})} \text{ is constant for all } (y_1, \dots, y_r), \quad (4)$$

and if (4) holds then the value of the constant is $\epsilon(\mathcal{A}) + \epsilon(\mathcal{B})$, from (1) and (3).

Given a complex-valued $s_1 \times \cdots \times s_r$ array $\mathcal{A} = (A[i_1, \dots, i_r])$, define $\mathcal{A}^* = (A^*[i_1, \dots, i_r])$ to be the $s_1 \times \cdots \times s_r$ array given by

$$A^*[i_1, \dots, i_r] := \overline{A[s_1 - 1 - i_1, \dots, s_r - 1 - i_r]} \text{ for all } (i_1, \dots, i_r). \quad (5)$$

It is easy to show that the generating functions of $\mathcal{A}$ and $\mathcal{A}^*$ are related by

$$A^*(y_1, \dots, y_r) = y_1^{s_1-1} \cdots y_r^{s_r-1}\overline{A(y_1^{-1}, \dots, y_r^{-1})}, \quad (6)$$

and that their aperiodic autocorrelation functions are identical. The corresponding array $(a^*[i_1, \dots, i_r])$ over $\mathbb{Z}_H$ is given by

$$a^*[i_1, \dots, i_r] = -a[s_1 - 1 - i_1, \dots, s_r - 1 - i_r] \text{ for all } (i_1, \dots, i_r). \quad (7)$$

Since addition of a constant in $\mathbb{Z}_H$ to all elements of an array over $\mathbb{Z}_H$ does not change its aperiodic autocorrelation function, all arrays in the set

$$E(\mathcal{A}) := \{(a[i_1, \dots, i_r] + c) \mid c \in \mathbb{Z}_H\} \cup \{(a^*[i_1, \dots, i_r] + c) \mid c \in \mathbb{Z}_H\}$$

(which has H elements if $(a^*[i_1, \dots, i_r]) = (a[i_1, \dots, i_r] + c)$ for some $c \in \mathbb{Z}_H$, and $2H$ elements otherwise) have identical aperiodic autocorrelation function. Therefore, if $(\mathcal{A}, \mathcal{B})$ is an $s_1 \times \cdots \times s_r$ Golay array pair over $\mathbb{Z}_H$ then so is every element of $E(\mathcal{A}) \times E(\mathcal{B})$.

It is possible that two arrays $\mathcal{A}$, $\mathcal{A}'$ over $\mathbb{Z}_H$ of the same size have identical aperiodic autocorrelation function, even though $E(\mathcal{A}) \neq E(\mathcal{A}')$. In this case we say that the pair $(\mathcal{A}, \mathcal{A}')$ has the *shared autocorrelation property*. Suppose that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are Golay array pairs, where $E(\mathcal{A}) \neq E(\mathcal{A}')$ and $E(\mathcal{B}) \neq E(\mathcal{B}')$. If the pair $(\mathcal{A}, \mathcal{A}')$ has the shared autocorrelation property, then so does the pair $(\mathcal{B}, \mathcal{B}')$; and moreover $(\mathcal{A}, \mathcal{B}')$ and $(\mathcal{A}', \mathcal{B})$ both form Golay array pairs by a "cross-over" of their autocorrelation functions, as illustrated in Figure 1.

The only known examples of cross-over Golay sequence pairs of length 2^m over $\mathbb{Z}_{2^h}$ occur for $m = 3$ and $h = 2$. For example, the sequences $[0, 0, 0, 2, 0, 0, 2, 0]$ and $[0, 1, 1, 2, 0, 3, 3, 2]$ form a length 8 cross-over Golay pair over $\mathbb{Z}_4$. All 512 ordered cross-over Golay sequence pairs of length 8 over $\mathbb{Z}_4$ can be derived from this pair, as shown in Theorem 12. Moreover the existence of the 1024 non-standard length 16 Golay sequences over $\mathbb{Z}_4$ found by Li and Chu [12] can be explained in terms of these length 8 cross-over pairs [6]. The study of which other non-standard Golay sequence pairs of length 2^m can be constructed from the length 8 cross-over pairs under known constructions was begun in [6], continued in [7], and is completed in this paper.

3 Projection of arrays to lower dimensions

In this section we describe projection mappings that reduce the dimension of an array, and give a graph-theoretic means of tracking the effect of successive projections. The importance of projection mappings is that they preserve the Golay array property.

We firstly define a mapping $\psi_{1,2}$ from the set of $s_1 \times \cdots \times s_r$ arrays (where $r \geq 2$) to the set of $s_1 s_2 \times s_3 \times \cdots \times s_r$ arrays. Given an $s_1 \times \cdots \times s_r$ complex-valued array $\mathcal{A} = (A[i_1, \dots, i_r])$, the mapping $\psi_{1,2}(\mathcal{A})$ is the array $(B[i, i_3, \dots, i_r])$ given by

$$B[i_1 + s_1 i_2, i_3, \dots, i_r] := A[i_1, \dots, i_r] \text{ for all } (i_1, \dots, i_r). \quad (8)$$

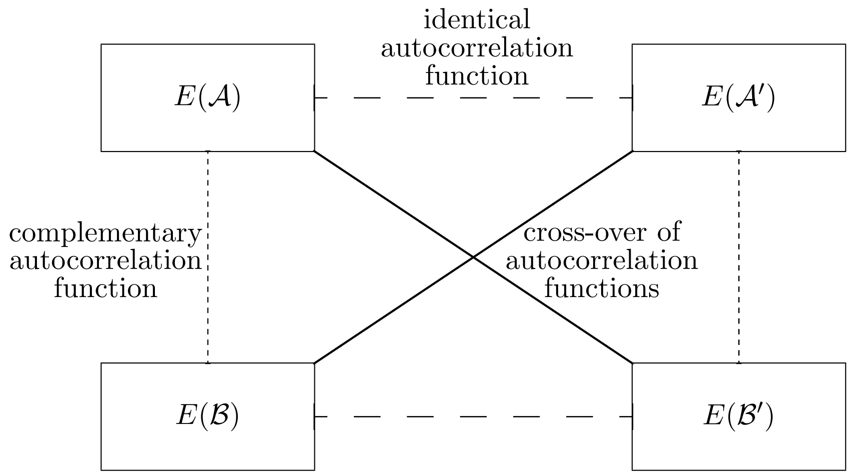

Figure 1: Cross-over of autocorrelation functions for Golay pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$, where $E(\mathcal{A}) \neq E(\mathcal{A}')$ and $E(\mathcal{B}) \neq E(\mathcal{B}')$

For distinct $j, k \in \{1, \ldots, r\}$, the array $\psi_{j,k}(\mathcal{A})$ is given similarly by removing the array argument i_j and replacing the array argument i_k by $i_j + s_j i_k$. For example, if $\mathcal{A} = (A[i_1, i_2, i_3, i_4])$ is an $8 \times 5 \times 7 \times 6$ array over $\mathbb{Z}_6$ then $\psi_{1,3}(\mathcal{A}) = (B[i_2, i, i_4])$ is the $5 \times 56 \times 6$ array over $\mathbb{Z}_6$ given by

$$B[i_2, i_1 + 8i_3, i_4] := A[i_1, i_2, i_3, i_4] \quad \text{for } 0 \leq i_1 < 8,\ 0 \leq i_2 < 5,\ 0 \leq i_3 < 7,\ 0 \leq i_4 < 6,$$

and $\psi_{4,2}(\mathcal{A}) = (C[i_1, i, i_3])$ is the $8 \times 30 \times 7$ array over $\mathbb{Z}_6$ given by

$$C[i_1, i_4 + 6i_2, i_3] := A[i_1, i_2, i_3, i_4] \quad \text{for } 0 \leq i_1 < 8,\ 0 \leq i_2 < 5,\ 0 \leq i_3 < 7,\ 0 \leq i_4 < 6.$$

We can interpret the action of $\psi_{j,k}$ on an array as replacing the $s_j \times s_k$ "slice" of the array formed from dimensions j and k by the sequence obtained when the elements of the slice are listed column by column. The definition of $\psi_{j,k}$ holds without modification for an array over $\mathbb{Z}_H$ because the mapping changes the locations but not the values of array elements. We call $\psi_{j,k}(\mathcal{A})$ a *projection* of the array $\mathcal{A}$ (from r to $r-1$ dimensions), and call $\psi_{j,k}$ a *projection mapping* that *joins* index j to index k.

The reason for our interest in projection mappings is that they preserve the Golay array property:

Theorem 1 (Jedwab and Parker [11, Theorem 11])**.** *For integer $r \geq 2$, suppose that $\mathcal{A}$ and $\mathcal{B}$ form an $s_1 \times \cdots \times s_r$ Golay array pair over an alphabet S. Then $\psi_{2,1}(\mathcal{A})$ and $\psi_{2,1}(\mathcal{B})$ form an $s_1 s_2 \times s_3 \times \cdots \times s_r$ Golay array pair over S.*

The $8 \times 30 \times 7$ array $(C[i_1, i, i_3])$ described above can instead be represented as the $8 \times 7 \times 30$ array $(C'[i_1, i_3, i])$, where

$$C'[i_1, i_3, i] = C[i_1, i, i_3] \quad \text{for } 0 \leq i_1 < 8,\ 0 \leq i < 30,\ 0 \leq i_3 < 7,$$

by reordering dimensions. However we do not consider arrays obtained by reordering dimensions to be distinct: they are different formal representations of the same object. By reordering dimensions in Theorem 1 we see that the arrays $\psi_{j,k}(\mathcal{A})$ and $\psi_{j,k}(\mathcal{B})$ also form a Golay array pair, for any distinct $j, k \in \{1, \ldots, r\}$.

We wish to apply successive projection mappings to a given r-dimensional array. In order to keep track of their effect, we represent the array indices by vertices $1, \ldots, r$ of a directed graph and represent successive projection mappings by arcs between vertices (according to the original index labellings). For example, suppose that three successive projection mappings applied to a $2 \times 2 \times 2 \times 2$ array join index 4 to index 3, then the original index 1 to the original index 2, and finally the joined indices 4 and 3 to the joined indices 1 and 2. The corresponding graph on four vertices 1, 2, 3, and 4 has an arc from vertex 4 to vertex 3, an arc from vertex 1 to vertex 2, and an arc from vertex 3 (the final vertex of the path from vertex 4 to vertex 3) to vertex 1 (the initial vertex of the path from vertex 1 to vertex 2):

$$\overset{4}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \tag{9}$$

The same graph would be obtained by joining index 4 to 3, then joining these joined indices to index 1, and then joining these joined indices to index 2. Algebraically, this equivalence is a consequence of the equality of the array arguments $(i_4+2i_3)+4(i_1+2i_2)$ and $((i_4+2i_3)+4i_i)+8i_2$.

