Exclusivity graphs from quantum graph states - and mixed graph generalisations

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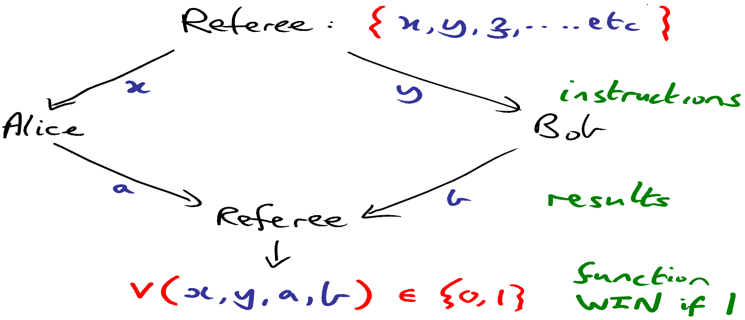
June 26, 2014

A game

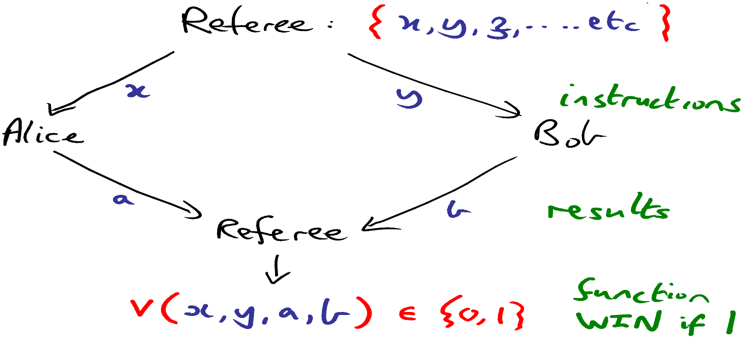
- Referee sends **instructions** to Alice and Bob.
- Alice and Bob **obey** instructions and **return results** to referee.
- Referee **computes function** of instructions and results. Returns 0 or 1. Win if result is 1.



A NONLOCAL game

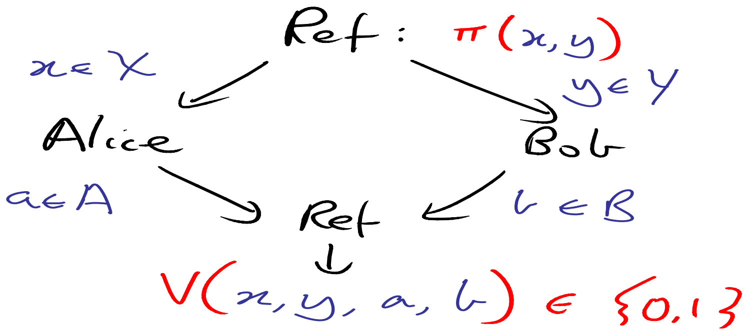
Alice and Bob are **nonlocal** to each other . . . i.e. cannot communicate instruction or result to each other.

Can Alice and Bob always win? . . . depends on game.



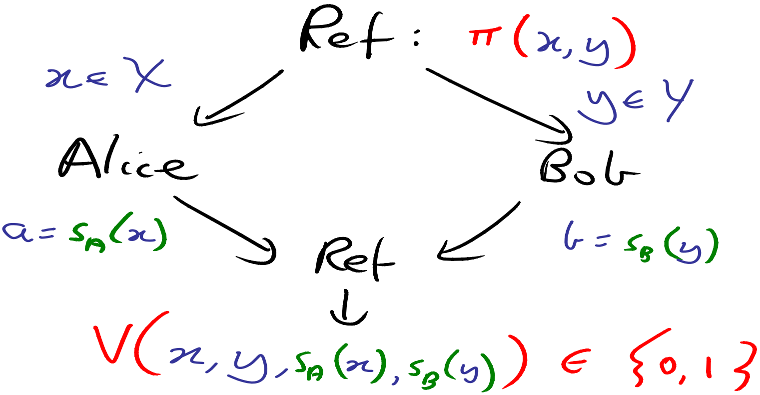
Nonlocality and pseudo-telepathy games

Referee sends input $x \in \mathcal{X}$ to Alice, $y \in \mathcal{Y}$ to Bob using prob. distr. $\pi(x, y)$.
Alice outputs $a \in A$, Bob outputs $b \in B$.
Referee computes $V(a, b, x, y) \in \{0, 1\}$. Declares 'win' if result is 1.



