Generalised complementary arrays

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Some results

For A a length-N complex sequence:

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For $V = (1, \beta, \beta^2, \ldots, \beta^{N-1})$, $|\beta| = 1$

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$(1+r^2+\ldots+r^{2(N-1)})^{-\frac{1}{2}}(1, ir, -r^2, \ldots, i^{N-1}r^{N-1})$, $r \in \mathbb{R}$

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then,

$$|\langle V, A\rangle|^2 \leq 2.$$

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then, $|\langle V, A\rangle|^2 \leq 2$.

Results generalised to arrays,

complementary sets,

... and further generalisations.

Length-4 BPSK sequence:

$$A = 1, 1, 1, -1$$

Aperiodic autocorrelation of A:

$$\begin{array}{ccccccc} & & & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & & & \end{array} \quad = -1$$

Aperiodic autocorrelation of A:

$$\begin{array}{cccccc} & & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & & \end{array} = 0$$

$-1,$

Aperiodic autocorrelation of A:

$$\begin{array}{ccccc} & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & \end{array} = 1$$

$-1, 0,$

Aperiodic autocorrelation of A:

$$\begin{array}{cccc} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{array} = 4$$

$-1, 0, 1$

Aperiodic autocorrelation of A:

$$
\begin{array}{ccccc}
1 & 1 & 1 & -1 & \\
 & 1 & 1 & 1 & -1
\end{array}
= 1
$$

$-1, 0, 1, 4$

Aperiodic autocorrelation of A:

$$\begin{array}{cccccccc} 1 & 1 & 1 & -1 & & & & \\ & & 1 & 1 & 1 & -1 & = & 0 \end{array}$$

$-1, 0, 1, 4, 1$

Aperiodic autocorrelation of A:

1 1 1 −1

1 1 1 −1 = −1

−1, 0, 1, 4, 1, 0

Aperiodic autocorrelation of A:

$$\mathrm{Aut}(A) =$$

$-1, 0, 1, 4, 1, 0, -1$

Aperiodic autocorrelation

$\equiv$

Polynomial multiplication

$A = 1, 1, 1, -1$

$\mathrm{Aut}(A) = -1, 0, 1, 4, 1, 0, -1$

$\equiv$

$A(z) = 1 + z + z^2 - z^3$

$A(z)\overline{A(z^{-1})} = -z^{-3} + z^{-1} + 4 + z - z^3$

Fourier transform(A)

$$\equiv$$

Evaluate $A(z)$ on unit circle

Fourier transform(A)

$\equiv$

Evaluate $A(z)$ on unit circle

$= A(z = \beta), \ |\beta| = 1.$

Fourier transform(A)

$$\equiv$$

Evaluate $A(z)$ on unit circle

e.g.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 1 \\ -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A(1) \\ A(i) \\ A(-1) \\ A(-i) \end{bmatrix}$$

Fourier transform(A)

$$\equiv$$

Evaluate $A(z)$ on unit circle

More generally,

$$\begin{bmatrix} 1 & \beta & \beta^2 & \beta^3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = A(\beta) \quad , \quad |\beta| = 1$$

Fourier power spectrum (A)

$$\equiv$$

Evaluate $A(z)\overline{A(z^{-1})}$ on unit circle

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Why?

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$$\equiv$$

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Why?

power $\text{spectrum}_{\beta}(A) = |A(\beta)|^2$

Fourier power spectrum(A)

$\equiv$

Evaluate $A(z)\overline{A(z^{-1})}$ on unit circle

Why?

power spectrum$_\beta(A) = |A(\beta)|^2$

$= A(\beta)A^*(\beta)$

Fourier power spectrum(A)

$$\equiv$$

Evaluate $A(z)\overline{A(z^{-1})}$ on unit circle

Why?

power spectrum$_\beta(A) = |A(\beta)|^2$

$$= A(\beta)A^*(\beta) = A(\beta)\overline{A(\beta^{-1})}.$$

Let

$$\lambda(z) = A(z)A^*(z)$$

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If $\lambda(z)$ is independent of z

i.e. $\lambda(z) = 0 \cdot z^{-k} + 0 \cdot z^{1-k} + \dots + 0 z^{-1} + c$
$+ 0 z + \dots + 0 z^{k-1} + 0 \cdot z^{k}$

Let $$\lambda(z) = A(z)A^*(z)$$

If $\lambda(z)$ is independent of z

then $\lambda(z) = c$, $c \in \mathbb{R}$.

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then

$$\text{power spectrum}_\beta(A) = \lambda(\beta) = c, \; \forall \beta$$

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$$\text{power spectrum}_\beta(A) = \lambda(\beta) = c, \ \forall \beta$$

i.e. CONSTANT POWER SPECTRUM

Let $\lambda(z) = A(z) A^*(z)$

If $\lambda(z)$ is independent of z

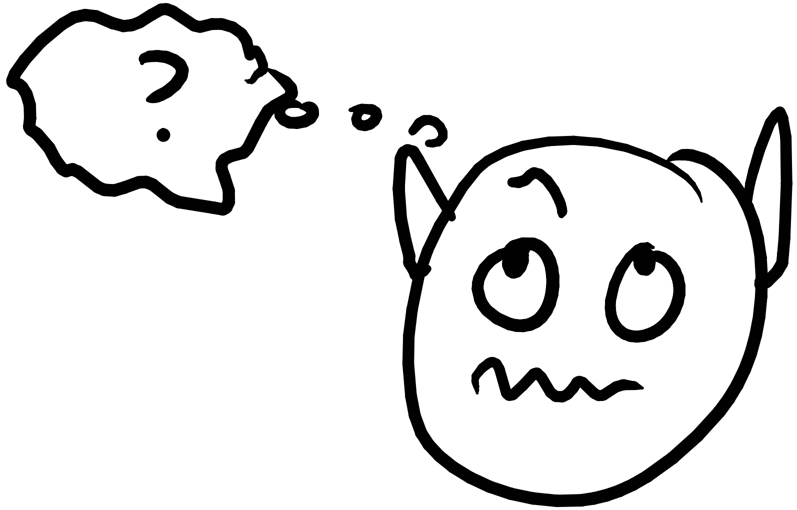
then $\lambda(z) = c$, $c \in \mathbb{R}$.

then

$$\text{power spectrum}_\beta(A) = \lambda(\beta) = c, \ \forall \beta$$

i.e. FLAT POWER SPECTRUM

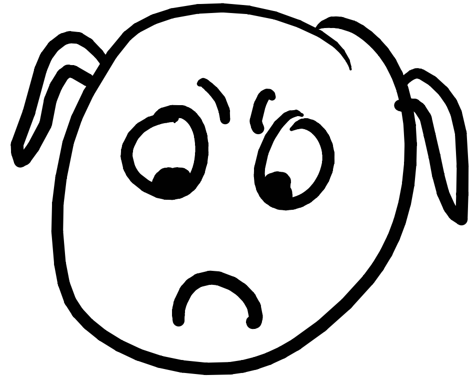
WISH



Find $A(z)$

Such that $A(z)A^*(z) = c$

PROBLEM

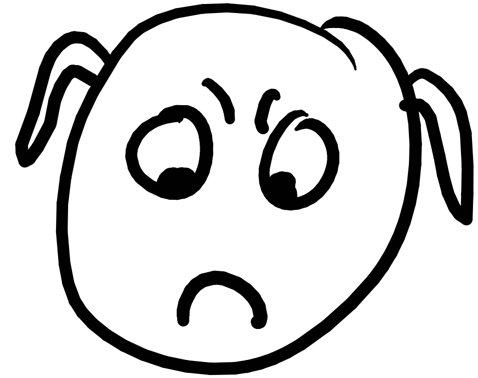


Find $A(z)$

such that $A(z)A^*(z) = c$

IMPOSSIBLE for $\text{degree}(A) > 0$

PROBLEM

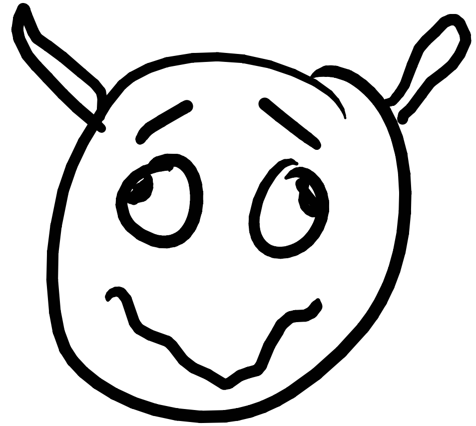


Find $A(z)$

such that $A(z)A^{*}(z) = C$

IMPOSSIBLE

SOLUTION

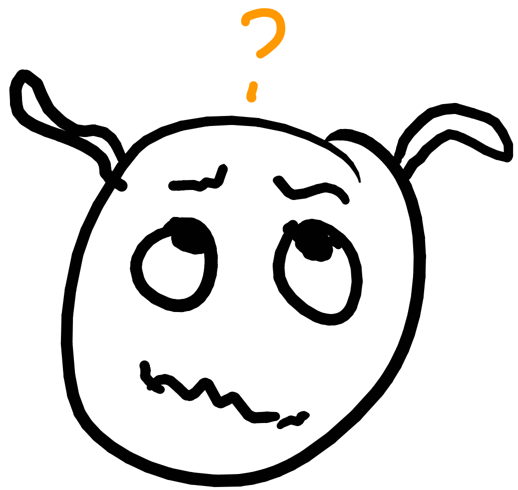


$\exists\ A(z)$ and $B(z)$

such that $A(z)A^*(z) + B(z)B^*(z)$

$= \lambda_A(z) + \lambda_B(z) = C$

USEFUL?

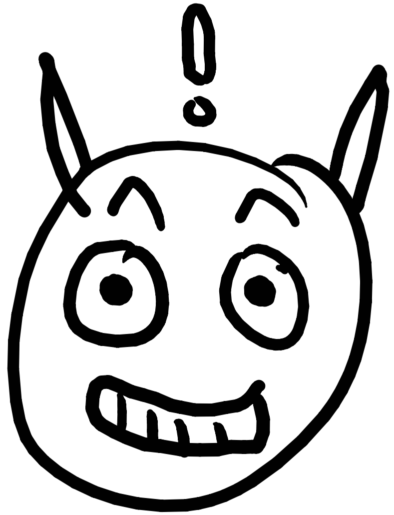


If $\exists A(z), B(z)$ such that

$$\lambda_{AB}(z) = \lambda_A(z) + \lambda_B(z) = c,$$

Then

?



YES!

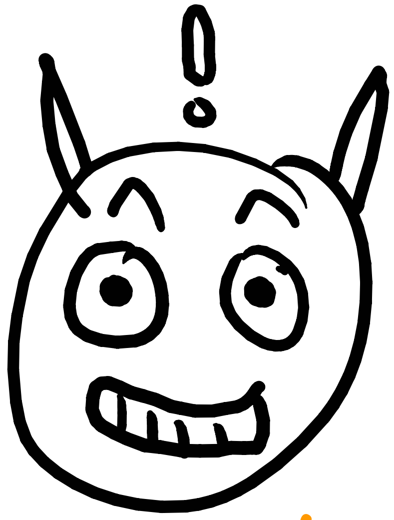
If $\exists A(z), B(z)$ such that

$$\lambda_{AB}(z) = \lambda_A(z) + \lambda_B(z) = c,$$

Then

$$\lambda_A(\beta) \leq c, \quad \forall \beta$$

YES!