In general, the graph representing the successive application of j projection mappings to an r-dimensional array comprises r vertices and a set of disjoint directed paths, each representing a set of joined indices; the total length of all paths is j. Applying a further projection mapping joins the final vertex of the path representing a first set of joined indices to the initial vertex of the path representing a second set of joined indices. Proposition 2 shows that the projected array corresponding to such a graph does not depend on the order in which arcs are added (as already suggested with reference to (9)). In particular, the sequence obtained by applying $r-1$ successive projection mappings to a given r-dimensional array is completely described by a directed path of the form

$$\overset{\sigma(1)}{\bullet} \longrightarrow \overset{\sigma(2)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\sigma(r)}{\bullet}$$

for some permutation σ of $\{1, \ldots, r\}$.

Proposition 2. *Let $(A[i_1, \ldots, i_r])$ be an $s_1 \times \cdots \times s_r$ array over an alphabet S, let σ be a permutation of $\{1, \ldots, r\}$, and let j be an integer satisfying $0 \le j \le r-1$. Then the application to $(A[i_1, \ldots, i_r])$ of any j successive projection mappings whose corresponding graph is*

$$\overset{\sigma(1)}{\bullet} \longrightarrow \overset{\sigma(2)}{\bullet} \longrightarrow \quad \cdots \quad \longrightarrow \overset{\sigma(j+1)}{\bullet} \quad \overset{\sigma(j+2)}{\bullet} \quad \cdots \overset{\sigma(r)}{\bullet}$$

results in the projected array $(B[i, i_{\sigma(j+2)}, \ldots, i_{\sigma(r)}])$ of size $\left(\prod_{k=1}^{j+1} s_{\sigma(k)}\right) \times s_{\sigma(j+2)} \times \cdots \times s_{\sigma(r)}$ over S given by

$$B[v_1 i_{\sigma(1)} + \cdots + v_{j+1} i_{\sigma(j+1)}, i_{\sigma(j+2)}, \ldots, i_{\sigma(r)}] = A[i_1, \ldots, i_r] \quad \text{for all } (i_1, \ldots, i_r), \tag{10}$$

where $v_i := \prod_{k=1}^{i-1} s_{\sigma(k)}$ (and $v_1 := 1$).

Proof. The proof is by induction on j. The base case $j = 0$ holds trivially; assume all cases up to $j-1$ hold. The graph representing the first $j-1$ projection mappings of case j must have the form

$$\overset{\sigma(1)}{\bullet} \longrightarrow \overset{\sigma(2)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\sigma(\ell)}{\bullet} \qquad \overset{\sigma(\ell+1)}{\bullet} \longrightarrow \overset{\sigma(\ell+2)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\sigma(j+1)}{\bullet} \quad \overset{\sigma(j+2)}{\bullet} \quad \cdots \overset{\sigma(r)}{\bullet}$$

for some integer ℓ satisfying $1 \le \ell \le j$. Regard this graph as arising from the composition of two sequences of projection mappings, the first corresponding to the path $\sigma(1) \to \sigma(2) \to \cdots \to \sigma(\ell)$ and the second corresponding to the path $\sigma(\ell+1) \to \sigma(\ell+2) \to \cdots \to \sigma(j+1)$. By the inductive hypothesis, the projected array $(C[i, i', i_{\sigma(j+2)}, \ldots, i_{\sigma(r)}])$ of size $\left(\prod_{k=1}^{\ell} s_{\sigma(k)}\right) \times \left(\prod_{k=\ell+1}^{j+1} s_{\sigma(k)}\right) \times s_{\sigma(j+2)} \times \cdots \times s_{\sigma(r)}$ corresponding to the composition mapping then satisfies

$$C\big[\big(v_1 i_{\sigma(1)} + \cdots + v_\ell i_{\sigma(\ell)}\big), \big(w_{\ell+1} i_{\sigma(\ell+1)} + \cdots + w_{j+1} i_{\sigma(j+1)}\big), i_{\sigma(j+2)}, \ldots, i_{\sigma(r)}\big] \\ = A[i_1, \ldots, i_r] \quad \text{for all } (i_1, \ldots, i_r), \tag{11}$$

where $w_i := \prod_{k=\ell+1}^{i-1} s_{\sigma(k)} = v_i / \prod_{k=1}^{\ell} s_{\sigma(k)}$ (and $w_{\ell+1} := 1$), by the definition of v_i. The array $(B[i, i_{\sigma(j+2)}, \ldots, i_{\sigma(r)}])$ now results from application of the projection mapping $\psi_{\ell,\ell+1}$ (using the original index labellings). Therefore (8) and (11) imply (10), since

$$\big(v_1 i_{\sigma(1)} + \cdots + v_\ell i_{\sigma(\ell)}\big) + \left(\prod_{k=1}^{\ell} s_{\sigma(k)}\right) \big(w_{\ell+1} i_{\sigma(\ell+1)} + \cdots + w_{j+1} i_{\sigma(j+1)}\big) = v_1 i_{\sigma(1)} + \cdots + v_{j+1} i_{\sigma(j+1)}.$$

This establishes case j and completes the induction. $\square$

We have shown that the application of a sequence of projection mappings to a given r-dimensional array can be completely described by a graph G on r vertices comprising a set of disjoint directed paths: repeated application of Proposition 2 (once for each directed path) shows that the corresponding projected array is independent of the sequence of projection mappings that leads to G.

4 Construction of Golay array pairs

In this section we generalise Dymond's construction [5] for a binary Golay array pair from two other binary Golay array pairs, and derive a second construction from it. We apply this second construction recursively, producing an explicit form for the resulting Golay array pair over $\mathbb{Z}_H$. This will be the main construction theorem in later sections.

We begin with Dymond's construction theorem for binary Golay array pairs, which is a generalisation to multiple dimensions of Turyn's 1974 composition theorem [19, Lemma 5] for binary Golay sequence pairs:

Theorem 3 (Dymond [5, Theorem 4.24])**.** *Let $\otimes$ represent the Kronecker product of arrays. Suppose that $\mathcal{A}$ and $\mathcal{B}$ form an $s_1 \times \cdots \times s_r$ binary Golay array pair, and that $\mathcal{C}$ and $\mathcal{D}$ form a $t_1 \times \cdots \times t_r$ binary Golay array pair (where any of the s_k and t_k can take the value 1). Then the arrays*

$$\mathcal{A} \otimes \left(\frac{\mathcal{C} + \mathcal{D}}{2}\right) + \mathcal{B} \otimes \left(\frac{\mathcal{C} - \mathcal{D}}{2}\right),$$
$$\mathcal{A} \otimes \left(\frac{\mathcal{C}^* - \mathcal{D}^*}{2}\right) - \mathcal{B} \otimes \left(\frac{\mathcal{C}^* + \mathcal{D}^*}{2}\right)$$

form an $s_1 t_1 \times \cdots \times s_r t_r$ binary Golay array pair.

Write $\mathbf{1}^{(r)}$ to represent $1 \times \cdots \times 1$, in which r copies of 1 appear, and likewise write $\boldsymbol{n}^{(r)}$ to represent the corresponding expression with r copies of any positive integer n. Consider specialising Theorem 3 to the case where $\mathcal{A}$ and $\mathcal{B}$ have size $s_1 \times \cdots \times s_r \times \mathbf{1}^{(v)}$, and $\mathcal{C}$ and $\mathcal{D}$ have size $\mathbf{1}^{(r)} \times t_1 \times \cdots \times t_v$, so that the Kronecker product simplifies to the tensor product. (We do not lose anything by this specialisation: Theorem 3 can be recovered by setting $v = r$ and then applying Theorem 1 to the resulting binary Golay array pair of size $s_1 \times \cdots \times s_r \times t_1 \times \cdots \times t_r$ in order to give an $s_1 t_1 \times \cdots \times s_r t_r$ binary Golay array pair.) Now generalise this tensor product construction from binary to H-phase input arrays, adding a condition on $\mathcal{C}$ and $\mathcal{D}$ to ensure that the constructed arrays are also H-phase arrays:

Theorem 4. *Abbreviate $i_1,\dots,i_r$ and $j_1,\dots,j_v$ to $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively, and abbreviate $s_1\times\cdots\times s_r$ and $t_1\times\cdots\times t_v$ to $\boldsymbol{s}$ and $\boldsymbol{t}$ respectively. Suppose that $(A[\boldsymbol{i}])$ and $(B[\boldsymbol{i}])$ form an H-phase Golay array pair of size $\boldsymbol{s}$. Suppose that $(C[\boldsymbol{j}])$ and $(D[\boldsymbol{j}])$ form an H-phase Golay array pair of size $\boldsymbol{t}$ and that*

$$\text{for each } \boldsymbol{j},\ \text{either } C[\boldsymbol{j}]=D[\boldsymbol{j}] \text{ or } C[\boldsymbol{j}]=-D[\boldsymbol{j}]. \tag{12}$$

Then the H-phase arrays $(F[\boldsymbol{i},\boldsymbol{j}])$ and $(G[\boldsymbol{i},\boldsymbol{j}])$ of size $\boldsymbol{s}\times\boldsymbol{t}$ given by

$$\left.\begin{aligned} F[\boldsymbol{i},\boldsymbol{j}] &:= A[\boldsymbol{i}]\cdot\left(\frac{C[\boldsymbol{j}]+D[\boldsymbol{j}]}{2}\right)+B[\boldsymbol{i}]\cdot\left(\frac{C[\boldsymbol{j}]-D[\boldsymbol{j}]}{2}\right),\\ G[\boldsymbol{i},\boldsymbol{j}] &:= A[\boldsymbol{i}]\cdot\left(\frac{C^*[\boldsymbol{j}]-D^*[\boldsymbol{j}]}{2}\right)-B[\boldsymbol{i}]\cdot\left(\frac{C^*[\boldsymbol{j}]+D^*[\boldsymbol{j}]}{2}\right) \end{aligned}\right\} \tag{13}$$

form a Golay array pair.

Proof. The arrays $(F[\boldsymbol{i},\boldsymbol{j}])$ and $(G[\boldsymbol{i},\boldsymbol{j}])$ have size $\boldsymbol{s}\times\boldsymbol{t}$ by construction, and are H-phase arrays by condition (12). We will show that they form a Golay array pair, modelling the proof on the generating function approach of [5, Theorem 4.24].