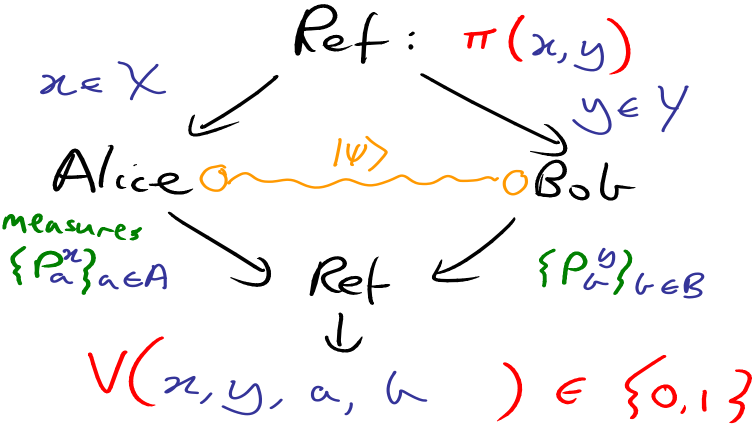
Classical version

Alice computes a from x using function $s_A : \mathcal{X} \to A$.
Bob computes b from y using function $s_B : \mathcal{Y} \to B$.

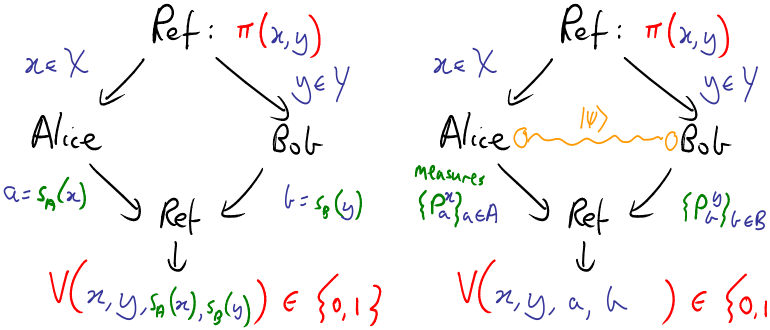


Quantum version

Alice/Bob share quantum state $|\psi\rangle$
Given x, Alice measures $\{P_a^x\}_{a\in A}$, outputs an $a \in A$.
Given y, Bob measures $\{P_b^y\}_{b\in B}$, outputs a $b \in B$.



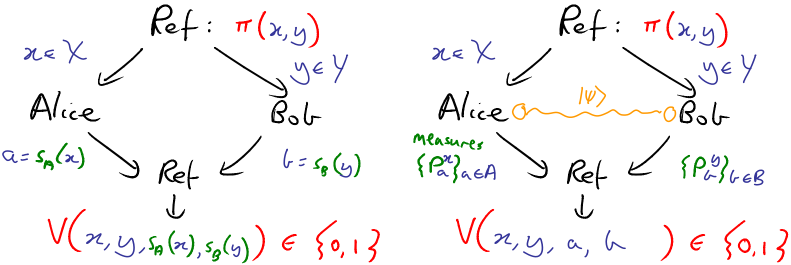
Classical vs quantum



Classical game: referee distributes **function inputs** to players.

Quantum game: referee distributes **measurement instructions** to players.

Classical vs. Quantum winning probs.



max. classical winning prob. is:

$$w_c = \max_{s_A, s_B} \sum_{x,y} \pi(x,y) V(x, y, s_A(x), s_B(y)).$$

max. quantum winning prob. is:

$$\begin{aligned} w_q &= \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x,y} \pi(x,y) V(x,y,a,b) \Pr(a,b|x,y) \\ &= \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x,y} \pi(x,y) V(x,y,a,b) \langle\psi| P_a^x \otimes P_b^y |\psi\rangle . \end{aligned}$$