If $\exists A(z), B(z)$ such that

$$\lambda_{AB}(z) = \lambda_A(z) + \lambda_B(z) = c,$$

constant, c, is positive

Then

$$\lambda_A(\beta) \leq c, \quad \forall \beta$$

Example

$$A(z) = 1 + z + z^2 - z^3$$

$$B(z) = 1 + z - z^2 + z^3$$

Example

$$A(z) = 1 + z + z^2 - z^3$$
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$$\chi_A(z) = -z^{-3} + z^{-1} + 4 + z + z^3$$
$$\chi_B(z) = z^{-3} - z^{-1} + 4 - z + z^3$$

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$$\lambda_{AB}(z) = \qquad\qquad 8$$

Example

$$A(z) = 1 + z + z^2 - z^3$$

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(A, B) a complementary pair

Example

$$A(z) = 1 + z + z^2 - z^3$$
$$B(z) = 1 + z - z^2 + z^3$$

$$\lambda_{AB}(z) = \qquad 8$$

(A, B) a complementary pair

Shapiro/Golay (1948/49)

Bivariate Version (Array version)

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$
$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

Bivariate Version (Array version)

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\lambda_{AB}(z_0, z_1) = A(z_0, z_1) A^*(z_0, z_1) + B(z_0, z_1) B^*(z_0, z_1)$$

$$= 8$$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{cccc} 1 & 1 & & \\ 1 & -1 & 1 & 1 \\ & & 1 & -1 \end{array} = -1 \quad + \quad \begin{array}{cccc} 1 & 1 & & \\ -1 & 1 & 1 & 1 \\ & & -1 & 1 \end{array} = 1$$

$$= 0$$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{cc} 1 & 1 \\ 1\;1 & -1\;1 \\ 1 & -1 \end{array} = 0 \quad + \quad \begin{array}{cc} 1 & 1 \\ -1\;1 & 1\;1 \\ -1 & 1 \end{array} = 0$$

$$= 0$$

0

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{ccccc} & 1 & 1 & & \\ 1 & 1\,1 & -1 & = & 1 \\ 1 & -1 & & & \end{array} \qquad\qquad \begin{array}{ccccc} & 1 & 1 & & \\ 1 & -1\,1 & 1 & = & -1 \\ -1 & 1 & & & \end{array}$$

$$= 0$$

$0,0$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{matrix} = 0 \qquad \begin{matrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{matrix} = 0$$

$$= 0$$

$$0, 0, 0$$

$$A(z_0,z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0,z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{cc} 1\,1 & 1\,1 \\ 1\,1 & -1\,-1 \end{array} = 4 \qquad \begin{array}{cc} 1\,1 & 1\,1 \\ -1\,-1 & 1\,1 \end{array} = 4$$

$$= 8$$

$0, 0, 0, 0$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{matrix} 1 & 1\;1 & 1 \\ 1 & 1\;{-1} & -1 \end{matrix} = 0 \qquad\qquad \begin{matrix} 1 & 1\;1 & 1 \\ -1 & -1\;1 & 1 \end{matrix} = 0$$

$$= 0$$

$0, 0, 0, 0, 8$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{cccc} & 1 & & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & & \end{array} = 1 \qquad \begin{array}{cccc} & 1 & & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & & \end{array} = -1$$

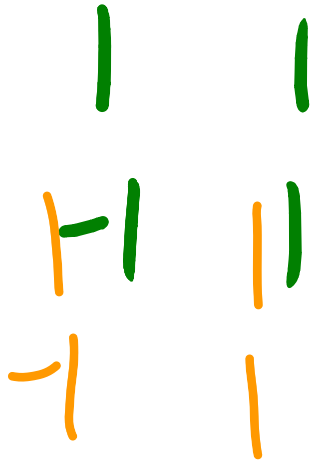
$$= 0$$

0, 0, 0, 0, 8, 0

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$= 0$



$= 0$

$= 0$

0, 0, 0, 0, 8, 0, 0

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\begin{array}{ccc} 1 & 1 & = -1 \\ 1 & (-1) & 1 \\ & 1 & -1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & = 1 \\ -1 & (1) & 1 \\ & -1 & 1 \end{array}$$

$= 0$

$0, 0, 0, 0, 8, 0, 0, 0$

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

$$\lambda_{AB}(z_0, z_1) = 8$$

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$$\lambda_{AB}(z_0, z_1) = 8$$

Complementary pair of arrays

$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$

$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$

Dymond

Parker, Tellambura

Matsufuji et al

Borwein, Ferguson

Fiedler et al.

$$\lambda_{AB}(z_0, z_1) = 8$$

Complementary pair of arrays

IMPORTANT

IMPORTANT

Array pair

$\Rightarrow$

Sequence pair

IMPORTANT

Example:

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

$$B(z_0, z_1) = 1 + z_0 - z_1 + z_0 z_1$$

IMPORTANT

Example:

$$A(z_0, z_1) = 1 + z_0 + z_1 - z_0 z_1$$

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Assign $z_1 = z_0^2$

IMPORTANT

Example:

$$A(z_0,z_1) = 1+z_0+z_1-z_0z_1$$

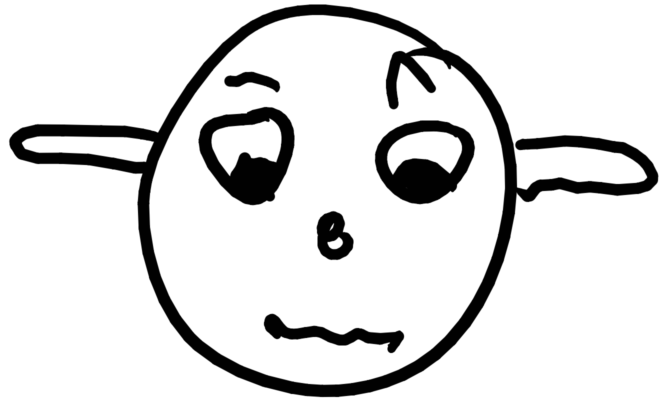
$$B(z_0,z_1) = 1+z_0-z_1+z_0z_1$$

Assign $z_1 = z_0^2 \Rightarrow$

$$A(z_0,z_0^2) = A(z_0) = 1+z_0+z_0^2-z_0^3$$

$$B(z_0,z_0^2) = B(z_0) = 1+z_0-z_0^2+z_0^3$$

IMPORTANT



Example:

$$\left.\begin{aligned} A(z_0, z_1) &= 1 + z_0 + z_1 - z_0 z_1 \\ B(z_0, z_1) &= 1 + z_0 - z_1 + z_0 z_1 \end{aligned}\right\} \text{array pair}$$

$$\Rightarrow \text{sequence pair} \left\{\begin{aligned} A(z_0) &= 1 + z_0 + z_0^2 - z_0^3 \\ B(z_0) &= 1 + z_0 - z_0^2 + z_0^3 \end{aligned}\right.$$

More generally:

Example:

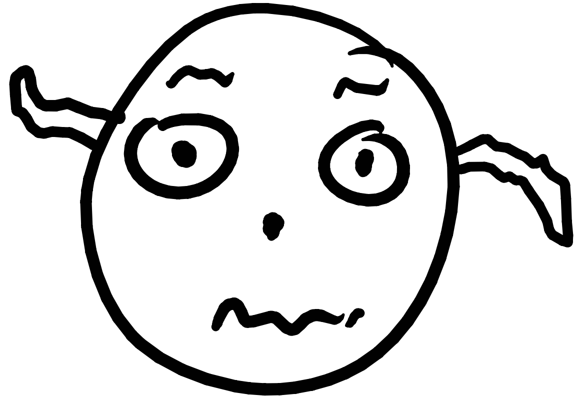
$$\left.\begin{aligned} A(z_0,z_1) &= 1+z_0+z_1-z_0z_1 \\ B(z_0,z_1) &= 1+z_0-z_1+z_0z_1 \end{aligned}\right\} \text{array pair}$$

Assign $z_1 = \beta z_0^k \Rightarrow$

$$\left.\begin{aligned} A(z_0,\beta z_0^k) &= A(z_0) \\ B(z_0,\beta z_0^k) &= B(z_0) \end{aligned}\right\} \text{sequence pair}$$

Why $z_1 = \beta z_0^k$, $|\beta| = 1$?

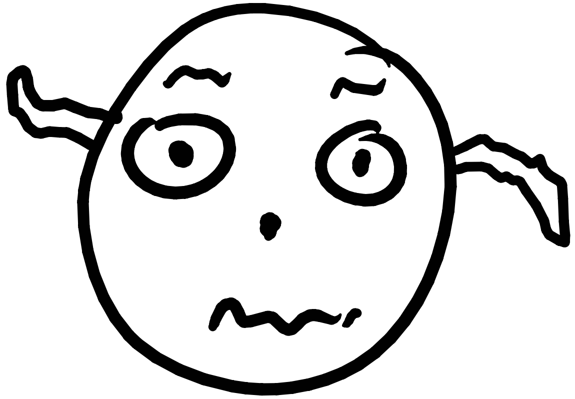
Why $z_1 = \beta z_0^{\,k}$, $|\beta| = 1$?



Because

$$\left(A\left(z_0, \beta z_0^{\,k}\right)\right)^* = A^*\left(z_0, \beta z_0^{\,k}\right)$$

Why $z_1 = \beta z_0{}^k$, $|\beta| = 1$?



Because

$$\left(A(z_0, \beta z_0{}^k)\right)^* = A^*(z_0, \beta z_0{}^k)$$

..... true in general

In general:

m-variate pairs (A, B)

$\Rightarrow$

m'-variate pairs (A', B'),

$m' < m$

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m-variate pairs (A, B)

$\Longrightarrow$

m'-variate pairs (A', B'),

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... but not vice versa!

In general:

more important

m-variate pairs (A,B)

$\Rightarrow$

m'-variate pairs (A',B'),

$m' < m$

... but not vice versa!

Example:

$$\left.\begin{array}{l} A = 1, 1, -1, -1, 1, 1, 1, -1, 1, -1 \\ B = 1, 1, 1, 1, 1, -1, 1, -1, -1, 1 \end{array}\right\} \text{sequence pair}$$

does **not** come from a pair of 2×5 arrays.

BAD Example!

(with J. Jedwab)

... but the length-10 sequence pair **does** come from a complementary pair of 4 x 3 arrays:

$$A = \begin{matrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{matrix} \qquad B = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{matrix}$$

Nice Example! (with J. Jedwab)

$$A'(z_0) = A(z_0, z_0^3), \quad B'(z_0) = B(z_0, z_0^3)$$

or

$$A''(z_1) = A(z_1^4, z_1), \quad B''(z_1) = B(z_1^4, z_1)$$

$$A = \begin{array}{ccc} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{array} \qquad B = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{array}$$

Summary

For $z = (z_0, z_1, \dots z_{m-1})$

Call $(A(z), B(z))$ a

λ_{AB}-pair,

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For $z = (z_0, z_1, \ldots z_{m-1})$

Call $(A(z), B(z))$ a

λ_{AB}-pair,

where

$$\lambda_{AB}(z) = A(z)A^*(z) + B(z)B^*(z)$$

Summary

For $z = (z_0, z_1, \ldots z_{m-1})$

Call $(A(z), B(z))$ a

λ_{AB}-pair,

where

$$\lambda_{AB}(z) = A(z)A^*(z) + B(z)B^*(z)$$

If $\lambda_{AB} = c$, a constant (+ve), then

(A, B) are a pair of complementary arrays

Constructions for (A,B)

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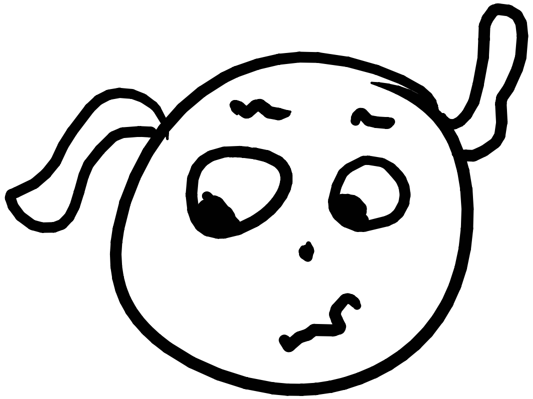
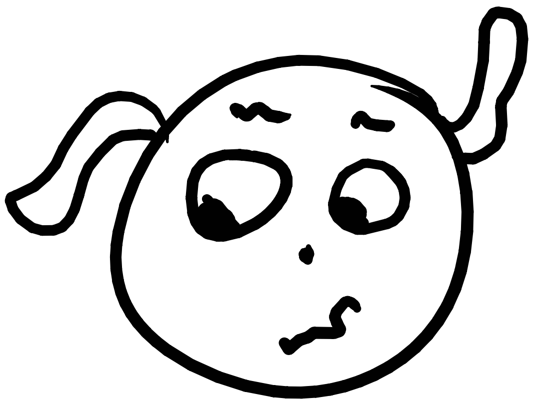


$$F(y,x) = C(y)A(x) + D^*(y)B(x)$$
$$G(y,x) = D(y)A(x) - C^*(y)B(x)$$

Constructions for (A,B)