Abbreviate $y_1,\dots,y_r$ and $z_1,\dots,z_v$ to $\boldsymbol{y}$ and $\boldsymbol{z}$ respectively, and abbreviate $y_1^{-1},\dots,y_r^{-1}$ and $z_1^{-1},\dots,z_v^{-1}$ to $\boldsymbol{y}^{-1}$ and $\boldsymbol{z}^{-1}$ respectively. From (13) we obtain the generating function equations

$$\begin{aligned} 2F(\boldsymbol{y},\boldsymbol{z}) &= A(\boldsymbol{y})\big(C(\boldsymbol{z})+D(\boldsymbol{z})\big)+B(\boldsymbol{y})\big(C(\boldsymbol{z})-D(\boldsymbol{z})\big),\\ 2G(\boldsymbol{y},\boldsymbol{z}) &= A(\boldsymbol{y})\big(C^*(\boldsymbol{z})-D^*(\boldsymbol{z})\big)-B(\boldsymbol{y})\big(C^*(\boldsymbol{z})+D^*(\boldsymbol{z})\big)\\ &= z_1^{t_1-1}\cdots z_v^{t_v-1}\left(A(\boldsymbol{y})\left(\overline{C(\boldsymbol{z}^{-1})}-\overline{D(\boldsymbol{z}^{-1})}\right)-B(\boldsymbol{y})\left(\overline{C(\boldsymbol{z}^{-1})}+\overline{D(\boldsymbol{z}^{-1})}\right)\right), \end{aligned}$$

by (6). Straightforward manipulation then shows that, for all $(\boldsymbol{y},\boldsymbol{z})$,

$$\begin{aligned} &F(\boldsymbol{y},\boldsymbol{z})\overline{F(\boldsymbol{y}^{-1},\boldsymbol{z}^{-1})}+G(\boldsymbol{y},\boldsymbol{z})\overline{G(\boldsymbol{y}^{-1},\boldsymbol{z}^{-1})}\\ &\quad= \frac{1}{2}\left(A(\boldsymbol{y})\overline{A(\boldsymbol{y}^{-1})}+B(\boldsymbol{y})\overline{B(\boldsymbol{y}^{-1})}\right)\left(C(\boldsymbol{z})\overline{C(\boldsymbol{z}^{-1})}+D(\boldsymbol{z})\overline{D(\boldsymbol{z}^{-1})}\right), \end{aligned}$$

which is constant by (4). Therefore $(F[\boldsymbol{i},\boldsymbol{j}])$ and $(G[\boldsymbol{i},\boldsymbol{j}])$ form a Golay array pair, by (4). □

Theorem 4 produces an H-phase Golay array pair of size $\boldsymbol{s}\times\boldsymbol{t}$ from H-phase Golay array pairs of size $\boldsymbol{s}$ and $\boldsymbol{t}$, subject to the condition (12) on the input pair of size $\boldsymbol{t}$. We now use Theorem 4 to derive an alternative construction which does not require this condition to hold, but instead produces an H-phase Golay array pair of size $\boldsymbol{s}\times\boldsymbol{t}\times 2$ from the same input array pairs:

Theorem 5. *Abbreviate $i_1,\dots,i_r$ and $j_1,\dots,j_v$ to $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively, and abbreviate $s_1\times\cdots\times s_r$ and $t_1\times\cdots\times t_v$ to $\boldsymbol{s}$ and $\boldsymbol{t}$ respectively. Suppose that $(A[\boldsymbol{i}])$ and $(B[\boldsymbol{i}])$ form an H-phase Golay array pair of size $\boldsymbol{s}$, and that $(C[\boldsymbol{j}])$ and $(D[\boldsymbol{j}])$ form an H-phase Golay array pair of size $\boldsymbol{t}$. Then the H-phase arrays $(F[\boldsymbol{i},\boldsymbol{j},x])$ and $(G[\boldsymbol{i},\boldsymbol{j},x])$ of size $\boldsymbol{s}\times\boldsymbol{t}\times 2$ given by*

$$\left.\begin{aligned} F[\boldsymbol{i},\boldsymbol{j},0] &:= A[\boldsymbol{i}]\cdot C[\boldsymbol{j}], & F[\boldsymbol{i},\boldsymbol{j},1] &:= B^*[\boldsymbol{i}]\cdot D[\boldsymbol{j}],\\ G[\boldsymbol{i},\boldsymbol{j},0] &:= B[\boldsymbol{i}]\cdot C[\boldsymbol{j}], & G[\boldsymbol{i},\boldsymbol{j},1] &:= -A^*[\boldsymbol{i}]\cdot D[\boldsymbol{j}] \end{aligned}\right\} \tag{14}$$

form a Golay array pair. The corresponding arrays over $\mathbb{Z}_H$ can be represented as

$$\left.\begin{aligned} f[\boldsymbol{i},\boldsymbol{j},x] &:= \Big(b^*[\boldsymbol{i}]+d[\boldsymbol{j}]-a[\boldsymbol{i}]-c[\boldsymbol{j}]\Big)x+a[\boldsymbol{i}]+c[\boldsymbol{j}],\\ g[\boldsymbol{i},\boldsymbol{j},x] &:= \left(a^*[\boldsymbol{i}]+d[\boldsymbol{j}]-b[\boldsymbol{i}]-c[\boldsymbol{j}]+\frac{H}{2}\right)x+b[\boldsymbol{i}]+c[\boldsymbol{j}]. \end{aligned}\right\} \tag{15}$$

Proof. We construct $(F[\boldsymbol{i},\boldsymbol{j},x])$ and $(G[\boldsymbol{i},\boldsymbol{j},x])$ as an H-phase Golay array pair by two applications of Theorem 4.

Since $(B^*[\boldsymbol{i}])$ has the same aperiodic autocorrelation function as $(B[\boldsymbol{i}])$, we see that $(A[\boldsymbol{i}])$ and $(B^*[\boldsymbol{i}])$ form a Golay array pair of size $\boldsymbol{s}$. Apply Theorem 4, taking the input $((A[\boldsymbol{i}]),(B[\boldsymbol{i}]))$ to be the Golay array pair $((A[\boldsymbol{i}]),(B^*[\boldsymbol{i}]))$ of size $\boldsymbol{s}$, and the input $((C[\boldsymbol{j}]),(D[\boldsymbol{j}]))$ to be the Golay array pair $([1,1],[1,-1])$ of size 2 that satisfies (12). This produces an H-phase Golay array pair $((F'[\boldsymbol{i},x]),(G'[\boldsymbol{i},x]))$ of size $\boldsymbol{s}\times 2$ given by

$$\begin{aligned} F'[\boldsymbol{i},0] &:= A[\boldsymbol{i}], \qquad & F'[\boldsymbol{i},1] &:= B^*[\boldsymbol{i}], \\ G'[\boldsymbol{i},0] &:= A[\boldsymbol{i}], \qquad & G'[\boldsymbol{i},1] &:= -B^*[\boldsymbol{i}]. \end{aligned}$$

Now apply Theorem 4 a second time, taking the input $((A[\boldsymbol{i}]),(B[\boldsymbol{i}]))$ to be the Golay array pair $((C[\boldsymbol{j}]),(D[\boldsymbol{j}]))$ of size $\boldsymbol{t}$, and the input $((C[\boldsymbol{j}]),(D[\boldsymbol{j}]))$ to be the Golay array pair $((F'[\boldsymbol{i},x]),(G'[\boldsymbol{i},x]))$ of size $\boldsymbol{s}\times 2$ that satisfies (12). Writing $F'[\boldsymbol{i},x] := A[\boldsymbol{i}](1-x)+B^*[\boldsymbol{i}]x$, we see from (5) that $F'^*[\boldsymbol{i},x] = A^*[\boldsymbol{i}]x + B[\boldsymbol{i}](1-x)$, and similarly $G'^*[\boldsymbol{i},x] = A^*[\boldsymbol{i}]x - B[\boldsymbol{i}](1-x)$. Therefore this second application of Theorem 4 produces the H-phase Golay array pair $((F[\boldsymbol{i},\boldsymbol{j},x]),(G[\boldsymbol{i},\boldsymbol{j},x]))$ of size $\boldsymbol{s}\times\boldsymbol{t}\times 2$ given in (14), as claimed.

From (14), the arrays over $\mathbb{Z}_H$ corresponding to $(F[\boldsymbol{i},\boldsymbol{j},x])$ and $(G[\boldsymbol{i},\boldsymbol{j},x])$ are defined by

$$\begin{aligned} f[\boldsymbol{i},\boldsymbol{j},0] &:= a[\boldsymbol{i}]+c[\boldsymbol{j}], \qquad & f[\boldsymbol{i},\boldsymbol{j},1] &:= b^*[\boldsymbol{i}]+d[\boldsymbol{j}], \\ g[\boldsymbol{i},\boldsymbol{j},0] &:= b[\boldsymbol{i}]+c[\boldsymbol{j}], \qquad & g[\boldsymbol{i},\boldsymbol{j},1] &:= a^*[\boldsymbol{i}]+d[\boldsymbol{j}]+H/2, \end{aligned}$$

and these definitions can be represented in the form (15), as claimed. □

Example 6. *The length* 8 *sequences* $(a[i]) := [0,0,0,2,0,0,2,0]$ *and* $(b[i]) := [2,1,1,0,2,3,3,0]$ *over* $\mathbb{Z}_4$ *form a (cross-over) Golay sequence pair, and likewise so do the length* 8 *sequences* $(c[j]) := [0,0,0,2,0,0,2,0]$ *and* $(d[j]) := [2,3,3,0,2,1,1,0]$. *Apply Theorem* 5 *with* $H=4$ *and* $\boldsymbol{s}=\boldsymbol{t}=8$ *to obtain the* $8\times 8\times 2$ *Golay array pair*

$$(f[i,j,x]) = \left[\underset{\substack{x=0\\(a[i]+c[j])}}{\begin{bmatrix} 0&0&0&2&0&0&2&0\\ 0&0&0&2&0&0&2&0\\ 0&0&0&2&0&0&2&0\\ 2&2&2&0&2&2&0&2\\ 0&0&0&2&0&0&2&0\\ 0&0&0&2&0&0&2&0\\ 2&2&2&0&2&2&0&2\\ 0&0&0&2&0&0&2&0 \end{bmatrix}} \quad \underset{\substack{x=1\\(b^*[i]+d[j])}}{\begin{bmatrix} 2&3&3&0&2&1&1&0\\ 3&0&0&1&3&2&2&1\\ 3&0&0&1&3&2&2&1\\ 0&1&1&2&0&3&3&2\\ 2&3&3&0&2&1&1&0\\ 1&2&2&3&1&0&0&3\\ 1&2&2&3&1&0&0&3\\ 0&1&1&2&0&3&3&2 \end{bmatrix}} \right],$$

$$(g[i,j,x]) = \left[\underset{\substack{x=0\\(b[i]+c[j])}}{\begin{bmatrix} 2&1&1&0&2&3&3&0\\ 2&1&1&0&2&3&3&0\\ 2&1&1&0&2&3&3&0\\ 0&3&3&2&0&1&1&2\\ 2&1&1&0&2&3&3&0\\ 2&1&1&0&2&3&3&0\\ 0&3&3&2&0&1&1&2\\ 2&1&1&0&2&3&3&0 \end{bmatrix}} \quad \underset{\substack{x=1\\(a^*[i]+d[j]+2)}}{\begin{bmatrix} 0&2&0&0&2&0&0&0\\ 1&3&1&1&3&1&1&1\\ 1&3&1&1&3&1&1&1\\ 2&0&2&2&0&2&2&2\\ 0&2&0&0&2&0&0&0\\ 3&1&3&3&1&3&3&3\\ 3&1&3&3&1&3&3&3\\ 2&0&2&2&0&2&2&2 \end{bmatrix}} \right],$$

where the horizontal and vertical direction of each 8×8 *array corresponds to the array arguments* i *and* j *respectively. Project this Golay array pair to a length* 128 *Golay sequence pair over* $\mathbb{Z}_4$ *by joining index* 2 *to index* 1 *to index* 3: *for each array of the pair this corresponds to listing the* *elements of both 8×8 arrays in turn, column by column. The resulting sequence pair is precisely that of* [7, *Example* 6].