Bell inequality for nonlocal game

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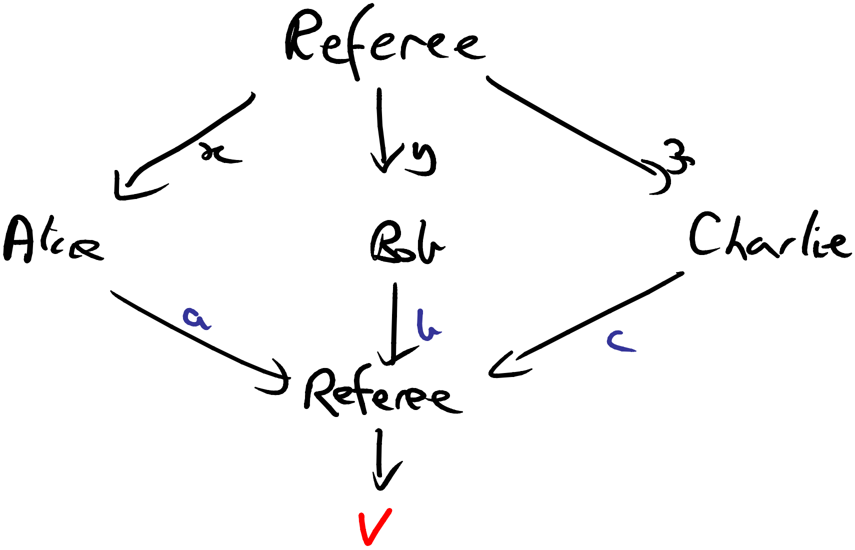
Bell inequality: $w_c \leq t$, $t \in [0,1]$.

Violated by quantum mechanics if: $w_q > t$.

Pseudo-telepathy game if: $w_c < w_q = 1$.

A 3-player Game



A 3-player Game

Choose $V = 1 - \log_{-1}(s_A(x)s_B(y)s_C(z)Q(x,y,z))$, where $s_A, s_B, s_C, Q \in \{1, -1\}$.

Instructions/measurements for Alice, Bob, Charlie from $\{I, X, Y, Z\}$. I returns 1 (don't measure).

Referee instruction set, each with prob. $\frac{1}{7}$: $\{XZZ, ZXI, YYZ, ZIX, YZY, IXX, XYY\}$. Choose Q so that $Q(XZZ) = Q(ZXI) = Q(YYZ) = Q(ZIX) = Q(YZY) = Q(IXX) = 1$, and $Q(XYY) = -1$.

Classical challenge: How do Alice, Bob, Charlie choose s_A, s_B, s_C so that $V = 1$ always?
Answer: **Impossible**

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Try some examples

We have: $Q(XZZ) = Q(ZXI) = Q(YYZ) = Q(ZIX) = Q(YZY) = Q(IXX) = 1$, and $Q(XYY) = -1$.

So how to choose

	X	Y	Z
s_A	?	?	?
s_B	?	?	?
s_C	?	?	?

so that $s_A s_B s_C Q = 1$ always? ... impossible. But quantum game always wins.

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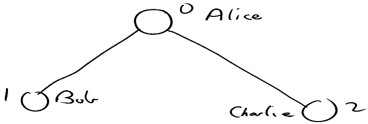
Game uses a 3-qubit quantum graph state

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = iXZ$, $i = \sqrt{-1}$. ... (Pauli matrices)

Let the graph state $|G\rangle$ be the unique joint eigenvector of operators $X \otimes Z \otimes Z$, $Z \otimes X \otimes I$, and $Z \otimes I \otimes X$.

Represent $|G\rangle$ by a 3-vertex graph, G, with adjacency matrix $\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} : \begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix}$



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$|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0 x_1 + x_0 x_2} = (1, 1, 1, -1, 1, -1, 1, 1)$.

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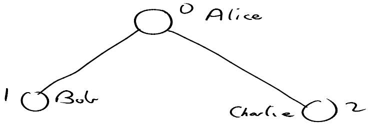

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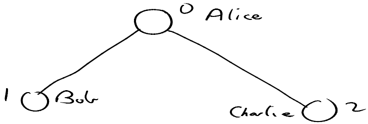

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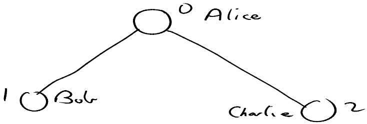

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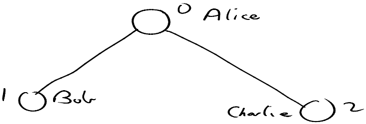

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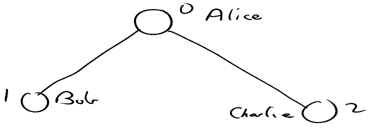

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Operator code

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So,

$$(X \otimes Z \otimes Z)|G\rangle = (Z \otimes X \otimes I)|G\rangle = (Z \otimes I \otimes X)|G\rangle = |G\rangle .$$

But also, for instance,

$$(X\otimes Z\otimes Z)(Z\otimes X\otimes I)|G\rangle = (Y\otimes Y\otimes Z)|G\rangle = |G\rangle ,$$

and

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Operator code $\leftrightarrow$ $\mathbb{F}_4$-additive code