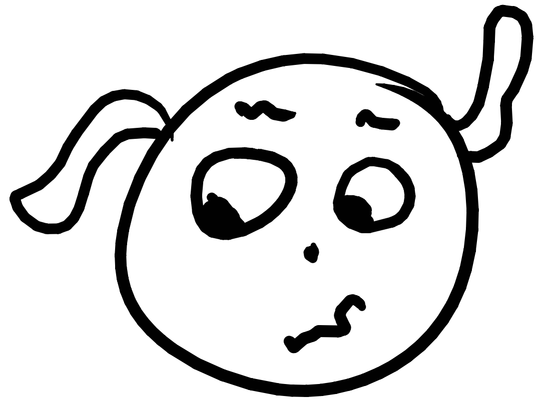


$$F(y,x) = C(y)A(x) + D^*(y)B(x)$$

$$G(y,x) = D(y)A(x) - C^*(y)B(x)$$

x, y disjoint vectors of variables

Constructions for (A, B)



$$F(y,x) = C(y)A(x) + D^*(y)B(x)$$
$$G(y,x) = D(y)A(x) - C^*(y)B(x)$$

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$$\lambda_{FG}(y,x) = \lambda_{CD}(y)\,\lambda_{AB}(x)$$

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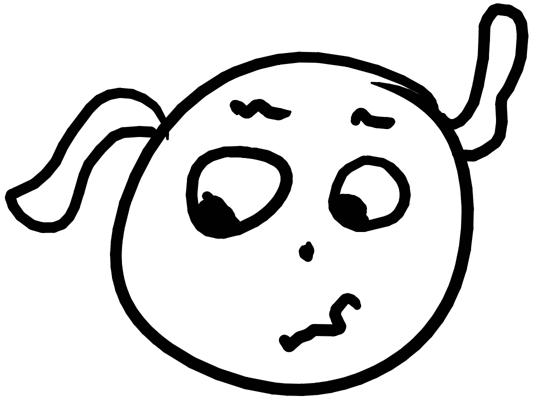

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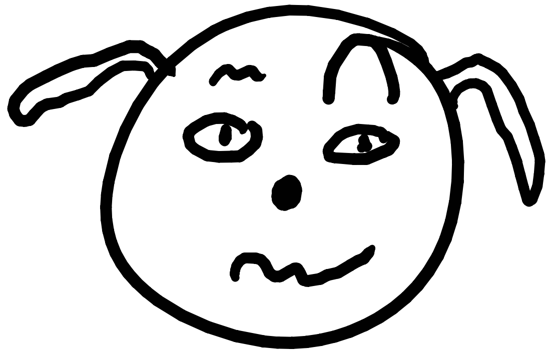
If λ_{CD} and λ_{AB} constants,

then λ_{FG} constant.

PROOF



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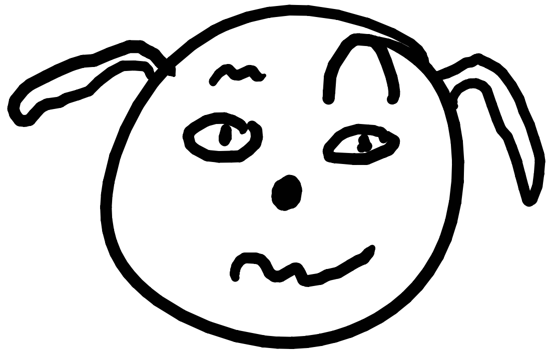

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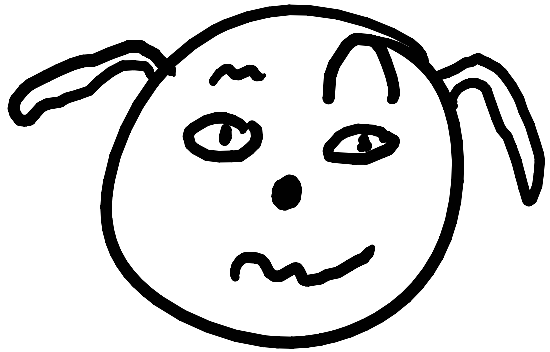
$$\begin{bmatrix} C & D^* \\ D & -C^* \end{bmatrix} \begin{bmatrix} C^* & D^* \\ D & -C \end{bmatrix} = \lambda_{CD} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

PROOF



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'unitary' matrix

Link to space-time codes

$$\begin{bmatrix} C & D^* \\ D & -C^* \end{bmatrix}$$ ~ complementary

$$\begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$$ ~ "Alamouti"
(space-time)

Link to space-time codes

$$\begin{bmatrix} C & D^* \\ D & -C^* \end{bmatrix} \sim \text{complementary}$$

$$= \quad !!!$$

$$\begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix} \sim \text{"Alamouti"}$$

(space-time)

Summary

$\left(A(x), B(x)\right)$ a λ_{AB}-pair,

where $\lambda_{AB}(x) = A(x)A^*(x) + B(x)B^*(x)$

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Summary

Complete Complementary Code with variable entries

(Suehiro), also Turyn

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$$\lambda_{AB}(x) = A(x)A^*(x) + B(x)B^*(x)$$

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now FORGET polynomials!

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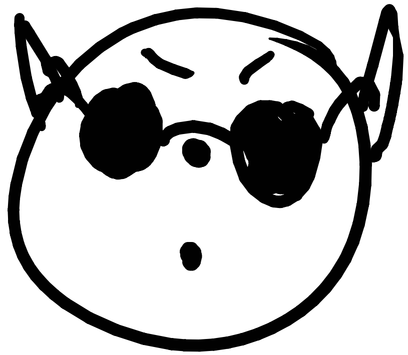

Let $A, B, C, D \in T$, a set

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for operations '$\circ$', '$+$', '$*$'

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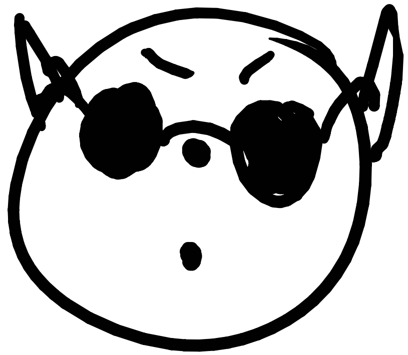

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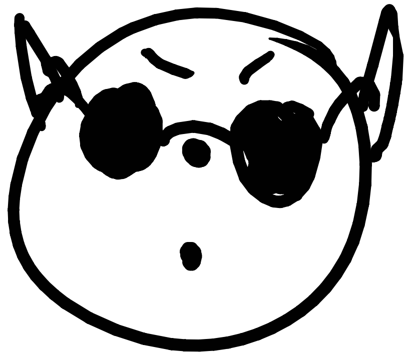

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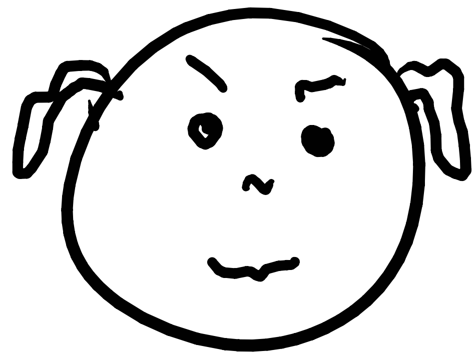
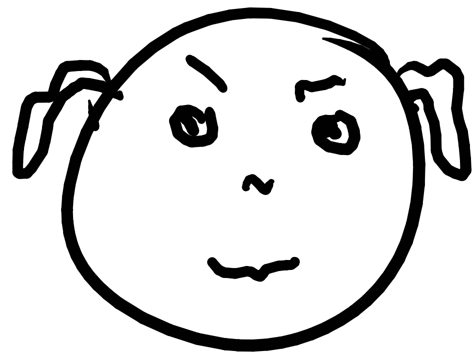
Choose:



Choose:

Γ : complex multivariate polynomials



Choose:

Γ : complex multivariate polynomials

'$\circ$' : multiplication

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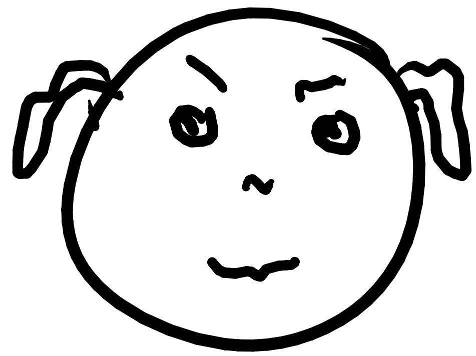

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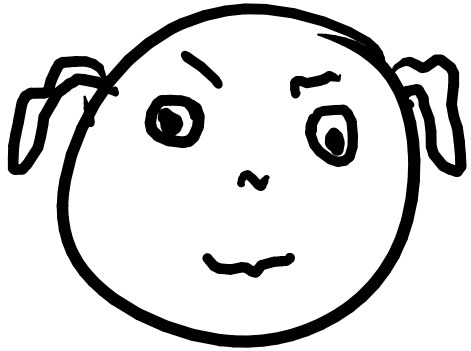
'$*$' : Type-I : $A^*(z_0, z_1, \dots z_{m-1})$

$$:= \overline{A(z_0^{-1}, z_1^{-1}, \dots z_{m-1}^{-1})} := \overline{A(z^{-1})}$$



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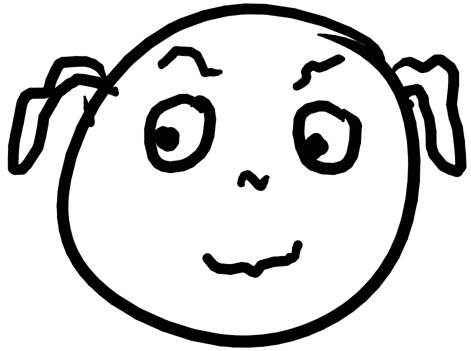

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: Type-II : $A^*(z) = \overline{A(z)}$

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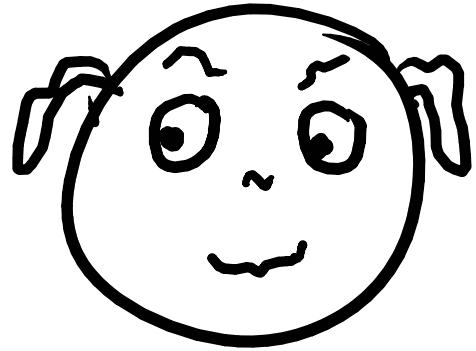
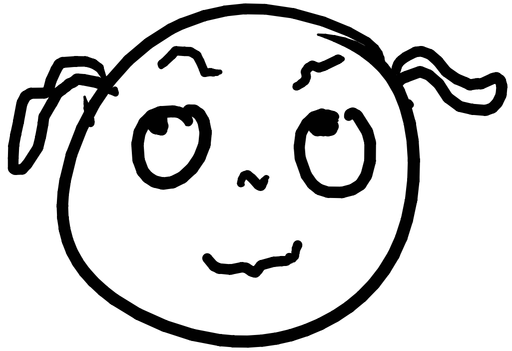
: Type-III : $A^*(z) = \overline{A(-z)}$



Three Types of Conjugation



'*' : Type-I : $A^*(z) = \overline{A(z^{-1})}$

: Type-II : $A^*(z) = \overline{A(z)}$

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Example

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

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Example

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$A^{+}(z) = \overline{A(z^{-1})} = \bar{a}_0 + \bar{a}_1 z^{-1} + \bar{a}_2 z^{-2} + \bar{a}_3 z^{-3} \quad : \mathrm{I}$$

$$A^{*}(z) = \overline{A(z)} = \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \bar{a}_3 z^3 \quad : \mathrm{II}$$

Example

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$$

$$A^*(z) = \overline{A(z^{-1})} = \bar{a}_0 + \bar{a}_1 z^{-1} + \bar{a}_2 z^{-2} + \bar{a}_3 z^{-3} \quad : \mathrm{I}$$

$$A^*(z) = \overline{A(z)} = \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \bar{a}_3 z^3 \quad : \mathrm{II}$$

$$A^*(z) = \overline{A(-z)} = \bar{a}_0 - \bar{a}_1 z + \bar{a}_2 z^2 - \bar{a}_3 z^3 \quad : \mathrm{III}$$

Three Types of Complementary Pair

'$*$' : Type-I : $A^*(z) = \overline{A(z^{-1})}$

: Type-II : $A^*(z) = \overline{A(z)}$

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Three Types of Complementary Pair

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} new

Example

$$A(z) = 2 + i z_0 - 3 z_1 + z_0 z_1 .$$

Then,

Type-I: $A^{+}(z) = 2 - i z_0^{-1} - 3 z_1^{-1} + z_0^{-1} z_1^{-1}$.