We now apply Theorem 5 recursively to obtain the main construction theorem of the paper.

Theorem 7. *Let $m \geq 1$ be an integer and abbreviate $x_1, \ldots, x_m$ to $\boldsymbol{x}$; for $k = 0, 1, \ldots, m$, abbreviate $i_{k,1}, \ldots, i_{k,r_k}$ to $\boldsymbol{i}_k$ and $s_{k,1} \times \cdots \times s_{k,r_k}$ to $\boldsymbol{s}_k$. Suppose that $(a_k[\boldsymbol{i}_k])$ and $(b_k[\boldsymbol{i}_k])$ form a Golay array pair of size $\boldsymbol{s}_k$ over $\mathbb{Z}_H$, for $k = 0, 1, \ldots, m$. Then the arrays $(f_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}])$ and $(g_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}])$ of size $\boldsymbol{s}_0 \times \cdots \times \boldsymbol{s}_m \times \boldsymbol{2}^{(m)}$ over $\mathbb{Z}_H$ given by*

$$\begin{aligned}
&f_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}] := \\
&\quad \sum_{k=1}^{m-1} \left(a_k[\boldsymbol{i}_k] + a_k^*[\boldsymbol{i}_k] - b_k[\boldsymbol{i}_k] - b_k^*[\boldsymbol{i}_k] + \frac{H}{2} \right) x_k x_{k+1} + \\
&\quad \sum_{k=1}^{m} \left(b_{k-1}^*[\boldsymbol{i}_{k-1}] + b_k[\boldsymbol{i}_k] - a_{k-1}[\boldsymbol{i}_{k-1}] - a_k[\boldsymbol{i}_k] \right) x_k + \sum_{k=0}^{m} a_k[\boldsymbol{i}_k], \\
&g_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}] := f'_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}] + \frac{H}{2} x_1,
\end{aligned}$$

form a Golay array pair, where $f'_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}]$ is $f_m[\boldsymbol{i}_0, \ldots, \boldsymbol{i}_m, \boldsymbol{x}]$ with $a_0[\boldsymbol{i}_0], b_0[\boldsymbol{i}_0]$ interchanged and with $a_0^*[\boldsymbol{i}_0], b_0^*[\boldsymbol{i}_0]$ interchanged.

Proof. For ease of presentation we will suppress the dependence of each a_k and b_k on $\boldsymbol{i}_k$, writing

$$f_1 = (b_0^* + b_1 - a_0 - a_1)x_1 + a_0 + a_1, \tag{16}$$

$$g_1 = (a_0^* + b_1 - b_0 - a_1 + H/2)x_1 + b_0 + a_1, \tag{17}$$

$$f_m = f_1 + \sum_{k=2}^{m} e_k, \tag{18}$$

$$g_m = g_1 + \sum_{k=2}^{m} e_k, \tag{19}$$

where

$$e_k = (a_{k-1} + a_{k-1}^* - b_{k-1} - b_{k-1}^* + H/2)x_{k-1}x_k + (b_{k-1}^* + b_k - a_{k-1} - a_k)x_k + a_k \quad \text{for } 2 \leq k \leq m. \tag{20}$$

We firstly prove by induction on m that

$$g_m^* - f_m = (a_m + a_m^* - b_m - b_m^* + H/2)x_m + b_m^* - a_m + (m \bmod 2)H/2 \quad \text{for } m \geq 1. \tag{21}$$

The array $(c[\boldsymbol{i}_0, x_1]) := (a_0[\boldsymbol{i}_0]x_1)$ satisfies $c^*[\boldsymbol{i}_0, x_1] = a_0^*[\boldsymbol{i}_0](1 - x_1)$ by (7), so from (16) and (17) we find

$$\begin{aligned}
g_1^* - f_1 &= (a_0 + b_1^* - b_0^* - a_1^* + H/2)(1 - x_1) + b_0^* + a_1^* - (b_0^* + b_1 - a_0 - a_1)x_1 - a_0 - a_1 \\
&= (a_1 + a_1^* - b_1 - b_1^* + H/2)x_1 + b_1^* - a_1 + H/2,
\end{aligned}$$

which is the base case $m = 1$ of (21). For $m \geq 2$, assume that (21) holds up to $m - 1$. By (18), (19) and (20),

$$\begin{aligned}
g_m^* - f_m &= g_{m-1}^* - f_{m-1} + e_m^* - e_m \\
&= g_{m-1}^* - f_{m-1} + (a_{m-1} + a_{m-1}^* - b_{m-1} - b_{m-1}^* + H/2)(1 - x_{m-1})(1 - x_m) \\
&\quad + (b_{m-1} + b_m^* - a_{m-1}^* - a_m^*)(1 - x_m) + a_m^* - (a_{m-1} + a_{m-1}^* - b_{m-1} - b_{m-1}^* + H/2)x_{m-1}x_m \\
&\quad - (b_{m-1}^* + b_m - a_{m-1} - a_m)x_m - a_m \\
&= \left(g_{m-1}^* - f_{m-1} - (a_{m-1} + a_{m-1}^* - b_{m-1} - b_{m-1}^* + H/2)x_{m-1} \right) \\
&\quad + (a_m + a_m^* - b_m - b_m^* + H/2)x_m + a_{m-1} - a_m - b_{m-1}^* + b_m^* + H/2.
\end{aligned}$$

Applying the inductive hypothesis to the bracketed expression involving $g_{m-1}^* - f_{m-1}$ gives (21), completing the induction.

Furthermore, writing $f_m^* - g_m = -(g_m^* - f_m)^*$, it follows from (21) that

$$f_m^* - g_m = g_m^* - f_m + H/2 \quad \text{for } m \geq 1. \tag{22}$$

We now prove by induction on m that f_m and g_m form a Golay array pair of size $\boldsymbol{s}_0 \times \cdots \times \boldsymbol{s}_m \times \boldsymbol{2}^{(m)}$ for $m \geq 1$, making use of (21) and (22). The base case $m = 1$ is given by applying Theorem 5, taking the input $((a[\boldsymbol{i}]), (b[\boldsymbol{i}]))$ to be the Golay array pair (a_0, b_0) of size $\boldsymbol{s}_0$ and the input $((c[\boldsymbol{j}]), (d[\boldsymbol{j}]))$ to be the Golay array pair (a_1, b_1) of size $\boldsymbol{s}_1$. For $m \geq 2$, assume by the inductive hypothesis that f_{m-1} and g_{m-1} form a Golay array pair. Apply Theorem 5, taking $((a[\boldsymbol{i}]), (b[\boldsymbol{i}]))$ to be (f_{m-1}, g_{m-1}) of size $\boldsymbol{s}_0 \times \cdots \times \boldsymbol{s}_{m-1} \times \boldsymbol{2}^{(m-1)}$ and $((c[\boldsymbol{j}]), (d[\boldsymbol{j}]))$ to be $(a_m, (b_m + (1+m \bmod 2)H/2))$ of size $\boldsymbol{s}_m$ to yield a Golay pair (f, g) of size $\boldsymbol{s}_0 \times \cdots \times \boldsymbol{s}_m \times \boldsymbol{2}^{(m)}$. (The second pair inherits the Golay property from the array pair (a_m, b_m), since $(b_m[\boldsymbol{i}_m])$ and $(b_m[\boldsymbol{i}_m] + c)$ have identical aperiodic autocorrelation functions for any constant $c \in \mathbb{Z}_H$.) Then from (15) and (21) we have

$$\begin{aligned} f - f_{m-1} &= \left[g_{m-1}^* - f_{m-1} + b_m - a_m + (1 + m \bmod 2)H/2\right] x_m + a_m \\ &= \left[(a_{m-1} + a_{m-1}^* - b_{m-1} - b_{m-1}^* + H/2)x_{m-1} + b_{m-1}^* - a_{m-1} + b_m - a_m\right] x_m + a_m \\ &= e_m \end{aligned}$$

by (20). Therefore $f = f_m$, by (18). Similarly

$$\begin{aligned} g - g_{m-1} &= \left[f_{m-1}^* - g_{m-1} + b_m - a_m + (m \bmod 2)H/2\right] x_m + a_m \\ &= e_m \end{aligned}$$

using (22), and so $g = g_m$ by (19). Therefore f_m and g_m form a Golay array pair of size $\boldsymbol{s}_0 \times \cdots \times \boldsymbol{s}_m \times 2^{(m)}$, completing the induction. □

Theorem 7 can take any Golay array pairs as inputs, but it is sufficient for our purposes to restrict the inputs to be Golay sequence pairs, either trivial pairs (of length 1) or cross-over pairs of length 8 over $\mathbb{Z}_4$.

5 Affine Offsets

Theorem 7 constructs a Golay array pair from $m + 1$ smaller Golay array pairs, which is the first stage of the three-stage construction process of the paper. We now show that, by taking "affine offsets", we can generate a set of Golay array pairs from a single Golay array pair. This generalises [6, Corollary 2] to multiple dimensions, and is the second stage of the process.

Lemma 8. *Suppose that $((a[i_1, \ldots, i_r]), (b[i_1, \ldots, i_r]))$ is an $s_1 \times \cdots \times s_r$ Golay array pair over $\mathbb{Z}_H$. Then the affine offset*

$$\left(\left(a[i_1 \ldots, i_r] + \sum_{k=1}^{r} e_k i_k + e_0\right), \ \left(b[i_1 \ldots, i_r] + \sum_{k=1}^{r} e_k i_k + e_0'\right)\right)$$

is also an $s_1 \times \cdots \times s_r$ Golay array pair over $\mathbb{Z}_H$, for all $e_0', e_0, e_1, \ldots, e_r \in Z_H$.