$|G\rangle$ is 'stabilized' by following 'code' of operators:

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Remember for our game $Q(XZZ) = Q(ZXI) = Q(ZIX) = Q(YYZ) = Q(YZY) = Q(IXX) = 1$, and $Q(XYY) = -1$

Operator code can be represented by self-dual $\mathbb{F}_4$-additive code:

$$\begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix} \leftrightarrow \begin{pmatrix} w & 1 & 1 \\ 1 & w & 0 \\ 1 & 0 & w \end{pmatrix},$$

$w^2 = w + 1, w \in \mathbb{F}_4.$

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Measuring graph states

Let $|+\rangle = \frac{1}{\sqrt{2}}(1,1)$, $|-\rangle = \frac{1}{\sqrt{2}}(1,-1)$ be orthogonal eigenvectors of X with eigenvalues 1 and -1, resp.

Let $|0\rangle = (1,0)$, $|1\rangle = (0,1)$ be orthogonal eigenvectors of Z with eigenvalues 1 and -1, resp.

Let $|y_+\rangle$, $|y_-\rangle$ be orthogonal eigenvectors of Y with eigenvalues 1 and -1, resp.

So $X|+\rangle = Z|0\rangle = Y|y_+\rangle = 1$, and
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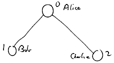
Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0 x_1 + x_0 x_2} =$ 

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$|G\rangle$ is 'stabilized' by operator code:

$$III, XZZ, ZXI, ZIX, YYZ, YZY, IXX, -XYY.$$

Measure XZZ on $|G\rangle$ means measure X on qubit 0, Z on qubit 1, Z on qubit 2.

Collapses $|G\rangle$ to one of:

$$|+\rangle \otimes |0\rangle \otimes |0\rangle, \qquad |-\rangle \otimes |1\rangle \otimes |0\rangle$$
$$|-\rangle \otimes |0\rangle \otimes |1\rangle, \qquad |+\rangle \otimes |1\rangle \otimes |1\rangle.$$

The four resultant vectors are pairwise orthogonal.

In all four cases, product of qubit eigenvalues is 1.

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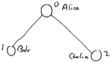
Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_0x_2} =$ 

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$$|+\rangle \otimes |0\rangle \otimes |0\rangle, \qquad |-\rangle \otimes |1\rangle \otimes |0\rangle$$
$$|-\rangle \otimes |0\rangle \otimes |1\rangle, \qquad |+\rangle \otimes |1\rangle \otimes |1\rangle.$$

The four resultant vectors are pairwise orthogonal.
In all four cases, product of qubit eigenvalues is 1.

Measuring XZZ on $|G\rangle$

If we measure XZZ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$,
then eigenvalue is $1 \times 1 \times 1 = 1$.
The event is: xzz.

If, instead, we obtain $|-\rangle \otimes |1\rangle \otimes |0\rangle$,
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The event is: $\underline{x}\underline{z}z$.

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Why four exclusive events??

For our example Alice, Bob, and Charlie measure XZZ and obtain one of:

$xzz, \underline{xz}z, \underline{x}z\underline{z}, x\underline{zz}$.

…e.g. xzz means that Alice, Bob, and Charlie all measure 1. But if, say, $\underline{xz}z$ is measured then Alice and Bob both measured -1, and Charlie measured 1. Classically it is **impossible** for both scenarios to be true, but quantumly it is possible.

We make a big graph whose vertices are all possible events and with edges between exclusive events, e.g. an edge between vertex xzz and vertex $\underline{xz}z$.

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A big graph H from a small graph G

Make a graph from all possible events resulting from measuring $|G\rangle$ with stabilizing operators.

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_0x_2}$. Operator code is:

$III, XZZ, ZXI, ZIX, YYZ, YZY, IXX, -XYY$.

Construct big graph $H(G)$ with 22 vertices:

$$
\begin{aligned}
&xzz, \underline{xz}z, \underline{x}z\underline{z}, x\underline{zz},\\
&zxI, \underline{zx}I,\\
&zIx, \underline{z}I\underline{x},\\
&yyz, \underline{yy}z, \underline{y}y\underline{z}, y\underline{y}\underline{z},\\
&yzy, \underline{y}\underline{z}y, \underline{y}z\underline{y}, y\underline{z}\underline{y},\\
&Ixx, I\underline{xz},\\
&\underline{x}yy, x\underline{y}y, xy\underline{y}, \underline{x}\underline{yy}.
\end{aligned}
$$