Type-II: $= 2 - i z_0 - 3 z_1 + z_0 z_1$.

Type-III: $= 2 + i z_0 + 3 z_1 + z_0 z_1$.

Remember:

$$\begin{bmatrix} F(x) \\ G(x) \end{bmatrix} = \underbrace{\begin{bmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{bmatrix}}_{P(y)} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix},$$

$$x = y|z$$

$$\lambda_{CD}(y) = C(y)C^*(y) + D(y)D^*(y).$$

x – length m''
y – length m' , $m'' = m' + m$.
z – length m

$$\begin{bmatrix} F(x) \\ G(x) \end{bmatrix} = \underbrace{\begin{bmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{bmatrix}}_{P(y)} \begin{bmatrix} A(z) \\ B(z) \end{bmatrix}, \quad x = y|z$$

$$\lambda_{co}(y) = C(y)C^*(y) + D(y)D^*(y).$$

transpose-conjugate

$$P(y)P^\dagger(y) = \lambda_{co}(y)\, I$$

$P(y)$ is "unitary", but is $P(e)$ unitary, $e \in \mathbb{C}^{m'}$?

For which $e \in \mathbb{C}^{m'}$ is $P(e)$ unitary, for types I, II, and III?

Example

Type-I

$$P(y) = \begin{pmatrix} 1+y^2 & y^{-1} \\ y & -1-y^{-2} \end{pmatrix}$$

Type-I:

$$P(y)P^{\dagger}(y)$$

$$= \begin{pmatrix} 1+y^2 & y^{-1} \\ y & -1-y^{-2} \end{pmatrix} \begin{pmatrix} 1+y^{-2} & y^{-1} \\ y & -1-y^{2} \end{pmatrix}$$

$$= (3+y^2+y^{-2})I = \lambda(y)I.$$

Type-I: $P(y)P^{t}(y)$ $\qquad \lambda(y) = (3+y^2+y^{-2})$

$$= \begin{pmatrix} 1+y^2 & y^{-1} \\ y & -1-y^{-2} \end{pmatrix} \begin{pmatrix} 1+y^{-2} & y^{-1} \\ y & -1-y^{2} \end{pmatrix}$$

$$= (3+y^2+y^{-2})I = \lambda(y)I. \quad \checkmark$$

Evaluations: $P(i)P^{t}(i) = \lambda(i)I. \quad \checkmark$

... but..

$P(3)P^{t}(3)$

$$= \begin{pmatrix} 10 & \frac{1}{3} \\ 3 & -\frac{10}{9} \end{pmatrix} \begin{pmatrix} 10 & 3 \\ \frac{1}{3} & -\frac{10}{9} \end{pmatrix} \neq \lambda(3)I. \quad \times$$

In general:

Let $E \subset \mathbb{C}^{n'}$ be the subset of n'-fold complex space where evaluation and conjugation commute:

$$E := \{ e \mid e \in \mathbb{C}^{n'},\ (p(e))^{\dagger} = p^{\dagger}(e) \}.$$

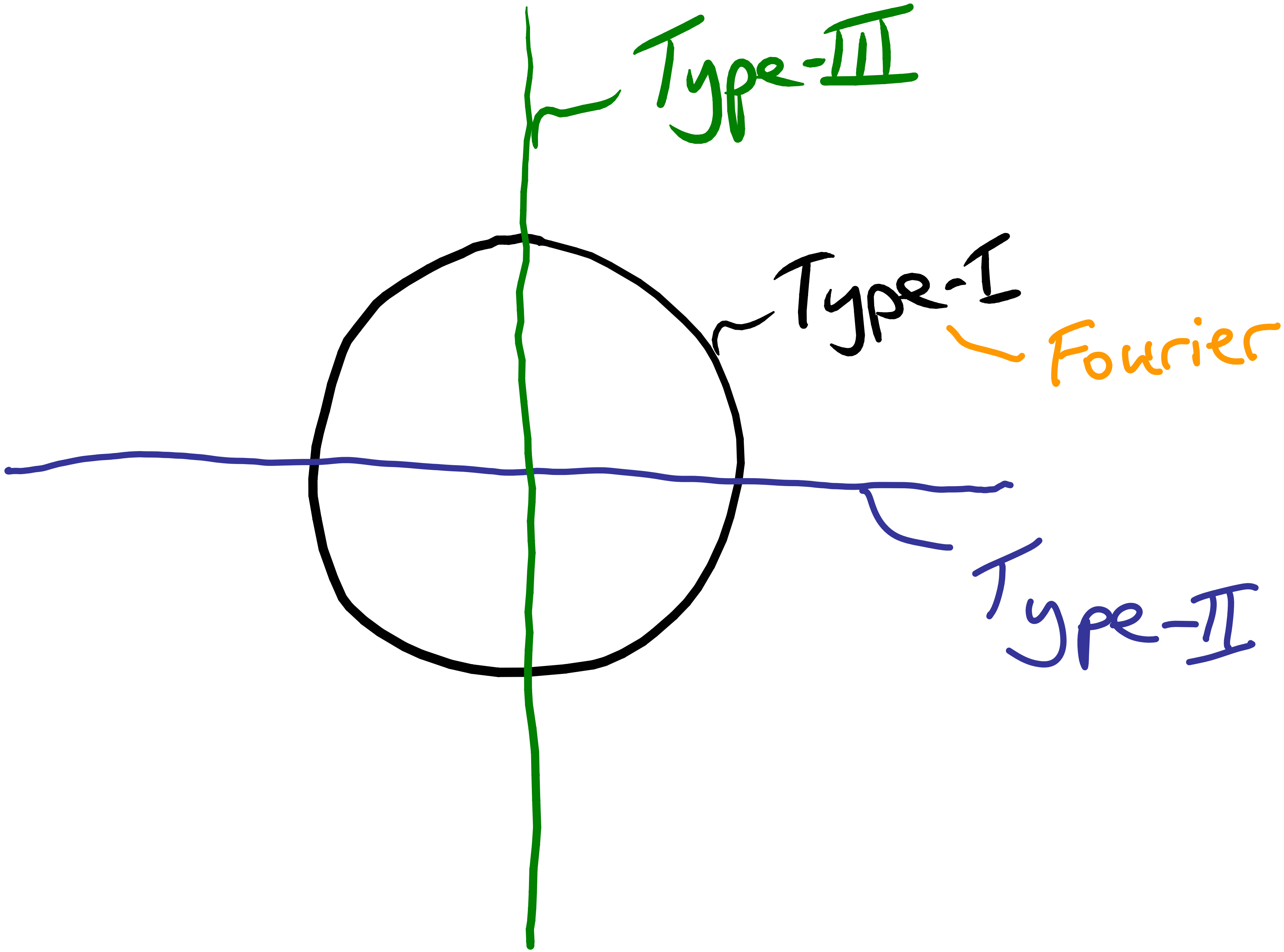
E is different for types-I, II, and III.

Type-I: $E = \{ e \mid |e_j| = 1, \forall j \}$

(m'-fold unit circle)

Type-II: $E = R^{m'}$ (m'-fold real axis)

Type-III: $E = I^{m'}$ (m'-fold imaginary axis).


Type-III
Type-I
Fourier
Type-II

For $A(z) = a_0 + a_1 z$.

Type-I: $z = \beta, \quad |\beta| = 1$

Type-II: $z = r, \quad r \in R$

Type-III: $z = ir, \quad r \in R$

$A(z) = a_0 + a_1 z.$

Type-I : $z = \beta, \quad |\beta| = 1$

Type-II : $z = r, \quad r \in \mathbb{R}$

Type-III : $z = ir, \quad r \in \mathbb{R}$

Type-I:

$$\begin{bmatrix} A(\beta) \\ A(-\beta) \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ 1 & -\beta \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$A(z) = a_0 + a_1 z.$

Type-I : $z = \beta, \quad |\beta| = 1$

Type-II : $z = r, \quad r \in \mathbb{R}$

Type-III : $z = ir, \quad r \in \mathbb{R}$

Type-II:

$$\begin{bmatrix} A(r) \\ A(-\frac{1}{r}) \end{bmatrix} = \begin{bmatrix} 1 & r \\ 1 & -\frac{1}{r} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$A(z) = a_0 + a_1 z.$

Type-I : $z = \beta, \quad |\beta| = 1$

Type-II : $z = r, \quad r \in \mathbb{R}$

Type-III : $z = ir, \quad r \in \mathbb{R}$

Type-III :

$$\begin{bmatrix} A(ir) \\ A\left(-\frac{i}{r}\right) \end{bmatrix} = \begin{bmatrix} 1 & ir \\ 1 & -\frac{i}{r} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$A(z) = a_0 + a_1 z.$

Type-I : $z = \beta, \quad |\beta| = 1$

Type-II : $z = r, \quad r \in \mathbb{R}$

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Unitary sets:

$$\mathcal{E}_{I} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \beta \\ 1 & -\beta \end{pmatrix} \,\middle|\, |\beta| = 1 \right\}$$

$$\mathcal{E}_{II} = \left\{ \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} 1 & r \\ r & -1 \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}$$

$$\mathcal{E}_{III} = \left\{ \frac{1}{\sqrt{1+r^2}} \begin{pmatrix} 1 & ri \\ r & -i \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}$$

$$A(z) = a_0 + a_1 z.$$

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Unitary sets:

$$\mathcal{E}_{I} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \beta \\ 1 & -\beta \end{pmatrix} \,\middle|\, |\beta| = 1 \right\}$$

$$\mathcal{E}_{II} = \left\{ \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \,\middle|\, \forall \phi \right\}$$

$$\mathcal{E}_{III} = \left\{ \begin{pmatrix} \cos\phi & i\sin\phi \\ \sin\phi & -i\cos\phi \end{pmatrix} \,\middle|\, \forall \phi \right\}$$

For $C(y), D(y)$, m'-variate,

normalise $\lambda(y)$ by dividing by:

Type-I: $2^{m'}$

Type-II: $\prod_{k=0}^{m'-1}(1+y_k^2)$

Type-III: $\prod_{k=0}^{m'-1}(1-y_k^2)$

More generally,

for $d_0 \times d_1 \times \ldots \times d_{m'-1}$ arrays,

normalise by:

Type-I : $\prod_{k=0}^{m'-1} d_k$

Type-II : $\prod_{k=0}^{m'-1} \left(1 + y_k^2 + y_k^4 + \ldots + y_k^{2(d_k-1)}\right)$

Type-III : $\prod_{k=0}^{m'-1} \left(1 - y_k^2 + y_k^4 - \ldots + (-1)^{d_k-1} y_k^{2(d_k-1)}\right)$.