Proof. Let ξ denote $\exp(2\pi\sqrt{-1}/H)$ (a primitive H-th root of unity), and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ be the complex-valued arrays corresponding to the four given arrays over $\mathbb{Z}_H$, in the order in which they are mentioned. Fix $(u_1, \ldots, u_r) \neq (0, \ldots, 0)$. By the definition of aperiodic autocorrelation,

$$\begin{aligned} C_{\mathcal{A}'}(u_1, \ldots, u_r) &= \sum_{i_1} \cdots \sum_{i_r} \xi^{a[i_1,\ldots,i_r] + \sum_{k=1}^r e_k i_k + e_0} \overline{\xi^{a[i_1+u_1,\ldots,i_r+u_r] + \sum_{k=1}^r e_k(i_k+u_k) + e_0}} \\ &= \xi^{-\sum_{k=1}^r e_k u_k} C_{\mathcal{A}}(u_1, \ldots, u_r), \end{aligned}$$

and the same relation holds when $\mathcal{A}'$ and $\mathcal{A}$ are replaced by $\mathcal{B}'$ and $\mathcal{B}$ respectively. Therefore

$$\begin{aligned} C_{\mathcal{A}'}(u_1,\ldots,u_r)+C_{\mathcal{B}'}(u_1,\ldots,u_r) &= \xi^{-\sum_{k=1}^r e_k u_k}(C_{\mathcal{A}}(u_1,\ldots,u_r)+C_{\mathcal{B}}(u_1,\ldots,u_r)) \\ &= 0, \end{aligned}$$

since $\mathcal{A}$, $\mathcal{B}$ form a Golay pair. Therefore $\mathcal{A}'$, $\mathcal{B}'$ form a Golay pair. □

6 Golay array pairs of volume 2^m from trivial Golay sequence pairs

In this section we construct a single m-dimensional Golay array pair from $m+1$ trivial input Golay array pairs using Theorem 7, and take affine offsets using Lemma 8. The third stage of the three-stage construction process is then to project these array pairs to lower dimensions.

We begin with a special case of this process, in which we consider just the projection to sequences. We will see in Corollary 10 that this recovers the "standard" Golay sequence pairs of length 2^m given in [4].

Theorem 9. *Let $m \geq 1$ be integer and let $e_0', e_0, e_1, \ldots, e_m \in \mathbb{Z}_H$. For any permutation π of $\{1,\ldots,m\}$, the sequences $(a[x])$ and $(b[x])$ of length 2^m over $\mathbb{Z}_H$ given by*

$$\left.\begin{aligned} a[x_m+2x_{m-1}+\cdots+2^{m-1}x_1] &:= \frac{H}{2}\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)}+\sum_{k=1}^{m} e_k x_{\pi(k)}+e_0, \\ b[x_m+2x_{m-1}+\cdots+2^{m-1}x_1] &:= \frac{H}{2}\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)}+\sum_{k=1}^{m} e_k x_{\pi(k)}+e_0'+\frac{H}{2}x_{\pi(1)} \end{aligned}\right\} \text{for all } (x_1,\ldots,x_m)\in\mathbb{Z}_2^m$$

form a Golay sequence pair.

As $e_0', e_0, e_1, \ldots, e_m$ and π range over all their possible values, the number of Golay sequences of length 2^m over $\mathbb{Z}_H$ of this form is

$$\begin{cases} H^{m+1}m!/2 & \text{for } m>1 \\ H^2 & \text{for } m=1, \end{cases}$$

and the corresponding number of ordered Golay sequence pairs is at least $H^{m+2}m!$

Proof. We construct the Golay sequence pair $((a[x]),(b[x]))$ according to the three-stage process.

Stage 1. Apply Theorem 7, taking each of the input pairs $((a_k[\boldsymbol{i}_k]),(b_k[\boldsymbol{i}_k]))$ for $k=0,1,\ldots,m$ to be the trivial length 1 Golay pair $([0],[0])$ over $\mathbb{Z}_H$. This gives a Golay array pair of size $\mathbf{1}^{(m+1)}\times\mathbf{2}^{(m)}$ over $\mathbb{Z}_H$. By removing all dimensions equalling 1 we can write this as a Golay array pair $((f'[x_1,\ldots,x_m]),(g'[x_1,\ldots,x_m]))$ of size $\mathbf{2}^{(m)}$ over $\mathbb{Z}_H$, where

$$\left.\begin{aligned} f'[x_1,\ldots,x_m] &:= \frac{H}{2}\sum_{k=1}^{m-1}x_kx_{k+1}, \\ g'[x_1,\ldots,x_m] &:= \frac{H}{2}\sum_{k=1}^{m-1}x_kx_{k+1}+\frac{H}{2}x_1 \end{aligned}\right\} \text{ for all } (x_1,\ldots,x_m)\in\mathbb{Z}_2^m.$$

Stage 2. Apply Lemma 8 with $r = m$ to show that the affine offset $((f[x_1,\dots,x_m]),(g[x_1,\dots,x_m]))$ is a Golay array pair of size $\mathbf{2}^{(m)}$ over $\mathbb{Z}_H$, where

$$\left.\begin{array}{rcl} f[x_1,\dots,x_m] & := & \frac{H}{2}\sum_{k=1}^{m-1} x_k x_{k+1} + \sum_{k=1}^{m} e_k x_k + e_0, \\ g[x_1,\dots,x_m] & := & \frac{H}{2}\sum_{k=1}^{m-1} x_k x_{k+1} + \sum_{k=1}^{m} e_k x_k + e_0' + \frac{H}{2}x_1 \end{array}\right\} \text{ for all } (x_1,\dots,x_m) \in \mathbb{Z}_2^m. \tag{23}$$

Stage 3. Apply $m-1$ successive projection mappings to the m-dimensional arrays $(f[x_1,\dots,x_m])$ and $(g[x_1,\dots,x_m])$, where the associated directed graph G on m vertices is

$$\overset{\pi^{-1}(m)}{\bullet} \longrightarrow \overset{\pi^{-1}(m-1)}{\bullet} \longrightarrow \cdots \longrightarrow \overset{\pi^{-1}(1)}{\bullet}$$

(see Section 3). By Proposition 2 with $r = m$, $j = m-1$, and $s_k = 2$ and $\sigma(k) = \pi^{-1}(m+1-k)$ for $1 \le k \le m$, the resulting sequences of length 2^m over $\mathbb{Z}_H$ are $(a'[x])$ and $(b'[x])$, where

$$\left.\begin{array}{rcl} a'[x_{\pi^{-1}(m)} + 2x_{\pi^{-1}(m-1)} + \cdots + 2^{m-1}x_{\pi^{-1}(1)}] & := & f[x_1,\dots,x_m], \\ b'[x_{\pi^{-1}(m)} + 2x_{\pi^{-1}(m-1)} + \cdots + 2^{m-1}x_{\pi^{-1}(1)}] & := & g[x_1,\dots,x_m] \end{array}\right\} \text{ for all } (x_1,\dots,x_m) \in \mathbb{Z}_2^m.$$

These forms are equivalent to

$$\left.\begin{array}{rcl} a'[x_m + 2x_{m-1} + \cdots + 2^{m-1}x_1] & := & f[x_{\pi(1)},\dots,x_{\pi(m)}], \\ b'[x_m + 2x_{m-1} + \cdots + 2^{m-1}x_1] & := & g[x_{\pi(1)},\dots,x_{\pi(m)}] \end{array}\right\} \text{ for all } (x_1,\dots,x_m) \in \mathbb{Z}_2^m,$$

so that $(a'[x]) = (a[x])$ and $(b'[x]) = (b[x])$ by (23). Furthermore, by Theorem 1, $(a'[x])$ and $(b'[x])$ form a Golay sequence pair since $(f[x_1,\dots,x_m])$ and $(g[x_1,\dots,x_m])$ form a Golay array pair.

It remains to determine the sequence and sequence pair counts, as $e_0', e_0, e_1, \dots, e_m \in \mathbb{Z}_H$ and π range over all their possible values. For $m = 1$ we have $a[x_1] := e_1x_1 + e_0$ and $b[x_1] := (e_1 + H/2)x_1 + e_0'$, giving H^2 distinct Golay sequences and H^3 ordered Golay sequence pairs, as required. For the rest of the proof, take $m > 1$.

Each permutation π corresponds to a unique projection graph G. Consider the multiset S of Golay sequences obtained under projection, as $(e_0, e_1, \dots, e_m)$ and G range over their $H^{m+1}m!$ values. Observe that $f[x_1,\dots,x_m]$ is invariant under the mapping

$$x_k \mapsto x_{m+1-k} \text{ and } e_k \mapsto e_{m+1-k} \quad \text{for } 1 \le k \le m, \tag{24}$$

and that this mapping relabels each vertex k of G as $m+1-k$ to give a distinct projection graph G' (in which all arc directions are reversed). Therefore the projected sequence obtained from $(e_0, e_1, \dots, e_m)$ and G is the same as the projected sequence obtained from $(e_0, e_m, \dots, e_1)$ and G', so the multiplicity of each sequence in S is 2. (This multiplicity is exact, rather than a lower bound, because by Proposition 2 two projected sequences in S are identical only if their unprojected m-dimensional arrays are identical under reordering of dimensions, and the only non-identity permutation mapping of the x_k and e_k under which $f[x_1,\dots,x_m]$ is invariant is (24).) Therefore the number of distinct Golay sequences in S is $H^{m+1}m!/2$.

For given $(e_0, e_1, \dots, e_m)$, the array $(f[x_1,\dots,x_m])$ forms a Golay sequence pair with $(g[x_1,\dots,x_m])$ for any of H values of $e_0' \in \mathbb{Z}_H$. Furthermore, $f[x_1,\dots,x_m]$ is invariant under the mapping (24) but $g[x_1,\ldots,x_m]$ maps to $g[x_1,\ldots,x_m] + \frac{H}{2}(x_1 + x_m)$, which is distinct from $g[x_1,\ldots,x_m]$ for any $e_0' \in \mathbb{Z}_H$ because $m > 1$. Therefore each distinct m-dimensional array $(f[x_1,\ldots,x_m])$ forms an ordered Golay array pair with at least $2H$ other arrays. By applying the same sequence of projection mappings to any such pair we see that the number of ordered Golay sequence pairs of length 2^m over $\mathbb{Z}_H$ is at least $2H$ times the corresponding number of distinct Golay sequences (using Proposition 2 to rule out the possibility that two distinct m-dimensional array pairs project to the same sequence pair in S), namely $2H \cdot H^{m+1}m!/2 = H^{m+2}m!$ □

In the language of Section 2, the final paragraph of the proof of Theorem 9 shows that $|E(\mathcal{A})| = 2H$ for each constructed Golay sequence $\mathcal{A} = (a[x])$. The use of "at least" in the final sentence of the proof is required, because two such sequences could have the shared autocorrelation property (as do the sequence pairs of Theorem 12).