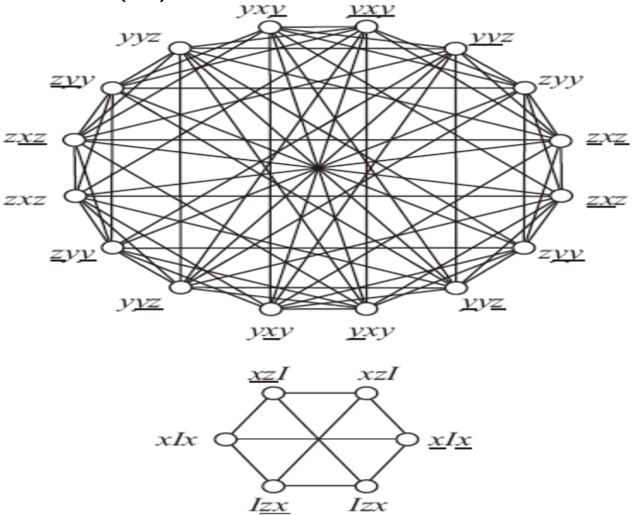
Edges between *mutually exclusive* events.

e.g. $xzz - \underline{xz}z$ and $xzz - \underline{y}\underline{z}y$.

Example big graph

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0 x_1 + x_1 x_2}$.

Then $H(G)$ is:



Another drawing for same graph

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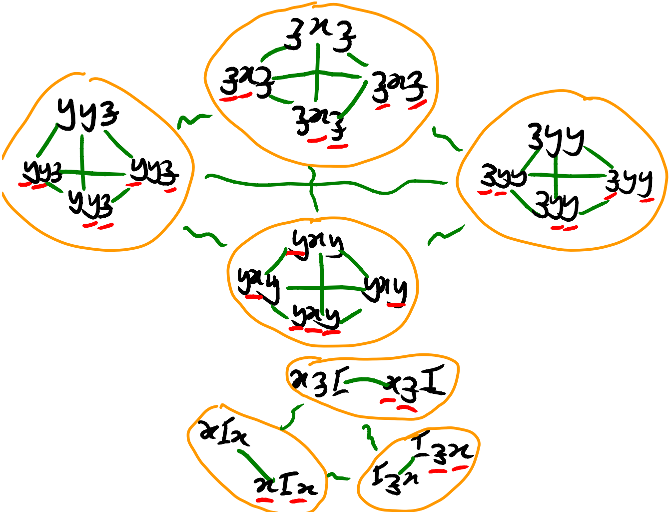
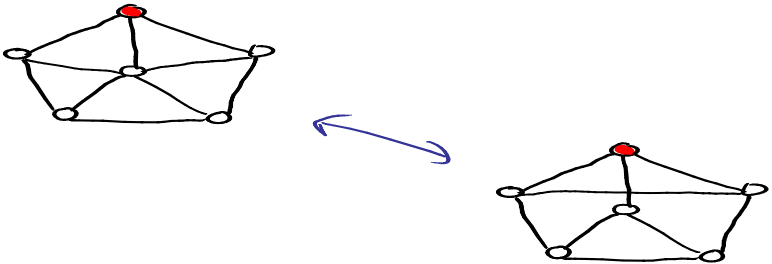


Big graph H invariant over LC orbit of G

Local complementation (LC) at a vertex, v, of G complements the edges between the neighbours of G, e.g.



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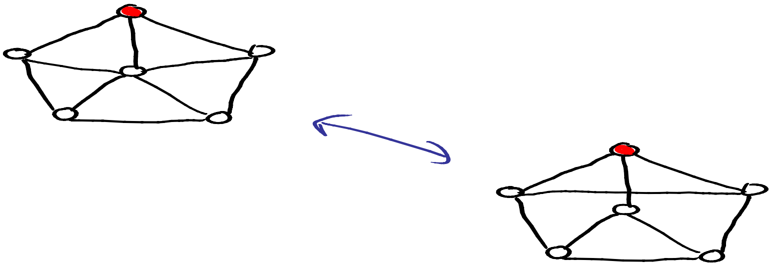