(c, D) is a **perfect** type-I, II or III pair, iff

$$\lambda = c, \quad : \text{Type-I}$$

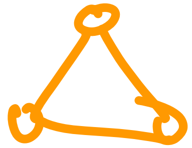
$$\lambda = c \prod_{k=0}^{m'-1} \left(1 + y_k^2 + y_k^4 + \cdots + y_k^{2(d_k-1)}\right) \quad : \text{Type-II}$$

$$\lambda = c \prod_{k=0}^{m'-1} \left(1 - y_k^2 + y_k^4 + \cdots + (-1)^{d_k-1} y_k^{2(d_k-1)}\right). \quad : \text{Type-III}$$

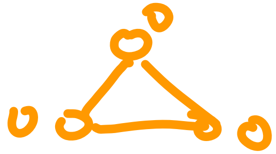
$$c \in \mathbb{R}.$$

Example:

$$C(y) = (-1)^{y_0y_1+y_0y_2+y_1y_2}$$



$$D(y) = C(y)(-1)^{y_0+y_1+y_2}$$



Type-II:

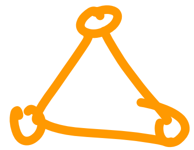
$C = 1, 1, 1, -1, 1, -1, -1, -1$

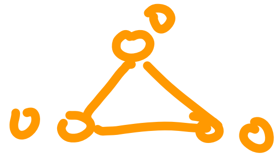
$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$

1

1

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$ 

$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$ 

Type-II:

$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

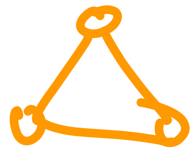
$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$

$$1, 1$$

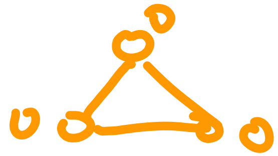
1, 2

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

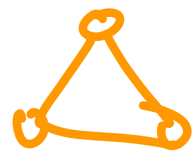
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$1, 1, 1$$

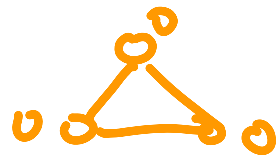
1, 2, 3

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

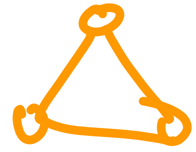
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, 1, 1, 1$$

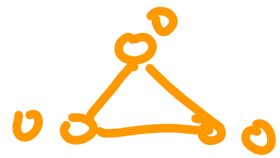
1, 2, 3, 0

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

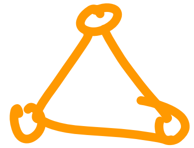
$C = 1, 1, 1, -1, 1, -1, -1, -1$

$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$
$1, -1, 1, 1, 1$

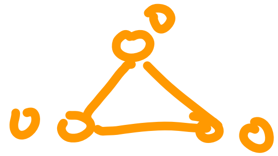
1, 2, 3, 0, 1

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

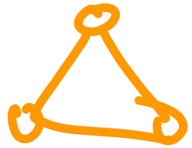
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1,$$
$$-1, 1, -1, 1, 1, 1$$

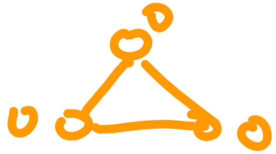
1, 2, 3, 0, 1, -2,

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

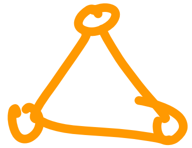
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, -1, 1, -1, 1, 1, 1$$

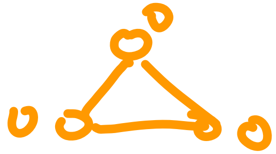
1, 2, 3, 0, 1, -2, -1

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

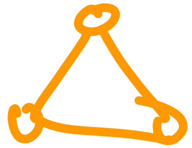
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, -1, -1, 1, -1, 1, 1, 1$$

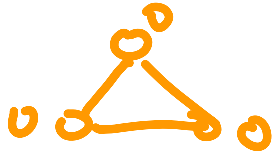
$$1, 2, 3, 0, 1, -2, -1, -8$$

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

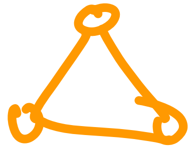
$C = 1, 1, 1, -1, 1, -1, -1, -1$

$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$
$-1, -1, -1, 1, -1, 1, 1$

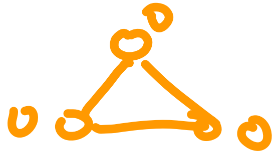
$1, 2, 3, 0, 1, -2, -1, -8, -1,$

Example:

$$C(y) = (-1)^{y_0y_1 + y_0y_2 + y_1y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

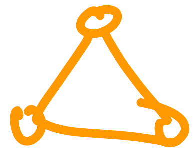
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, -1, -1, 1, -1, 1$$

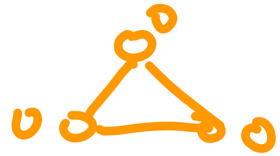
$$1, 2, 3, 0, 1, -2, -1, -8, -1, -2$$

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)\,(-1)^{y_0 + y_1 + y_2}$$



Type-II:

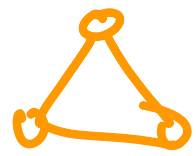
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, -1, -1, 1, -1$$

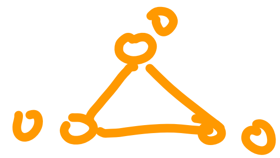
1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1,

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

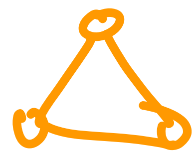
$C = 1, 1, 1, -1, 1, -1, -1, -1$

$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$
$-1, -1, -1, 1$

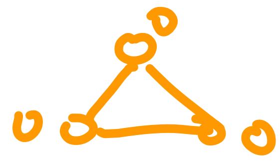
1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1, 0,

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

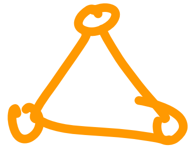
$$C = 1, 1, 1, -1, 1, -1, -1, -1$$

$$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$$
$$-1, -1, -1$$

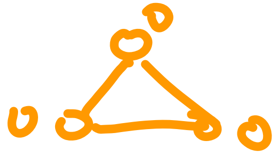
1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1, 0, 3,

Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



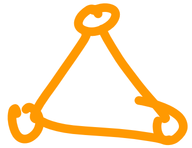
Type-II:

$C = 1, 1, 1, -1, 1, -1, -1, -1$

$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$
$-1, -1$

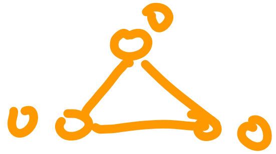
1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1, 0, 3, 2,

Example:



$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$

$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



Type-II:

$C = 1, 1, 1, -1, 1, -1, -1, -1$

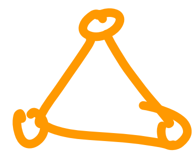
$CC^* = 1, 1, 1, -1, 1, -1, -1, -1$

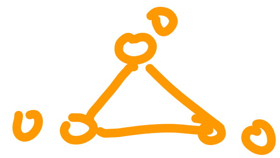
-1

$1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1, 0, 3, 2, 1$

Example:

$$\left.\begin{aligned} C(y) &= (-1)^{y_0y_1+y_0y_2+y_1y_2} \\ D(y) &= C(y)(-1)^{y_0+y_1+y_2} \end{aligned}\right\} \text{perfect Type-II pair}$$





Type-II:

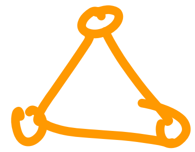
$C = 1, 1, 1, -1, 1, -1, -1, -1$

$D = 1, -1, -1, -1, -1, -1, -1, 1$

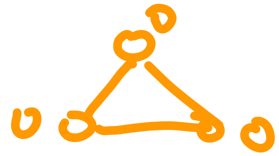
$$\begin{aligned} CC^* &= 1, 2, 3, 0, 1, -2, -1, -8, -1, -2, 1, 0, 3, 2, 1 \\ DD^* &= \underline{1, -2, -1, 0, 1, 2, 3, 8, 3, 2, 1, 0, -1, -2, 1} \\ &\ 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2 \end{aligned}$$

For

$$C(y) = (-1)^{y_0y_1+y_0y_2+y_1y_2}$$



$$D(y) = C(y)(-1)^{y_0+y_1+y_2}$$

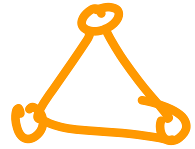


Type-II:

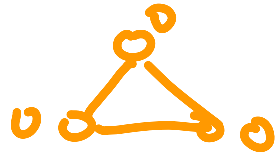
$$CC^* + DD^* = 2(1+y^2+y^4+y^6+y^8+y^{10}+y^{12}+y^{14})$$

For

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



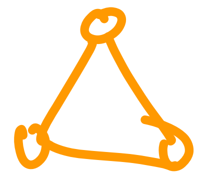
Type-II:

$$CC^* + DD^* = 2(1 + y^2 + y^4 + y^6 + y^8 + y^{10} + y^{12} + y^{14})$$

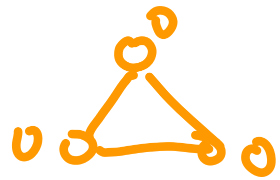
$$\Rightarrow \frac{CC^* + DD^*}{(1 + y^2 + y^4 + \dots + y^{14})} = 2$$

So

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



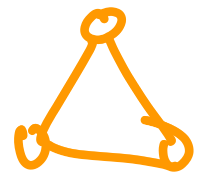
$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



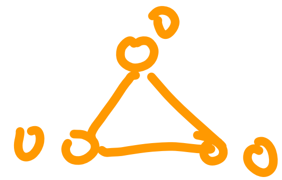
is a perfect Type-II complementary sequence pair of length 8.

So

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_1 y_2}$$



$$D(y) = C(y)(-1)^{y_0 + y_1 + y_2}$$



is a perfect Type-II complementary sequence pair of length 8.

....more generally, (C, D) is a perfect Type-II $2 \times 2 \times 2$ array pair.

Spectral Properties?

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|\hat{C}|^2 + |\hat{D}|^2$$

$$(I \otimes I \otimes I) \left\{ \begin{array}{cc} C & D \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, & \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \end{array} \right\} = \left\{ \begin{array}{cc} \hat{C} & \hat{D} \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, & \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \end{array} \right\} : \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

Spectral Properties?

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$(H \otimes I \otimes I)\left\{ C, D \right\} = \left\{ \hat{C}, \hat{D} \right\} : \quad |\hat{C}|^2 + |\hat{D}|^2 \rightarrow \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

Spectral Properties?

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & & & & & & \\ 1 & -1 & & & & & 0 & \\ & & 1 & 1 & & & & \\ & & 1 & -1 & & & & \\ & & & & 1 & 1 & & \\ & & & & 1 & -1 & & \\ & 0 & & & & & 1 & 1 \\ & & & & & & 1 & -1 \end{bmatrix} \left\{ \overset{C}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}}, \overset{D}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} \right\} = \sqrt{2} \left\{ \overset{\hat{C}}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}, \overset{\hat{D}}{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}} \right\} : \overset{|\hat{C}|^2 + |\hat{D}|^2}{\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}}$$

Spectral Properties?

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$(H \otimes H \otimes I) \left\{ \overset{C}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}}, \overset{D}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} \right\} = \frac{1}{\sqrt{2}} \left\{ \overset{\hat{C}}{\begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix}}, \overset{\hat{D}}{\begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix}} \right\} : \quad \overset{|\hat{C}|^2 + |\hat{D}|^2}{\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}}$$

Spectral Properties?

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{1}{2} \begin{bmatrix} \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{matrix} \end{bmatrix} \left\{ \overset{C}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}}, \overset{D}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}} \right\} = \left\{ \overset{\hat{C}}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}, \overset{\hat{D}}{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}} \right\} : \overset{|\hat{C}|^2 + |\hat{D}|^2}{\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}}$$

Spectral Properties?

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$(H \otimes H \otimes H) \left\{ \begin{array}{c} C \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \end{array}, \begin{array}{c} D \\ \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \end{array} \right\} = \frac{1}{2\sqrt{2}} \left\{ \begin{array}{c} \hat{C} \\ \begin{pmatrix} 0 \\ 4 \\ 4 \\ 0 \\ 4 \\ 0 \\ 0 \\ -4 \end{pmatrix} \end{array}, \begin{array}{c} \hat{D} \\ \begin{pmatrix} -4 \\ 0 \\ 0 \\ 4 \\ 0 \\ 4 \\ 4 \\ 0 \end{pmatrix} \end{array} \right\} : \begin{array}{c} |\hat{C}|^2 + |\hat{D}|^2 \\ \downarrow \\ \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} \end{array}$$

Spectral Properties?