Now the *algebraic normal form* of a sequence $(f[x])$ of length 2^m over $\mathbb{Z}_H$ is the unique function $f'(x_1,\ldots,x_m) : \mathbb{Z}_2^m \to \mathbb{Z}_H$ satisfying

$$f'(x_1,\ldots,x_m) = f[x_m + 2x_{m-1} + \cdots + 2^{m-1}x_1] \quad \text{for all } (x_1, x_2, \ldots, x_m) \in \mathbb{Z}_2^m.$$

(The sequences $[0,0,0,0,1,1,1,1]$, $[0,0,1,1,0,0,1,1]$, and $[0,1,0,1,0,1,0,1]$ of length 8 over $\mathbb{Z}_H$ have algebraic normal form x_1, x_2, and x_3 respectively; note some authors use a different labelling convention for these sequences.) The algebraic normal form of the Golay sequence pair $(a[x])$ and $(b[x])$ of Theorem 9 is immediate, as stated in Corollary 10; the case $H = 2^h$ of this corollary was given by Davis and Jedwab [4, Theorem 3], and Paterson [17] showed that the case $H \neq 2^h$ holds without modification to the construction in [4]:

Corollary 10. *Let $m \geq 1$ be integer and let $e_0', e_0, e_1, \ldots, e_m \in \mathbb{Z}_H$. For any permutation π of $\{1,\ldots,m\}$, the sequences of length 2^m over $\mathbb{Z}_H$ having algebraic normal form*

$$\left.\begin{array}{rcl} a(x_1,\ldots,x_m) & := & \dfrac{H}{2}\displaystyle\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} e_k x_{\pi(k)} + e_0, \\ b(x_1,\ldots,x_m) & := & \dfrac{H}{2}\displaystyle\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} e_k x_{\pi(k)} + e_0' + \dfrac{H}{2}x_{\pi(1)} \end{array}\right\}$$

form a Golay sequence pair, called a standard Golay sequence pair in [7].

(In general, let $(a[x])$ be a sequence of length 2^m over $\mathbb{Z}_H$, obtained from an array $(f[x_1,\ldots,x_m])$ of size $\mathbf{2}^{(m)}$ over $\mathbb{Z}_H$ by applying $m-1$ successive projection mappings. Stage 3 of the proof of Theorem 9, together with Corollary 10, illustrates how to find the algebraic normal form $a(x_1,\ldots,x_m)$ of $(a[x])$: interpret the array arguments $x_1,\ldots,x_m$ as variables of the algebraic normal form, and represent the projection mappings using a permutation π.)

Theorem 9 deals with the special case in which the m-dimensional Golay array pair (23) is projected to 1 dimension. We now extend the argument used in Stage 3 of the proof of Theorem 9, in order to describe the projection to j dimensions for any j satisfying $1 \leq j \leq m$. (In Theorem 11, $0!$ takes its usual value 1, and the binomial coefficient $\binom{r}{i}$ for integer r and i is 1 for $i = 0$ and is 0 for nonzero $i > r$.)

Theorem 11. *Let m and j be integers satisfying $m > 1$ and $1 \leq j \leq m$. The number of j-dimensional Golay arrays of volume 2^m over $\mathbb{Z}_H$ that can be derived from affine offsets and projection mappings, after taking all input array pairs in Theorem 7 to be trivial, is*

$$\frac{1}{2}\left(H^{m+1}n_{m-j}(m) + H^{\lceil m/2\rceil+1}t_{m-j}(m)\right),$$

and the corresponding number of ordered Golay array pairs is at least $2H$ *times this number, where*

$$\begin{aligned} n_i(m) &:= i!\binom{m}{i}\binom{m-1}{i} \quad \textit{for } i \geq 0, \\ t_i(m) &:= \begin{cases} n_{i/2}(\lfloor m/2 \rfloor) \cdot 2^{i/2} & \textit{for } i \textit{ even}, \\ 0 & \textit{for } i \textit{ odd}. \end{cases} \end{aligned}$$

Proof. From Stage 2 of the proof of Theorem 9, the arrays $(f[x_1, \ldots, x_m])$ and $(g[x_1, \ldots, x_m])$ of size $\mathbf{2}^{(m)}$ over $\mathbb{Z}_H$ given by

$$\left.\begin{aligned} f[x_1, \ldots, x_m] &:= \frac{H}{2}\sum_{k=1}^{m-1} x_k x_{k+1} + \sum_{k=1}^{m} e_k x_k + e_0, \\ g[x_1, \ldots, x_m] &:= \frac{H}{2}\sum_{k=1}^{m-1} x_k x_{k+1} + \sum_{k=1}^{m} e_k x_k + e_0' + \frac{H}{2}x_1 \end{aligned}\right\} \text{ for all } (x_1, \ldots, x_m) \in \mathbb{Z}_2^m$$

form a Golay array pair, for any choice of affine offset variables $e_0', e_0, e_1, \ldots, e_m \in \mathbb{Z}_H$. Let $G_i(m)$ be the set of all graphs comprising m distinguished vertices and i arcs arranged as disjoint directed paths. Consider applying any $m - j$ successive projection mappings to the m-dimensional Golay array $(f[x_1, \ldots, x_m])$. Each resulting j-dimensional array of volume 2^m over $\mathbb{Z}_H$ is a Golay array by Theorem 1, and can be represented by the pair $((e_0, e_1, \ldots, e_m), G)$ for some $G \in G_{m-j}(m)$ (see Section 3).

Let S be the multiset of j-dimensional Golay arrays arising in this way, as $((e_0, e_1, \ldots, e_m), G)$ ranges over all $H^{m+1}|G_{m-j}(m)|$ values in $\mathbb{Z}_H^{m+1} \times G_{m-j}(m)$. We now show that each array in S occurs with multiplicity 1 or 2. As before, $f[x_1, \ldots, x_m]$ is invariant under the mapping

$$x_k \mapsto x_{m+1-k} \text{ and } e_k \mapsto e_{m+1-k} \quad \text{for } 1 \leq k \leq m.$$

For a given $G \in G_{m-j}(m)$, let $G' \in G_{m-j}(m)$ be the graph obtained by relabelling each vertex k as $m + 1 - k$, for $1 \leq k \leq m$. Since this relabelling corresponds to the mapping $x_k \mapsto x_{m+1-k}$, we see that the array represented by the pair $((e_0, e_1, \ldots, e_m), G)$ is the same as the array represented by the pair $((e_0, e_m, \ldots, e_1), G')$. The multiplicity in S of the array represented by the pair $((e_0, e_1, \ldots, e_m), G)$ is therefore 1 if

$$(e_1, \ldots, e_m) = (e_m, \ldots, e_1) \text{ and } G = G', \tag{25}$$

and 2 otherwise. (These multiplicities are exact, by a similar argument to that used in the proof of Theorem 9.)

We claim firstly that $|G_i(m)| = n_i(m)$, so that the total number of pairs $((e_0, e_1, \ldots, e_m), G) \in Z_H^{m+1} \times G_{m-j}(m)$ is $H^{m+1}n_{m-j}(m)$. We claim secondly that the number of $G \in G_i(m)$ satisfying $G = G'$ is $t_i(m)$, so that the number of pairs $((e_0, e_1, \ldots, e_m), G)$ satisfying (25) is $H^{\lceil m/2 \rceil + 1}t_{m-j}(m)$. It follows that the number of distinct arrays in S (each of which is a j-dimensional Golay array of volume 2^m over $\mathbb{Z}_H$) is

$$\frac{1}{2}\left(H^{m+1}n_{m-j}(m) - H^{\lceil m/2 \rceil + 1}t_{m-j}(m)\right) + H^{\lceil m/2 \rceil + 1}t_{m-j}(m),$$

as required. The corresponding number of ordered Golay array pairs is at least $2H$ times this number, by a similar argument to that used in the proof of Theorem 9 (replacing "sequence of length 2^m" by "j-dimensional array of volume 2^m").

For the first claim, each graph in $G_i(m)$ has exactly m distinguished vertices and i arcs arranged as disjoint directed paths. Consider enumerating the graphs in $G_i(m)$ by choosing i of the m vertices to have an outgoing arc and then assigning to each of these i arcs in turn any destination vertex that has no incoming arc so that no loop or cycle is created. This gives $|G_i(m)| = \binom{m}{i}(m-1)(m-2)\cdots(m-i) = n_i(m)$, establishing the first claim.

For the second claim, each graph $G \in G_i(m)$ satisfying $G = G'$ has i arcs that are arranged in pairs joining vertex k_1 to k_2 and vertex $m+1-k_1$ to $m+1-k_2$. If i is odd then clearly there are no such graphs, so take i to be even. In the case that m is odd, vertex $(m+1)/2$ cannot have any incoming or outgoing arc. We can therefore enumerate all such graphs as follows:

1. choose a graph in $G_{i/2}(\lfloor m/2 \rfloor)$ on the first $\lfloor m/2 \rfloor$ vertices (where $\lfloor m/2 \rfloor \geq 1$ since $m > 1$);
2. for each of the $i/2$ arcs of this graph independently, choose to leave the destination vertex k unchanged or to replace it by $m+1-k$, so that each arc now joins a vertex $k_1 \leq \lfloor m/2 \rfloor$ to a vertex k_2;
3. for each arc joining a vertex $k_1 \leq \lfloor m/2 \rfloor$ to a vertex k_2, add the arc joining vertex $m+1-k_1$ to $m+1-k_2$.

The total number of such graphs is therefore $n_{i/2}(\lfloor m/2 \rfloor) \cdot 2^{i/2} = t_i(m)$ for i even, establishing the second claim. □

In the sequence case $j = 1$, the counts in Theorem 11 for $m > 1$ match those in Theorem 9 (noting that $t_{m-1}(m) = 0$ for $m > 1$, reflecting the fact that no graph $G \in G_{m-1}(m)$ satisfies $G = G'$ for $m > 1$). We can obtain explicit forms for the j-dimensional Golay array pairs counted in Theorem 11, by representing each graph $G \in G_{m-j}(m)$ using a permutation σ and applying Proposition 2 to the array pair (23). The Golay array count given in Theorem 11 has been verified computationally for $3 \leq m \leq 6$ and $1 \leq j \leq m$.

7 Golay array pairs from length 8 cross-over pairs over $\mathbb{Z}_4$

In this section we use Theorem 7 to obtain Golay array pairs of length 2^n over $\mathbb{Z}_{2^h}$ that are different from those counted in Theorem 11. Since Theorem 7 captures the result of applying a construction method (Theorem 5) recursively, we require at least one input Golay array pair of length 2^m over $\mathbb{Z}_{2^h}$ not contained in Theorem 11. The only source of such array pairs currently known is the 512 cross-over Golay sequence pairs of length 8 over $\mathbb{Z}_4$:

Theorem 12 (Fiedler and Jedwab [6])**.** *Each of the sequence pairs in the set*

$$P := \{(\mathcal{A},\mathcal{B}),\ \ (\mathcal{A},\mathcal{B}^*),\ \ (\mathcal{A}^*,\mathcal{B}),\ \ (\mathcal{A}^*,\mathcal{B}^*),\ \ (\mathcal{B},\mathcal{A}),\ \ (\mathcal{B},\mathcal{A}^*),\ \ (\mathcal{B}^*,\mathcal{A}),\ \ (\mathcal{B}^*,\mathcal{A}^*)\}$$

is a cross-over Golay sequence pair of length 8 over $\mathbb{Z}_4$, where

$$\begin{aligned} \mathcal{A} &:= [0,0,0,2,0,0,2,0], \\ \mathcal{B} &:= [0,1,1,2,0,3,3,2]. \end{aligned}$$

All 512 ordered cross-over Golay sequence pairs of length 8 over $\mathbb{Z}_4$ occur as affine offsets of the 8 pairs in P, using Lemma 8.