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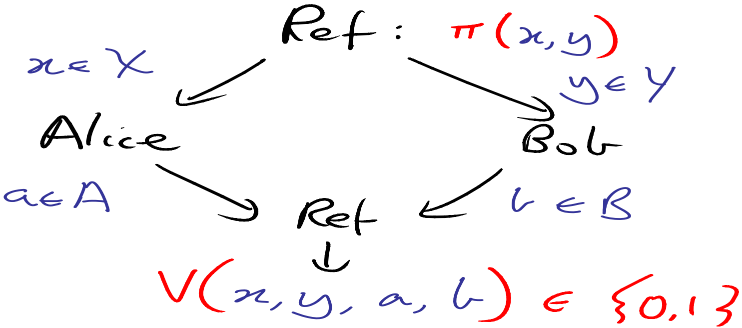
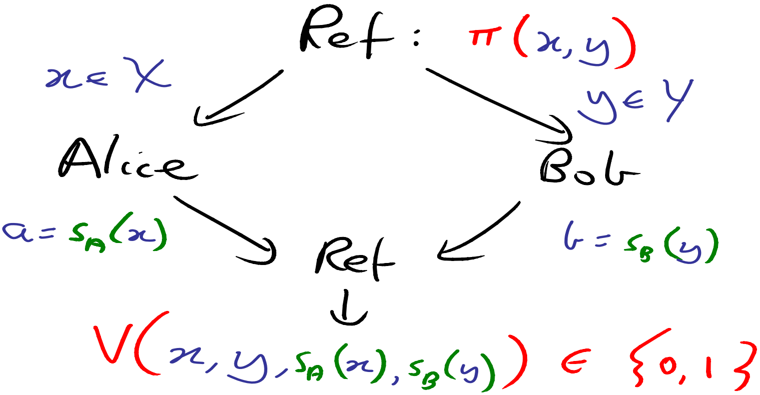
Reminder: Nonlocality and pseudo-telepathy games

Referee sends input $x \in \mathcal{X}$ to Alice, $y \in \mathcal{Y}$ to Bob using prob. distr. $\pi(x, y)$.
Alice outputs $a \in A$, Bob outputs $b \in B$.
Referee computes $V(a, b, x, y) \in \{0, 1\}$. Declares 'win' if result is 1.



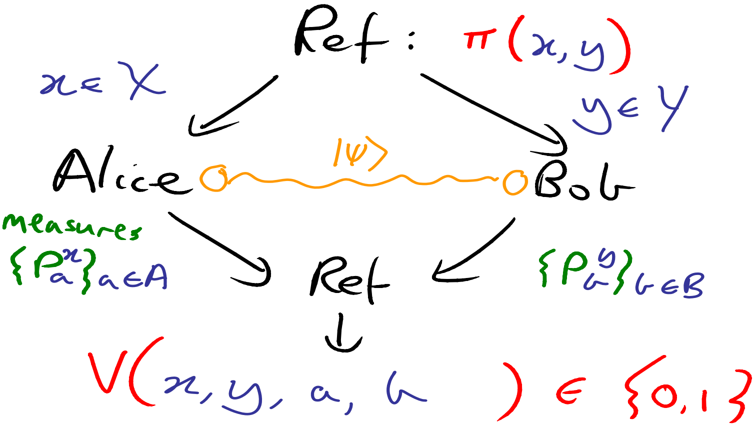
Classical version

Alice computes a from x using function $s_A : \mathcal{X} \to A$.
Bob computes b from y using function $s_B : \mathcal{Y} \to B$.

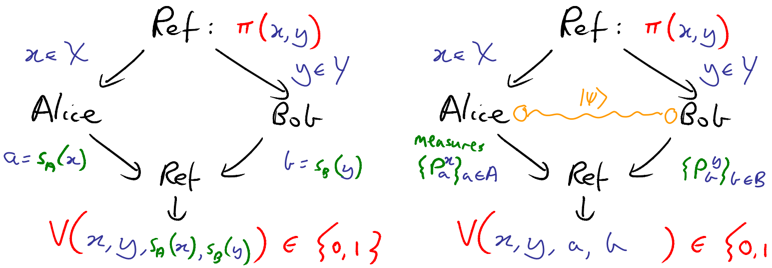


Quantum version

Alice/Bob share state $|\psi\rangle$
Given x, Alice measures $\{P_a^x\}_{a \in A}$, outputs an $a \in A$.
Given y, Bob measures $\{P_b^y\}_{b \in B}$, outputs a $b \in B$.



Classical vs. Quantum winning probs.



max. classical winning prob. is:

$$w_c = \max_{s_A, s_B} \sum_{x,y} \pi(x,y) V(x,y,s_A(x),s_B(y)).$$

max. quantum winning prob. is:

$$w_q = \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x,y} \pi(x,y) V(x,y,a,b) \langle\psi| P_a^x \otimes P_b^y |\psi\rangle .$$

Bell inequality for nonlocal game

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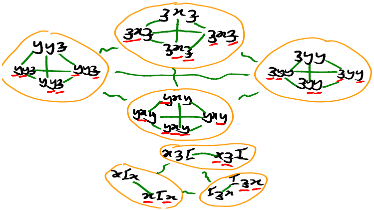
Bell inequality: $w_c \leq t$, $t \in [0,1]$.