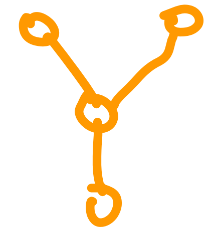
$$(\cos\theta_0, \sin\theta_0) \otimes (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2) \left\{ C, D \right\}$$

$$= \left\{ \hat{c}, \hat{d} \right\}, \text{ s.t.}$$

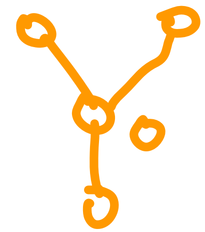
$$|\hat{c}|^2 + |\hat{d}|^2 = 2.$$

Another Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_0 y_3}$$



$$D(y) = C(y)(-1)^{y_0}$$



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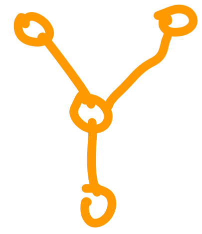

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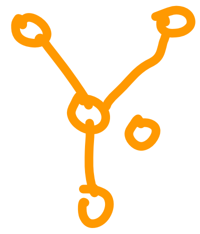


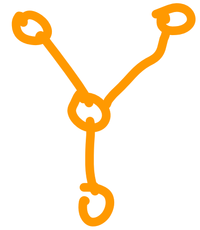
Type-III:

$$CC^* + DD^* =$$

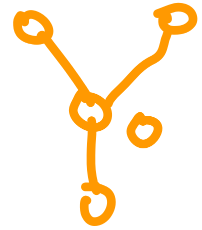
2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2,
0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2

Another Example:

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_0 y_3}$$



$$D(y) = C(y)(-1)^{y_0}$$



Perfect Type-III pair

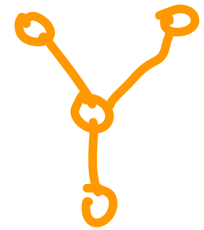
Type-III:

$$CC^* + DD^* =$$

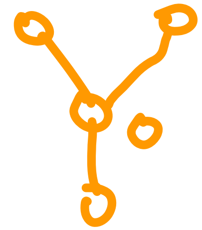
2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2,
0, 2, 0, -2, 0, 2, 0, -2, 0, 2, 0, -2

For

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_0 y_3}$$



$$D(y) = C(y)(-1)^{y_0}$$

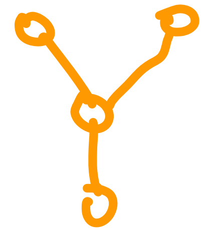


Type-III:

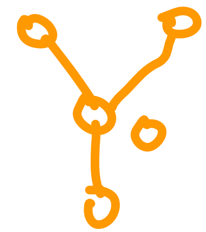
$$CC^* + DD^* = 2\left(1 - y^2 + y^4 - y^6 + \cdots + y^{28} - y^{30}\right)$$

For

$$C(y) = (-1)^{y_0 y_1 + y_0 y_2 + y_0 y_3}$$



$$D(y) = C(y)(-1)^{y_0}$$



Type-III:

$$CC^* + DD^* = 2(1 - y^2 + y^4 - y^6 + \cdots + y^{28} - y^{30})$$

$$\Rightarrow \frac{CC^* + DD^*}{(1 - y^2 + y^4 - y^6 + \cdots + y^{28} - y^{30})} = 2.$$

Summary

$x = y | z$

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} = \begin{pmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}$$

Summary

$x = y|z$

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} = \begin{pmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}$$

Then,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

If,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

$$\lambda_{FG}(e'') = \lambda_{CD}(e')\,\lambda_{AB}(e).$$

$e \in C^{n}$, $e' \in C^{n'}$, $e'' \in C^{n''}$

If,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

$$\lambda_{FG}(e'') = \lambda_{CD}(e')\,\lambda_{AB}(e).$$

$e \in C^{m}, e' \in C^{m'}, e'' \in C^{m''}$

Type-I: e, e', e'' on m, m', m''-fold unit circles

$\Rightarrow$ mapping to complex unitary transforms.

If,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

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then

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$e \in \mathbb{C}^{m}, e' \in \mathbb{C}^{m'}, e'' \in \mathbb{C}^{m''}$

Type-II: e, e', e'' on m, m', m''-fold real axes

$\Rightarrow$ mapping to complex unitary transforms.

If,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

$$\lambda_{FG}(e'') = \lambda_{CD}(e')\,\lambda_{AB}(e).$$

$e \in C^{n}, e' \in C^{n'}, e'' \in C^{n''}$

If,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

$$\lambda_{FG}(e'') = \lambda_{CD}(e')\,\lambda_{AB}(e).$$

$e \in \mathbb{C}^{m}, e' \in \mathbb{C}^{m'}, e'' \in \mathbb{C}^{m''}$

Type = $\underline{\mathrm{III}}$: e, e', e'' on m, m', m''-fold imaginary axes

$\Rightarrow$ mapping to complex unitary transforms.

Given,

$$\lambda_{FG}(x) = \lambda_{CD}(y)\,\lambda_{AB}(z).$$

then

λ_{FG} perfect if $\lambda_{CD}, \lambda_{AB}$ perfect

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λ_{FG} perfect if $\lambda_{CD}, \lambda_{AB}$ perfect, i.e.:

Type-I: $\lambda_{CD} = c'$ and $\lambda_{AB} = c$, constants.

Type-II: $\lambda_{CD} = c' \prod_{k=0}^{m'-1} \left(1 + y_k^2 + y^4 + \ldots + y_k^{2(d_k-1)}\right)$, λ_{AB} similar.

Given,

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λ_{FG} **perfect** if $\lambda_{CD}, \lambda_{AB}$ perfect, i.e.:

Type-I: $\lambda_{CD} = c'$ and $\lambda_{AB} = c$, constants.

Type-II: $\lambda_{CD} = c' \prod_{k=0}^{m'-1} \left(1 + y_k^2 + y^4 + \ldots + y_k^{2(d_k-1)}\right)$, λ_{AB} similar.

Type-III: $\lambda_{CD} = c' \prod_{k=0}^{m'-1} \left(1 + y_k^2 - y^4 + \ldots + (-1)^{d_k-1} y_k^{2(d_k-1)}\right)$, λ_{AB} similar.

Pair Recursion

$$\begin{pmatrix} F_j(z_j) \\ G_j(z_j) \end{pmatrix} = \begin{pmatrix} U_j(y_j) \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix} V_j(y_j)^{(\dagger)} \end{pmatrix} \begin{pmatrix} F_{j-1}(z_{j-1}) \\ G_{j-1}(z_{j-1}) \end{pmatrix}$$

Pair Recursion

$$z_j = y_j / z_{j-1}$$

$$\begin{pmatrix} F_j(z_j) \\ G_j(z_j) \end{pmatrix} = \left(U_j(y_j) \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix} V_j(y_j) \right)^{(t)} \begin{pmatrix} F_{j-1}(z_{j-1}) \\ G_{j-1}(z_{j-1}) \end{pmatrix}$$

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If $F_{-1} = G_{-1} = 1$, then,

$$\lambda_{FG,n-1} = \prod_{j=0}^{n-1} \lambda_{C D j}$$

Pair Recursion

$$z_j = y_j / z_{j-1}$$

symmetries

$$\begin{pmatrix} F_j(z_j) \\ G_j(z_j) \end{pmatrix} = \left(U_j(y_j) \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix} V_j(y_j) \right)^{\dagger} \begin{pmatrix} F_{j-1}(z_{j-1}) \\ G_{j-1}(z_{j-1}) \end{pmatrix}$$

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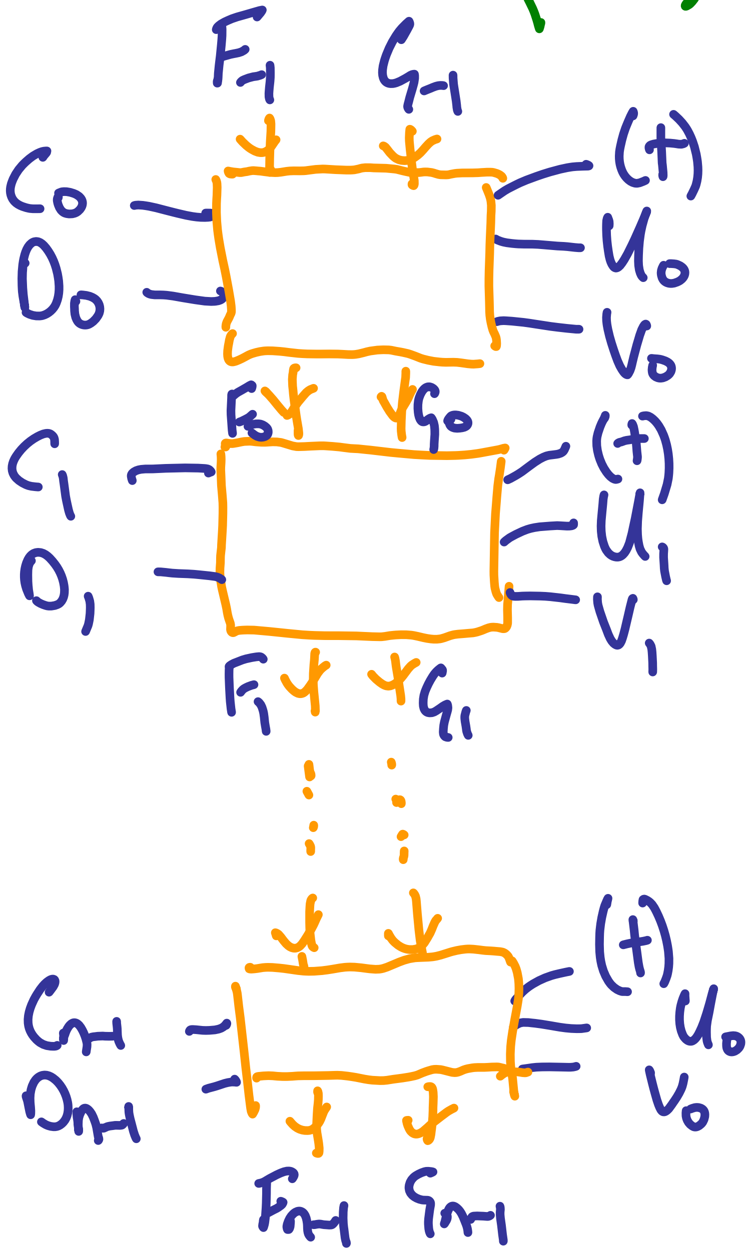
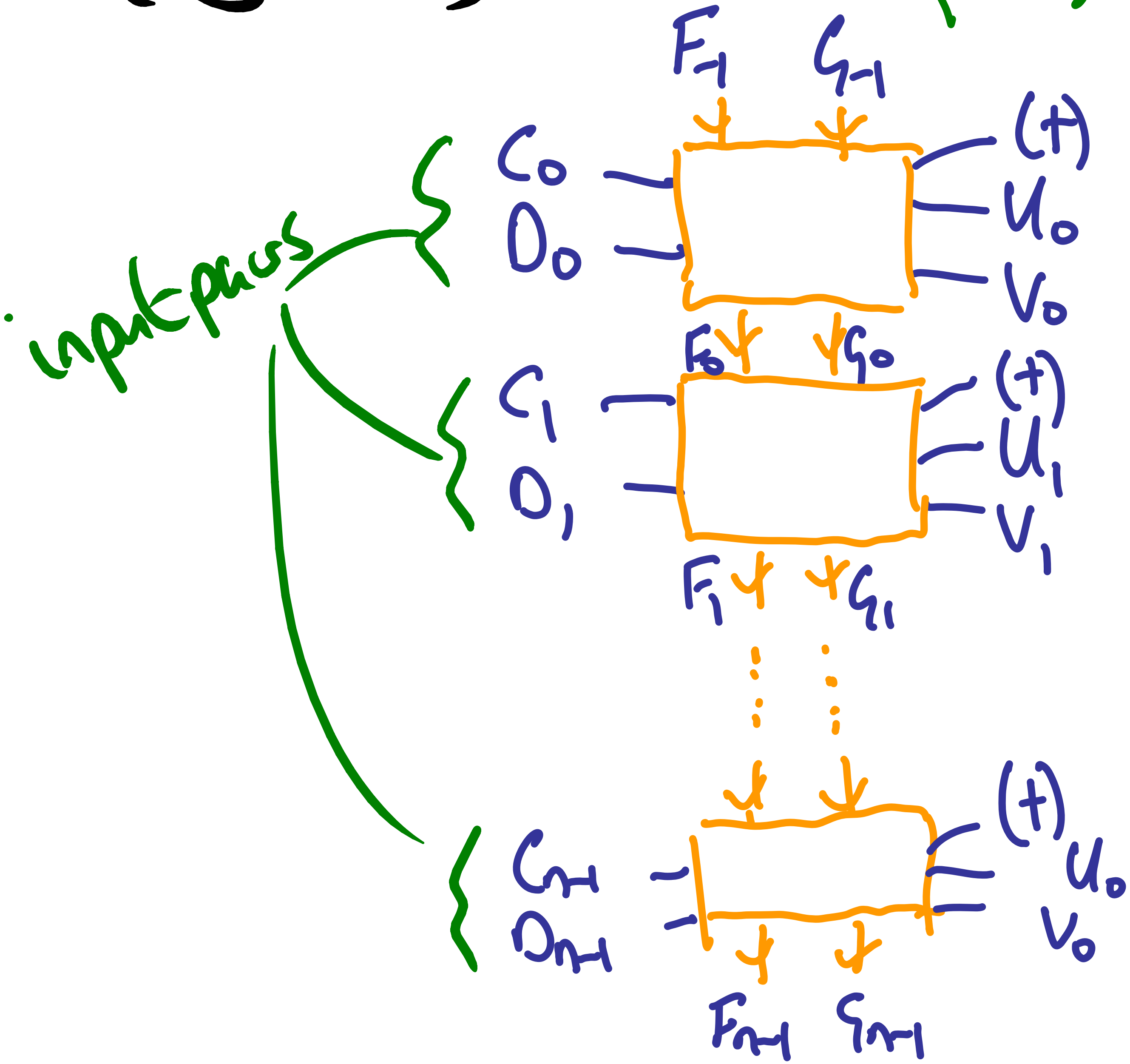
Pair Recursion
(algorithm)