Theorem 12 simplifies the results of [6], showing how $4 \cdot 4^2 = 64$ standard Golay sequences give rise to $8 \cdot 4^3 = 512$ non-standard Golay sequence pairs. In Theorem 14 we will take $c \geq 1$ of the $m+1$ input Golay array pairs for Theorem 7 to be from the set P, and the rest of the input pairs to be trivial. In preparation, we show that certain related input arrays for Theorem 7 lead to related output arrays; the proof consists of straightforward algebraic manipulation.

Lemma 13. *Let the output array (f_m) of Theorem 7 for the input Golay array pairs*

$$((a_k[\boldsymbol{i}_k]),(b_k[\boldsymbol{i}_k])) \text{ for } k=0,1,\dots,m$$

be $(h_m[\boldsymbol{i}_0,\dots,\boldsymbol{i}_m,x_1,\dots,x_m])$. Then the output array (f_m) of Theorem 7 for the input Golay array pairs

$$((a_{m-k}[\boldsymbol{i}_{m-k}]),(b^*_{m-k}[\boldsymbol{i}_{m-k}])) \text{ for } k=0,1,\dots,m$$

is $(h_m[\boldsymbol{i}_0,\dots,\boldsymbol{i}_m,x_m,\dots,x_1])$.

Theorem 14. *Let m, j and c be integers satisfying $m \geq 1$ and $1 \leq c \leq m+1$ and $1 \leq j \leq m+c$. The number of j-dimensional Golay arrays of volume $8^c 2^m$ over $\mathbb{Z}_4$ that can be derived from affine offsets and projection mappings, after taking c of the $m+1$ input array pairs in Theorem 7 to be from the set P of Theorem 12 and the remaining $m+1-c$ input array pairs to be trivial, is*

$$\frac{1}{2}\left[\binom{m+1}{c}8^c 4^{m+c+1} n_{m+c-j}(m+c) + \binom{\lceil m/2\rceil}{c/2}8^{c/2}4^{\lceil m/2\rceil+c/2+1}t_{m+c-j}(m+c)\right],$$

and the corresponding number of ordered Golay array pairs is at least 8 times this number, where

$$\begin{aligned} n_i(m) &:= i!\binom{m}{i}\binom{m-1}{i} \quad \text{for } i \geq 0,\\ t_i(m) &:= \begin{cases} n_{i/2}(\lfloor m/2\rfloor)\cdot 2^{i/2} & \text{for } i \text{ even},\\ 0 & \text{for } i \text{ odd.}\end{cases}\end{aligned}$$

Proof.

Stage 1. The sequence of Golay array pairs $(a_0,b_0),\dots(a_m,b_m)$ used as input to Theorem 7 is determined by the indices k of the c non-trivial array pairs, and by which of the 8 possible values in P each non-trivial array pair takes. Therefore the number of allowed values for this sequence of pairs is $\binom{m+1}{c}8^c$. Apply Theorem 7 to any of these choices of input array pairs, then remove any dimensions equalling 1 and relabel the remaining array arguments i_k as $i_1,\dots,i_c$ in order to give a Golay array pair $((f'[i_1,\dots,i_c,x_1,\dots,x_m]),(g'[i_1,\dots,i_c,x_1,\dots,x_m]))$ of size $\mathbf{8}^{(c)}\times\mathbf{2}^{(m)}$ over $\mathbb{Z}_4$.

Stage 2. For any choice of affine offset variables $e'_0,e'_1,\dots,e'_c,e_0,e_1,\dots,e_m\in\mathbb{Z}_4$, apply Lemma 8 with $r=m+c$ to give the Golay array pair $((f[i_1,\dots,i_c,x_1,\dots,x_m]),(g[i_1,\dots,i_c,x_1,\dots,x_m]))$, where

$$\left.\begin{aligned} f[i_1,\dots,i_c,x_1,\dots,x_m] &:= f'[i_1,\dots,i_c,x_1,\dots,x_m]+\sum_{k=1}^{c}e'_k i_k+\sum_{k=1}^{m}e_k x_k+e_0,\\ g[i_1,\dots,i_c,x_1,\dots,x_m] &:= g'[i_1,\dots,i_c,x_1,\dots,x_m]+\sum_{k=1}^{c}e'_k i_k+\sum_{k=1}^{m}e_k x_k+e'_0\end{aligned}\right\} \quad (26)$$

$$\text{for all } (i_1,\dots,i_c,x_1,\dots,x_m)\in\mathbb{Z}_8^c\times\mathbb{Z}_2^m.$$

Stage 3. Define $G_i(m)$ as in the proof of Theorem 11, recalling that $|G_i(m)|=n_i(m)$ and that, for $m>1$, $t_i(m)$ is the number of graphs in $G_i(m)$ that are invariant under the vertex relabelling $k\mapsto m+1-k$ for $1\leq k\leq m$. Consider applying any $p:=m+c-j$ successive projection mappings to the $(m+c)$-dimensional Golay array $(f[i_1,\dots,i_c,x_1,\dots,x_m])$. Each resulting j-dimensional array of volume 8^c2^m over $\mathbb{Z}_4$ is a Golay array by Theorem 1, and can be represented by the triple

$$(((a_0,b_0),\dots,(a_m,b_m)),(e'_1,\dots,e'_c,e_0,e_1,\dots,e_m),G) \quad (27)$$

for some $G\in G_p(m+c)$, where we label the vertices of $G_p(m+c)$ as $x_1,\dots,x_m$ and $i_1,\dots i_c$.

Let S be the multiset of j-dimensional Golay arrays arising in this way, as the triple (27) ranges over all its

$$\binom{m+1}{c} 8^c \cdot 4^{m+c+1} n_p(m+c) \tag{28}$$

allowed values. By Lemma 13, $f'[i_1,\ldots,i_c,x_1,\ldots,x_m]$ is invariant under the mapping

$$\begin{aligned}(a_k,b_k) \mapsto (a_{m-k},b^*_{m-k}) \ \text{ for } 0 \le k \le m; \quad i_k &\mapsto i_{c+1-k} \ \text{ for } 1 \le k \le c;\\ x_k &\mapsto x_{m+1-k} \ \text{ for } 1 \le k \le m\end{aligned}$$

(where the given mapping of the relabelled array arguments $i_1,\ldots,i_c$ is equivalent to the mapping "$i_k \mapsto i_{m-k}$ for $0 \le k \le m$" for the original array arguments $i_0,\ldots,i_m$). Therefore $f[i_1,\ldots,i_c,x_1,\ldots,x_m]$ is invariant under the mapping

$$\begin{aligned}(a_k,b_k) \mapsto (a_{m-k},b^*_{m-k}) \ \text{ for } 0 \le k \le m; \quad i_k \mapsto i_{c+1-k} \text{ and } e'_k &\mapsto e'_{c+1-k} \ \text{ for } 1 \le k \le c;\\ x_k \mapsto x_{m+1-k} \text{ and } e_k &\mapsto e_{m+1-k} \ \text{ for } 1 \le k \le m,\end{aligned}$$

and (by reference to the proof of Theorem 9) under no other non-identity permutation mapping of the x_k, e_k, i_k and e'_k. Therefore the multiplicity in S of the array represented by the triple (27) is 1 if

$$\left.\begin{aligned}&((a_0,b_0),\ldots,(a_m,b_m)) = ((a_m,b^*_m),\ldots,(a_0,b^*_0)) \text{ and}\\ &(e'_1,\ldots,e'_c,e_1,\ldots,e_m) = (e'_c,\ldots,e'_1,e_m,\ldots,e_1) \text{ and } G = G',\end{aligned}\right\} \tag{29}$$

and 2 otherwise, where $G' \in G_p(m+c)$ is the graph obtained from G under the vertex relabelling

$$x_k \mapsto x_{m+1-k} \text{ for } 1 \le k \le m; \quad i_k \mapsto i_{c+1-k} \text{ for } 1 \le k \le c. \tag{30}$$

We claim that the number of triples (27) satisfying (29) is

$$\begin{cases} 0 & \text{for } c \text{ odd},\\ \binom{\lceil m/2 \rceil}{c/2} 8^{c/2} 4^{\lceil m/2 \rceil + c/2+1} t_p(m+c) & \text{for } c \text{ even}.\end{cases}$$

Since the total number of allowed triples (27) is given by (28), and $p = m+c-j$, the number of distinct arrays in S (each of which is a j-dimensional Golay array of volume $8^c 2^m$ over $\mathbb{Z}_4$) is then as required. The corresponding number of ordered Golay array pairs is at least 8 times this number, by a similar argument to that used in the proof of Theorems 9 and 11.

To prove the claim, note that the first equality of (29) forces the indices of the c non-trivial array pairs taken from the set P to occur in pairs $\{k, m-k\}$ (where $k \ne m/2$ for m even, since no array pair (a,b) from the set P satisfies $b = b^*$). Therefore when c is odd there are no triples (27) satisfying (29). We therefore take c to be even, and determine the required number of triples (27) as the product of three terms, one for each of the equalities in (29). The number of allowed sequences $((a_0,b_0),\ldots,(a_m,b_m))$ satisfying the first equality of (29) is $\binom{\lceil m/2 \rceil}{c/2} 8^{c/2}$, which gives the first term. The number of $(m+c+1)$-tuples $(e'_1,\ldots,e'_c,e_0,e_1,\ldots,e_m)$ satisfying the second equality of (29) is $4^{\lceil m/2 \rceil + c/2+1}$, which gives the second term. For the third term, relabel the vertices $i_1,\ldots,i_{c/2},x_1,\ldots,x_m,i_{c/2+1},\ldots,i_c$ of the graph $G \in G_p(m+c)$ as $1,\ldots,m+c$ in that order. The mapping (30) then becomes

$$k \mapsto m+c+1-k \ \text{ for } 1 \le k \le m+c,$$

and so the number of graphs $G \in G_p(m+c)$ satisfying $G = G'$ is $t_p(m+c)$, which gives the third term of the product. Multiplication of these three terms establishes the claim. □

We can obtain explicit forms for the j-dimensional Golay array pairs counted in Theorem 14, by representing the graph $G \in G_{m+c-j}(m+c)$ using a permutation σ and applying Proposition 2 to the array pair (26); this array pair can itself be obtained explicitly by substituting the input Golay array pairs (a_k, b_k) taken from the set P into Theorem 7. The Golay array count given in Theorem 14 has been verified computationally for $1 \le j \le m+c$, where $(m,c) \in \{(1,1),(2,1),(3,1),(1,2),(2,2)\}$.