Violated by quantum mechanics if: $w_q > t$.

Pseudo-telepathy game if: $w_c < w_q = 1$.

Pseudo-telepathy using big graph

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0 x_1 + x_1 x_2}$. So $H(G)$ is:



$n = 3$ players. Set of instructions is $\{I, X, Y, Z\}$ for each player. Ref sends one of $XZI, ZXZ, YYZ, IZX, XIX, ZYY, YXY$ as defined by $\pi(\{I, X, Y, Z\}^3)$. $Q = 1$ except $Q(YXY) = -1$.
V is product of 3 measurement results $\times$ Q.
Win is 1.

Pseudo-telepathy using big graph

Optimal classical strategy: choose max. size independent set in $H(G)$.

e.g. $zxz, yyz, yx\underline{y}, xzI, xIx, Izx$ - size 6. Then:

s_A: $X \to 1, Y \to 1, Z \to 1$

s_B: $X \to 1, Y \to 1, Z \to 1$

s_C: $X \to 1, Y \to -1, Z \to 1.$

Product of function results $\times$ Q is 1 but if referee sends ZYY then

$s_A(Z)s_B(Y)s_C(Y)Q(ZYY) = 1 \times 1 \times -1 \times 1 = -1.$

But joint quantum measurement gives either $1 \times 1 \times 1 \times 1$ or $-1 \times -1 \times 1 \times 1$ or $-1 \times 1 \times -1 \times 1$ or $1 \times -1 \times -1 \times 1$, so result **always** 1.

So $w_c = \frac{6}{2^3-1} = \frac{6}{7}$, $w_q = \frac{2^3-1}{2^3-1} = 1$.

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Important properties of big graph

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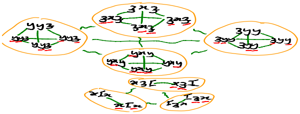

Max independent set size: $\alpha(H(G)) = 6$.

Lovasz number: $\vartheta(H(G)) = 2^n - 1 = 2^3 - 1$.

$$= \max \sum_{i=0}^{n-1} |\langle\psi| |v_i\rangle|^2,$$

max taken over all unit vectors ψ and all orthogonal representations $\{v_i\}$ of $H(G)$ - orth. representation maps adjacent vertices in $H(G)$ to orth. vectors.

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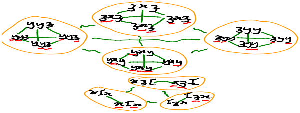

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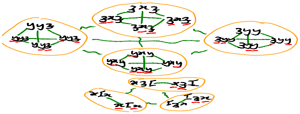

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Proof that $\vartheta(H(G)) \geq 2^n - 1$

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_1x_2}$.
Let $S = \{ZXZ, YYZ, YXY, XZI, XIX, IZX, ZYY\}$
and $s_i \in S$.
Eigendecomposition: $s_i = \sum_j \lambda_{ij} | \, |s_{i,j}\rangle \langle s_{i,j}| \, |$.

$$\begin{aligned}
2^n - 1 &= \sum_{i=1}^{2^n-1} \langle G|s_i|G\rangle \\
&= \sum_{i=1}^{2^n-1} \sum_j \lambda_{ij} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\rangle \\
&= \sum_{i=1}^{2^n-1} \sum_{j:\lambda_{ij}=1} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\rangle \\
&= \sum_{i=1}^{2^n-1} \sum_{j:\lambda_{ij}=1} |\langle G|s_{(i,j)}\rangle|^2 \\
&\leq \vartheta(H),
\end{aligned}$$

Fractional packing number of $H(G)$

Fractional packing number of H is given by:

$$\alpha^*(H(G)) = \max \sum_{i \in V(H)} w_i,$$

where max is over $0 \leq w_i \leq 1$ restricted by $\sum_{i \in C_j} w_i \leq 1$, for all cliques, $C_j \in H(G)$.