$$\begin{pmatrix} F_j(z_j) \\ G_j(z_j) \end{pmatrix} = \left(U_j(y_j) \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix} V_j(y_j) \right)^{(t)} \begin{pmatrix} F_{j+1}(z_{j+1}) \\ G_{j+1}(z_{j+1}) \end{pmatrix}$$



Pair Recursion (algorithm)

$$\begin{pmatrix} F_j(z_j) \\ G_j(z_j) \end{pmatrix} = \left(U_j(y_j) \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix} V_j(y_j) \right)^{(t)} \begin{pmatrix} F_{j-1}(z_{j-1}) \\ G_{j-1}(z_{j-1}) \end{pmatrix}$$



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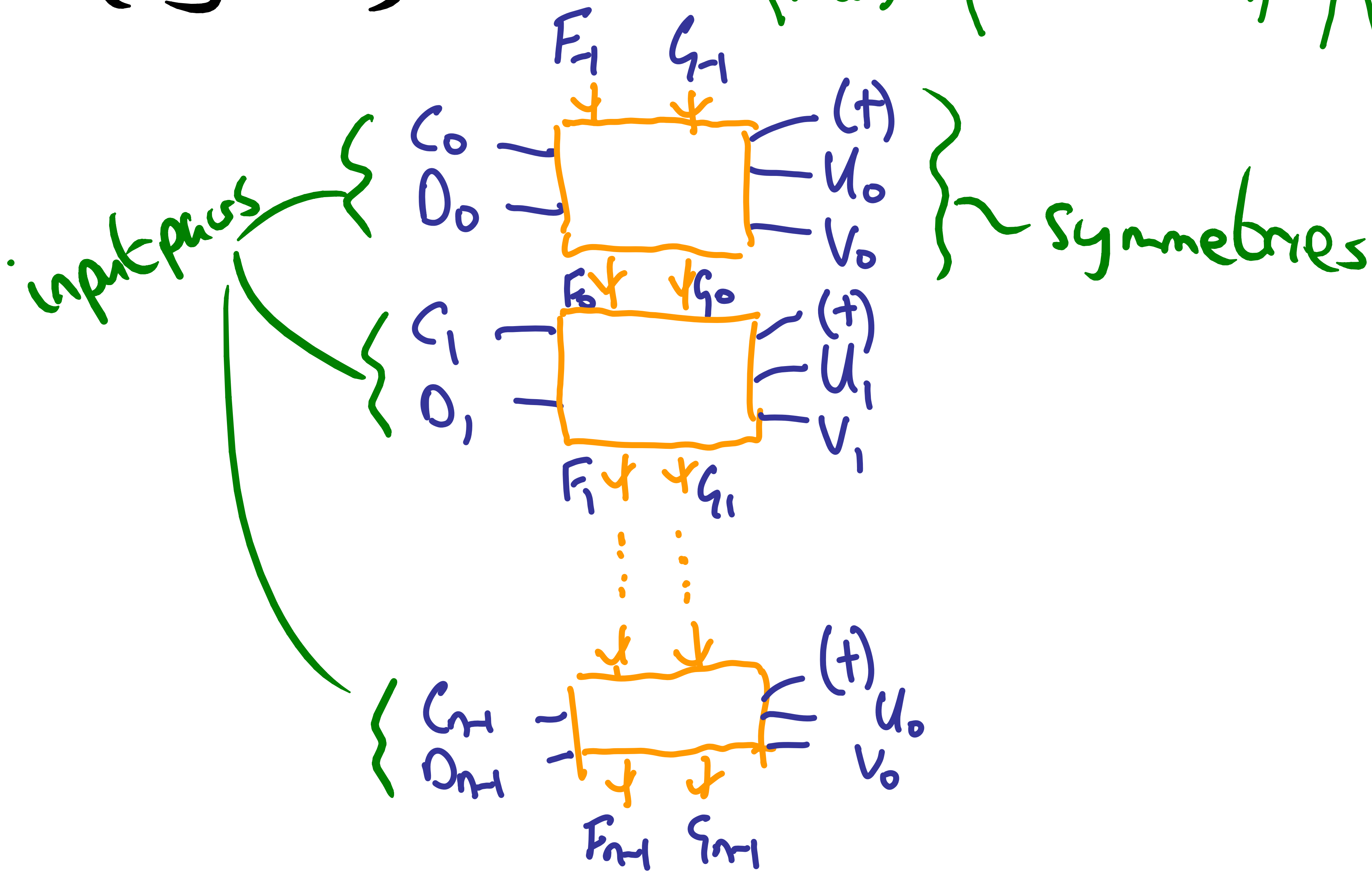
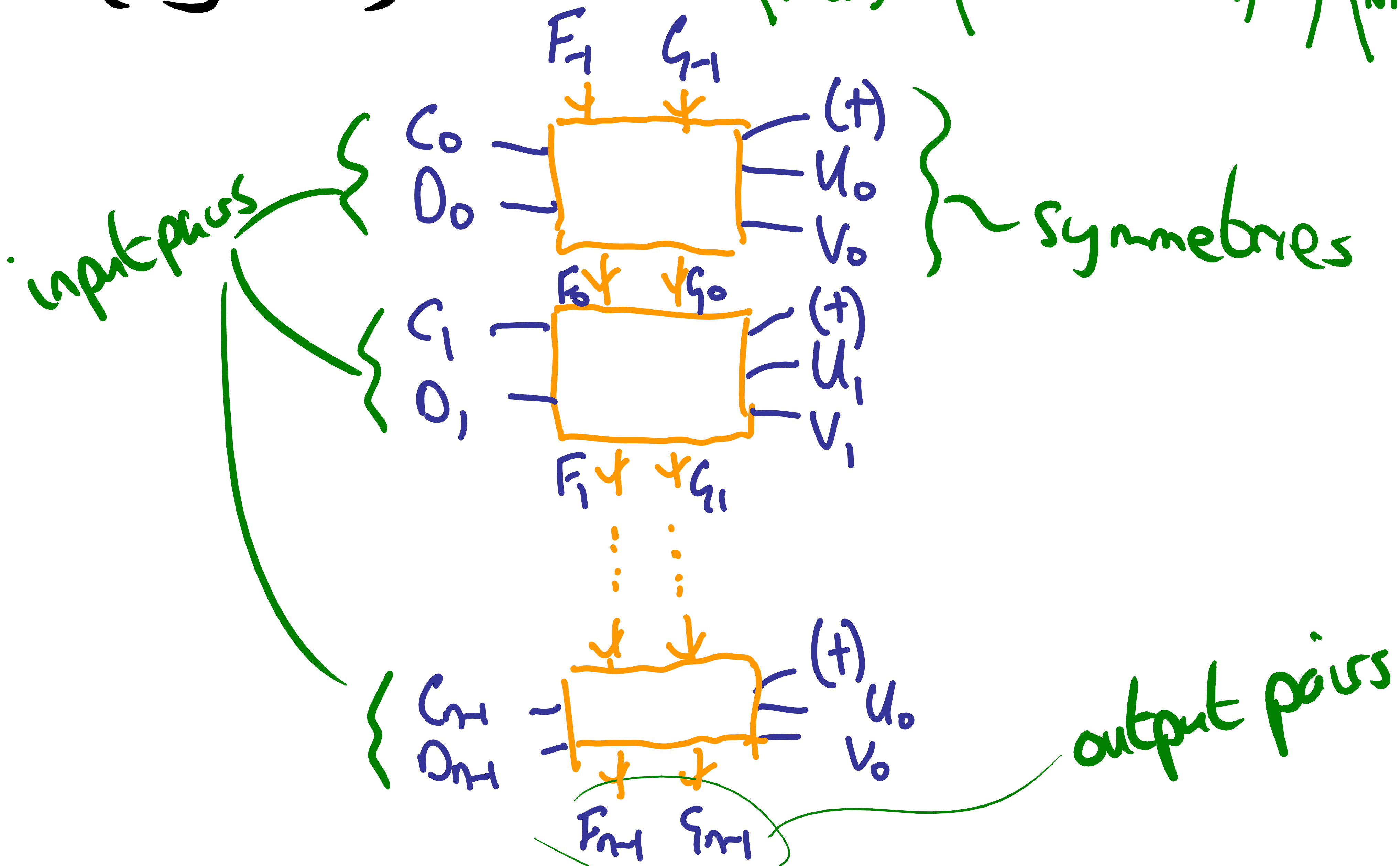


Pair Recursion (algorithm)

$$\begin{pmatrix}F_j(z_j)\\ G_j(z_j)\end{pmatrix} = \left(U_j(y_j)\begin{pmatrix}C_j(y_j) & D_j^*(y_j)\\ D_j(y_j) & -C_j^*(y_j)\end{pmatrix}V_j(y_j)\right)^{(t)}\begin{pmatrix}F_{j-1}(z_{j-1})\\ G_{j-1}(z_{j-1})\end{pmatrix}$$



Pair Recursion for $\{1,-1\}$ arrays

$$M_j = \begin{pmatrix} C_j(y_j) & D_j^{\dagger}(y_j) \\ D_j(y_j) & -C_j^{\dagger}(y_j) \end{pmatrix}$$

Pair Recursion for $\{1,-1\}$ arrays

$$M_j = \begin{pmatrix} C_j(y_j) & D_j^*(y_j) \\ D_j(y_j) & -C_j^*(y_j) \end{pmatrix}$$

I: $$\begin{pmatrix} F_j \\ C_j \end{pmatrix} = \pm\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \left(M_j \begin{pmatrix} 1 & 0 \\ 0 & y_j^{d_{j-1}} \end{pmatrix} \right)^{(T)} \begin{pmatrix} 1 & 1 \\ \pm 1 & \mp 1 \end{pmatrix} \begin{pmatrix} F_{j-1} \\ C_{j-1} \end{pmatrix}$$

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$$\mathrm{I}: \begin{pmatrix} F_j \\ G_j \end{pmatrix} = \frac{\pm 1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \left(M_j \begin{pmatrix} 1 & 0 \\ 0 & y_j^{d_{j-1}} \end{pmatrix} \right)^{(T)} \begin{pmatrix} 1 & 1 \\ \pm 1 & \mp 1 \end{pmatrix} \begin{pmatrix} F_{j-1} \\ G_{j-1} \end{pmatrix}$$

$$\mathrm{II}: \begin{pmatrix} F_j \\ G_j \end{pmatrix} = \frac{\pm 1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} M_j^{+} \begin{pmatrix} 1 & 1 \\ \pm 1 & \mp 1 \end{pmatrix} \begin{pmatrix} F_{j-1} \\ G_{j-1} \end{pmatrix}$$

Pair Recursion for $\{1,-1\}$ arrays

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II: $$\begin{pmatrix} F_j \\ G_j \end{pmatrix} = \frac{\pm 1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} M_j^{\dagger} \begin{pmatrix} 1 & 1 \\ \pm 1 & \mp 1 \end{pmatrix} \begin{pmatrix} F_{j-1} \\ G_{j-1} \end{pmatrix}$$

III: $$\begin{pmatrix} F_j \\ G_j \end{pmatrix} = \frac{\pm 1}{\sqrt{2}} M_j^{\dagger} \begin{pmatrix} \pm 1 & 1 \\ \mp 1 & 1 \end{pmatrix} \begin{pmatrix} F_{j-1} \\ G_{j-1} \end{pmatrix}.$$

Example

Let $F_0 = 1 + z_0 + z_0^2$, $G_0 = 1 - z_0 - z_0^2$

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Let $F_0 = 1 + z_0 + z_0^2$, $G_0 = 1 - z_0 - z_0^2$

Type-III: $F_0 F_0^* + G_0 G_0^* = 2(1 - z_0^2 + z_0^4)$

Example

Let $F_0 = 1 + z_0 + z_0^2$, $G_0 = 1 - z_0 - z_0^2$

Type-$\underline{\text{III}}$: $F_0 F_0^* + G_0 G_0^* = 2(1 - z_0^2 + z_0^4)$

... so (F_0, G_0) are type-$\underline{\text{III}}$ perfect.