The case $j = 1$ of Theorem 14, giving Golay sequence pairs, is of particular interest:

Corollary 15. *Let m and c be integers satisfying $m \ge 1$ and $1 \le c \le m+1$. The number of (non-standard) Golay sequences of length 2^{m+3c} over $\mathbb{Z}_4$ that can be derived from affine offsets and projection mappings, after taking c of the $m+1$ input array pairs in Theorem 7 to be from the set P of Theorem 12 and the remaining $m+1-c$ input array pairs to be trivial, is*

$$2^{2m+5c+1}\binom{m+1}{c}(m+c)!,$$

and the corresponding number of ordered Golay sequence pairs is at least 8 times this number.

The case $c = 1$ of Corollary 15 gives a count of $2^{2m+6}(m+1)(m+1)!$ non-standard Golay sequences over $\mathbb{Z}_4$ and at least $2^{2m+9}(m+1)(m+1)!$ non-standard Golay pairs over $\mathbb{Z}_4$, in agreement with [7, Corollary 11]. The subcase of the case $c = 1$, in which the single cross-over Golay sequence pair is either the input (a_0, b_0) or the input (a_m, b_m) of Theorem 7, is related to the constructions for "near-complementary" sequences given by Parker and Tellambura [16, Theorem 6] and Schmidt [18, Theorem 7] (see [7, Section 8] for details).

We can combine the counts for Golay sequences and Golay sequence pairs given in Theorem 9 and Corollary 15:

Corollary 16. *Let $n > 3$ be an integer. There are at least*

$$\sum_{c=0}^{\lfloor (n+1)/4 \rfloor} 2^{2n-c+1}\binom{n-3c+1}{c}(n-2c)!$$

Golay sequences of length 2^n over $\mathbb{Z}_4$, and at least 8 times this number of Golay sequence pairs of length 2^n over $\mathbb{Z}_4$.

Proof. Write $n = m+3c$, where m and c are any integers satisfying $m \ge 1$ and $0 \le c \le m+1$ (and $m > 3$ if $c = 0$). There are $2^{2n-c+1}\binom{n-3c+1}{c}(n-2c)!$ Golay sequences of length 2^n over $\mathbb{Z}_4$, and at least 8 times as many Golay sequence pairs: use Corollary 15 for $c \ge 1$, and Theorem 9 for $c = 0$. We claim that any two Golay sequences arising from different values of c and c' are distinct. The required lower bound on the number of Golay sequences and Golay sequence pairs then follows by summing over the allowed values of c, noting that $n = m+3c \ge (c-1)+3c = 4c-1$; the condition $m = n-3c \ge 1$ imposes an extra constraint only when $n = 3$, which is excluded.

It remains to prove the claim. Let $\mathcal{F} = (f[i_1,\ldots,i_c,x_1,\ldots,x_m])$ be a Golay array of size $\mathbf{8}^{(c)} \times \mathbf{2}^{(m)}$ over $\mathbb{Z}_4$, obtained by taking exactly $c \ge 0$ of the $m+1$ input array pairs in Theorem 7 to be from the set P of Theorem 12 and the others to be trivial, followed by addition of an affine offset (as described in (26) for $c \ge 1$ and in (23) for $c = 0$). Similarly let $\mathcal{G}$ be a Golay array of size $\mathbf{8}^{(c')} \times \mathbf{2}^{(m')}$ over $\mathbb{Z}_4$ formed by taking exactly c' of $m'+1$ input array pairs in Theorem 7 to be from the set P and the others to be trivial, where $c' > c$ and $n = m+3c = m'+3c'$.

Suppose, for a contradiction, that the same Golay sequence of length 2^n is obtained under projection from $\mathcal{F}$ and under projection from $\mathcal{G}$. It follows from Proposition 2 that $\mathcal{G}$ can be obtained from $\mathcal{F}$ by applying $2(c'-c)$ successive projection mappings. Set $r := 3(c'-c-1)$ and reorder dimensions so that we can write $\mathcal{G} = (g[i_1,\ldots,i_{c'},x_{r+4},\ldots,x_m])$, where by Proposition 2

$$g[i_1,\ldots,i_c,4x_1+2x_2+x_3,\ldots,4x_{r+1}+2x_{r+2}+x_{r+3},x_{r+4},\ldots,x_m] = f[i_1,\ldots,i_c,x_1,\ldots,x_m]. \quad (31)$$

Now Theorem 7 associates an input array pair (a_k, b_k) with array arguments i_k and x_k (prior to removal of any dimensions equalling 1 and subsequent relabelling of the i_k). In the construction of $\mathcal{G}$ from Theorem 7, let the array argument $i := 4x_1 + 2x_2 + x_3$ be associated with an input array pair $(a, b) \in P$ and the array argument x_j, where $j \geq r + 4$. Write $\overline{g}[i, x_j]$ for the 8×2 array obtained from $g[i_1, \ldots, i_{c'}, x_{r+4}, \ldots, x_m]$ by setting all of the i_k except i to 0, and all of the x_k except x_j to 0. From Theorem 7 and (26) we have

$$\overline{g}[i, x_j] = (b[i] - a[i])x_j + a[i] + e'i + e_j x_j + e_0 \quad \text{for } 0 \leq i < 8 \text{ and } 0 \leq x_j < 2$$

for some affine offset variables e', e_j, e_0, so that

$$\overline{g}[i, 1] - \overline{g}[i, 0] = b[i] - a[i] + e_j \quad \text{for } 0 \leq i < 8. \tag{32}$$

Now let $\overline{f}[x_1, x_2, x_3, x_j]$ be the $2 \times 2 \times 2 \times 2$ array obtained from $f[i_1, \ldots, i_c, x_1, \ldots, x_m]$ by setting all of the i_k to 0, and all of the x_k except x_1, x_2, x_3, x_j to 0. By (31),

$$\overline{g}[4x_1 + 2x_2 + x_3, x_j] = \overline{f}[x_1, x_2, x_3, x_j]$$

and so

$$\overline{g}[4x_1 + 2x_2 + x_3, 1] - \overline{g}[4x_1 + 2x_2 + x_3, 0] = \overline{f}[x_1, x_2, x_3, 1] - \overline{f}[x_1, x_2, x_3, 0].$$

But the left hand side is quadratic in x_1, x_2 and x_3 for any cross-over pair $(a, b) \in P$ (as can easily be verified from (32) and the algebraic normal forms given in [7, Theorem 2]), whereas the right hand side is linear in x_1, x_2 and x_3 (by examination of Theorem 7). This gives the required contradiction. $\square$

The smallest value of n for which the minimum Golay sequence count in Corollary 16 exceeds the previously known minimum count [7, Table 1] is $n = 7$, for which the value $c = 2$ gives an additional $2^{14} \cdot 3!$ Golay sequences. The minimum Golay sequence and sequence pair counts in Corollary 16 are known by exhaustive search to be exact for lengths 2, 4, 8 and 16, but it is not currently known whether they are exact for larger lengths of the form 2^n.

8 Summary

We have argued that the natural viewpoint for a Golay complementary sequence is as a projection of a multi-dimensional Golay array. We have given a greatly simplified and completely elementary process for constructing Golay array and sequence pairs:

1. construct suitable Golay array pairs from lower-dimensional Golay array pairs using Theorem 7;
2. take affine offsets of these Golay array pairs using Lemma 8; and
3. take projections of the resulting Golay array pairs to lower dimensions, using a directed graph representation and Proposition 2.

We have used this process to construct new infinite families of Golay sequences of length 2^m over $\mathbb{Z}_4$. All Golay arrays and sequences of volume 2^m over $\mathbb{Z}_{2^h}$ obtainable under any known method can be constructed in this way (see Theorems 11 and 14). In particular we have found the full generalisation of [7, Example 6] that was sought in [7].

Howard, Calderbank and Moran [10] have recently given an alternative explanation for the existence of binary Golay sequences of length 2^m in terms of the $(2m + 1)$-dimensional discrete Heisenberg-Weyl group over the field $\mathbb{Z}_2$. Although this makes a connection between such Golay sequences and higher-dimensional structures, it does not mention Golay arrays, does not construct new Golay sequences, and is restricted to a binary alphabet. Our methods relate m-dimensional Golay arrays to Golay arrays of all lower dimensions, apply to any alphabet $\mathbb{Z}_H$ (where H is necessarily even), and construct and enumerate infinite new families of Golay array and sequence pairs.

We can also now answer a question posed in [6]: why do standard Golay sequences of length 2^m over $\mathbb{Z}_H$ occur as complete cosets of the first-order Reed-Muller code (that is, their algebraic normal form in the variables $x_1, \ldots, x_m$ contains a term $\sum_{k=1}^{m} e_k x_k + e_0$ for arbitrary $e_k \in \mathbb{Z}_H$), whereas the non-standard examples of Li and Chu [12] and their derived Golay sequences do not? The structural explanation sought in [6] is found in the form of the inputs to Theorem 7. We have seen in Theorem 9 and Corollary 10 that standard Golay sequence pairs can be constructed from trivial input array pairs, and that the affine offsets provided by Lemma 8 give complete first-order cosets under the mapping to algebraic normal form. In contrast, the known non-standard pairs are constructed from at least one cross-over Golay sequence pair of length 8 over $\mathbb{Z}_4$, which crucially cannot be represented as a $2 \times 2 \times 2$ (or even 4×2) Golay array. Application of Lemma 8, followed by the mapping to algebraic normal form, then gives an incomplete subset of the first-order Reed-Muller code.

The algebraic normal form of any length 8 sequence is expressed using three $\mathbb{Z}_2$ variables, and it is easy to check from [7, Theorem 2] that, for any cross-over Golay sequence pair (a, b) of length 8 given in Theorem 12, the expression $a + a^* - b - b^*$ is linear in these three variables. It follows from the form of Theorem 7 that all the non-standard Golay sequences counted in Corollary 15 (which are constructed using Theorem 7) have a cubic algebraic normal form. However, even in the case $c = 1$, this algebraic normal form is unwieldy (and rather uninformative) when written explicitly, as seen in [7, Theorem 10].

We conclude with some open questions:

1. How can the three-stage construction process of this paper be used to simplify or extend known results on the construction of Golay sequences in other contexts, such as 16-QAM modulation [1], a ternary alphabet $\{1, 0, -1\}$ [3], or quaternary sequences whose length is not a power of 2 [2]?

2. What underlies the shared autocorrelation property that gives rise to the length 8 quaternary cross-over Golay sequence pairs of Theorem 12? Are there further examples of Golay pairs of length 2^m over $\mathbb{Z}_{2^h}$ having the shared autocorrelation property? If so, this would allow the construction of further infinite families of non-standard Golay sequences and pairs via a new cross-over of autocorrelation functions.

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