If no of vertices of G is n then

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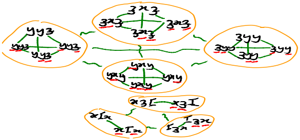

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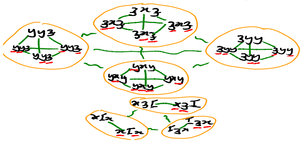

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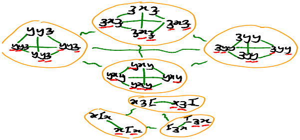

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The property $\alpha(H(G)) < \vartheta(H(G))$ explains why we have a nonlocality game for $|G\rangle$.
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...and we know that $\vartheta(H(G)) \geq 2^n - 1$ and $\alpha^*(H(G)) = 2^n - 1$.

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A generalisation to mixed graph states

A graph state, $|G\rangle$, is a joint eigenvector, i.e. it is **stabilised** by each operator row of the operator code, e.g. for our 3-qubit example, the operator code is generated by: $\begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix}$

$|G\rangle$ only exists because operators **fully commute** with each other. For instance,
$(X \otimes Z \otimes Z)(Z \otimes I \otimes X) = (Z \otimes I \otimes X)(X \otimes Z \otimes Z)$
... and the same for any pair of rows.

Always true when **symmetric** matrix with X on the diagonal and $\{I, Z\}$ off it.

... but what about when matrix is not symmetric??

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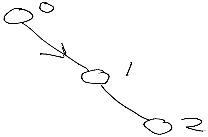
Example mixed graph

Consider $\begin{pmatrix} X & Z & I \\ I & X & Z \\ I & Z & X \end{pmatrix}$: mixed graph:



$(X \otimes Z \otimes I)(I \otimes X \otimes Z) = -(I \otimes X \otimes Z)(X \otimes Z \otimes I)$
so first two rows **anti-commute**.
So $|G\rangle$ doesn't exist. So embedd non-commuting matrix in larger commuting matrix. For example, embedd 3×3 in 4×4:

$$\left(\begin{array}{ccc|c} X & Z & I & Z \\ I & X & Z & X \\ I & Z & X & I \\ Z & I & I & X \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} X & Z & I & Z \\ Z & X & Z & I \\ I & Z & X & I \\ Z & I & I & X \end{array}\right)$$

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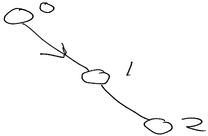

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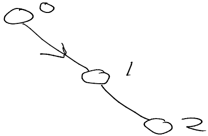

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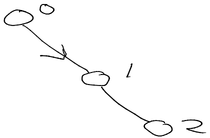

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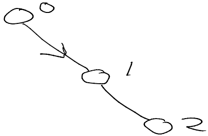

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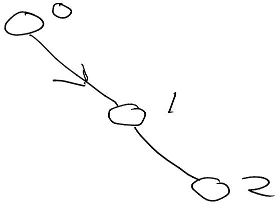
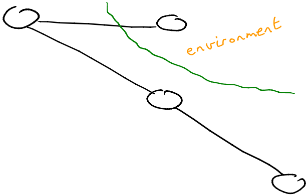
mixed graph extended to graph

$$\begin{pmatrix} X & Z & I \\ I & X & Z \\ I & Z & X \end{pmatrix}$$: mixed graph:



. . . extended to . . .

$$\left(\begin{array}{ccc|c} X & Z & I & Z \\ I & X & Z & X \\ I & Z & X & I \\ Z & I & I & X \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} X & Z & I & Z \\ Z & X & Z & I \\ I & Z & X & I \\ Z & I & I & X \end{array}\right)$$: graph:



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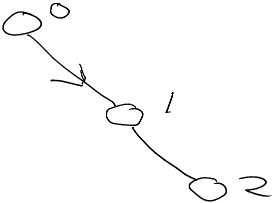

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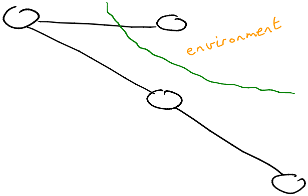
...extended to ...

$$\left(\begin{array}{ccc|c} X & Z & I & Z \\ I & X & Z & X \\ I & Z & X & I \\ Z & I & I & X \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} X & Z & I & Z \\ Z & X & Z & I \\ I & Z & X & I \\ Z & I & I & X \end{array}\right)$$: graph:



The game?

Alice, Bob, Charlie receive instructions from the non-commuting operator code: $\left(\begin{array}{ccc|c} X & Z & I & Z \\ I & X & Z & X \\ I & Z & X & I \\ Z & I & I & X \end{array}\right)$.

Instructions are $XZI, IXZ, XYZ, IZX, XIX, IYY, XXY$ with $Q = 1$ apart from $Q(XYZ) = -1$.

...

...to be continued ...