Example

Let $F_0 = 1 + z_0 + z_0^2$, $G_0 = 1 - z_0 - z_0^2$

Type-III: $F_0 F_0^* + G_0 G_0^* = 2(1 - z_0^2 + z_0^4)$

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Let $C_1 = D_1 = 1 + z_1$ ... type-III perfect.

Example

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... so (F_0, G_0) are type-III perfect.

Let $C_1 = D_1 = 1 + z_1$ ... type-III perfect.

Then, for a certain symmetry:

$$\frac{F_1}{\sqrt{2}} = 1 - z_0 - z_0^2 + z_1 + z_0 z_1 + z_0^2 z_1$$

$$\frac{G_1}{\sqrt{2}} = 1 + z_0 + z_0^2 + z_1 - z_0 z_1 - z_0^2 z_1$$

Example

Let $F_0 = 1 + z_0 + z_0^2$, $G_0 = 1 - z_0 - z_0^2$

Type-III: $F_0 F_0^* + G_0 G_0^* = 2(1 - z_0^2 + z_0^4)$

... so (F_0, G_0) are type-III perfect.

Let $C_1 = D_1 = 1 + z_1$ ... type-III perfect.

Then, for a certain symmetry:

$$\left.\begin{aligned} \frac{F_1}{\sqrt{2}} &= 1 - z_0 - z_0^2 + z_1 + z_0 z_1 + z_0^2 z_1 \\ \frac{G_1}{\sqrt{2}} &= 1 + z_0 + z_0^2 + z_1 - z_0 z_1 - z_0^2 z_1 \end{aligned}\right\}$$

... a type-III perfect 2×3 array.

Observe

$$F_0 = 1 + z_0 + z_0^2 \qquad G_0 = 1 - z_0 - z_0^2$$

Observe

$$F_0 = 1 + z_0 + z_0^2, \quad G_0 = 1 - z_0 - z_0^2$$

$$\lambda_{F,G,0} = 2\left(1 - z_0^2 + z_0^4\right)$$

Observe

$$F_0 = 1 + z_0 + z_0^2, \quad G_0 = 1 - z_0 - z_0^2$$

$$\lambda_{F_{G},0} = 2\left(1 - z_0^2 + z_0^4\right)$$

$$C_1 = 1 + z_1, \quad D_1 = 1 + z_1$$

Observe

$$F_0 = 1 + z_0 + z_0^2, \quad G_0 = 1 - z_0 - z_0^2$$

$$\lambda_{FG,0} = 2(1 - z_0^2 + z_0^4)$$

$$C_1 = 1 + z_1, \quad D_1 = 1 + z_1$$

$$\lambda_{CD,1} = 2(1 - z_1^2)$$

Observe

$$F_0 = 1 + z_0 + z_0^2, \quad G_0 = 1 - z_0 - z_0^2$$

$$\lambda_{FG,0} = 2(1 - z_0^2 + z_0^4)$$

$$C_1 = 1 + z_1, \quad D_1 = 1 + z_1$$

$$\lambda_{CD,1} = 2(1 - z_1^2)$$

$\Rightarrow$

$$\frac{F_1}{\sqrt{2}} = 1 - z_0 - z_0^2 + z_1 + z_0 z_1 + z_0^2 z_1, \quad \frac{G_1}{\sqrt{2}} = 1 + z_0 + z_0^2 + z_1 - z_0 z_1 - z_0^2 z_1$$

Observe

$$F_0 = 1 + z_0 + z_0^2, \quad G_0 = 1 - z_0 - z_0^2$$

$$\lambda_{FG,0} = 2(1 - z_0^2 + z_0^4)$$

$$C_1 = 1 + z_1, \quad D_1 = 1 + z_1$$

$$\lambda_{CD,1} = 2(1 - z_1^2)$$

$\Rightarrow$

$$\frac{F_1}{\sqrt{2}} = 1 - z_0 - z_0^2 + z_1 + z_0 z_1 + z_0^2 z_1, \quad \frac{G_1}{\sqrt{2}} = 1 + z_0 + z_0^2 + z_1 - z_0 z_1 - z_0^2 z_1$$

$$\lambda_{FG,1} = \lambda_{CD,1} \lambda_{FG,0} = 4(1 - z_1^2)(1 - z_0^2 + z_0^4).$$

Set Recursion

$$F_j = \begin{pmatrix} F_{j,0}(z_j) \\ F_{j,1}(z_j) \\ \vdots \\ F_{j,S-1}(z_j) \end{pmatrix} = \left(U_j(y_j) M_j V_j(y_j)\right)^{(t)} \begin{pmatrix} F_{j-1,0}(z_{j-1}) \\ F_{j-1,1}(z_{j-1}) \\ \vdots \\ F_{j-1,S-1}(z_{j-1}) \end{pmatrix}$$

Set Recursion

$$F_j = \begin{pmatrix} F_{j,0}(z_j) \\ F_{j,1}(z_j) \\ \vdots \\ F_{j,S-1}(z_j) \end{pmatrix} = \left(U_j(y_j) M_j V_j(y_j) \right)^{(t)} \begin{pmatrix} F_{j-1,0}(z_{j-1}) \\ F_{j-1,1}(z_{j-1}) \\ \vdots \\ F_{j-1,S-1}(z_{j-1}) \end{pmatrix}$$

If $F_{-1,s} = 1, \forall s$, then

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what is M_j?

If $F_{-1,s} = 1$, $\forall s$, then

$$\lambda_{F,j} = \prod_{j=0}^{-1} \lambda_{co,j}$$

Can we make

$$M = \begin{pmatrix} C & D^* \\ D & -C^* \end{pmatrix}$$

bigger?

... with more variable entries?

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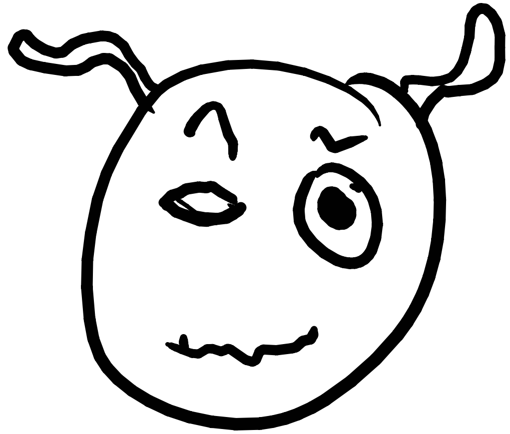
No. See Tarokh.

... but see complex orthogonal designs...

... with more variable entries?



One answer:

$$M_j = \begin{pmatrix} C_0 & D_0^{*} \\ D_0 & -C_0 \end{pmatrix} \otimes \begin{pmatrix} C_1 & D_1^{*} \\ D_1 & -C_1 \end{pmatrix} \otimes \cdots\cdots \otimes \begin{pmatrix} C_{t-1} & D_{t-1}^{*} \\ D_{t-1} & -C_{t-1} \end{pmatrix}$$

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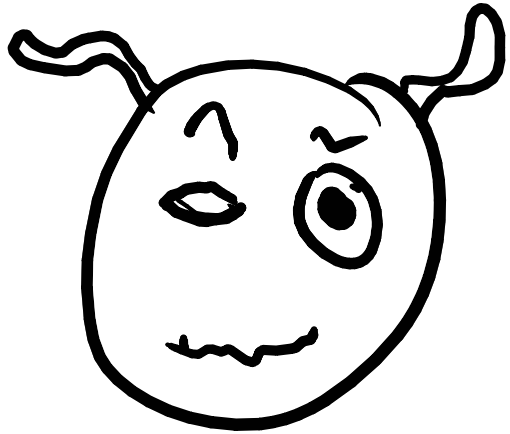

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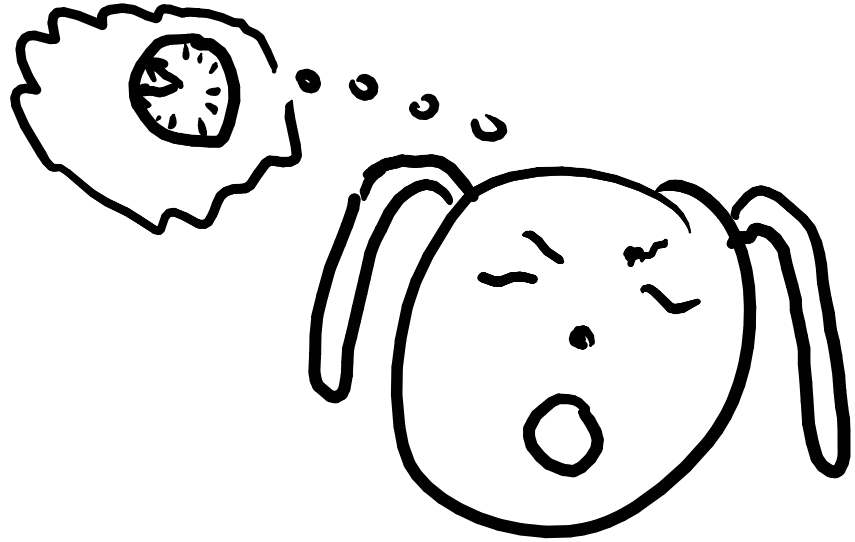
$2t$ variables in a matrix of size 2^t.

Set Recursion - special case

$$F_j = \begin{pmatrix} F_{j,0}(z_j) \\ F_{j,1}(z_j) \\ \vdots \\ F_{j,S-1}(z_j) \end{pmatrix} = \big(U_j(y_j) M_j V_j(y_j) \big)^{(t)} \begin{pmatrix} F_{j-1,0}(z_{j-1}) \\ F_{j-1,1}(z_{j-1}) \\ \vdots \\ F_{j-1,S-1}(z_{j-1}) \end{pmatrix}$$

$$M_j = \bigotimes_{k=0}^{t-1} \begin{pmatrix} C_{j,k} & D^*_{j,k} \\ D_{j,k} & -C^*_{j,k} \end{pmatrix}.$$



. . . . uh oh . . .

TOO MUCH

NOTATION !

Type-IV : Rayleigh Quotient Pairs

Definition:

M unitary Hermitian if

$$M^{\dagger} = M, \quad MM^{\dagger} = I.$$

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Further, let each row of M be a

$d_0 \times d_1 \times \ldots \times d_{m-1}$ array.

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$|RQ| \leq 1$, with $RQ(A,M) = \pm 1$ iff A is an **eigenvector** (eigenarray) of M.

Definition: **Rayleigh Quotient Pair** (A,B) with respect to M:

$$RQ_2((A,B),M) := \frac{\langle A, MA\rangle + \langle B, MB\rangle}{\langle A, A\rangle + \langle B, B\rangle}$$

Definition: Rayleigh Quotient Pair (A,B) with respect to M:

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$|RQ_2| \leq 1$, with $RQ_2((A,B),M) = \pm 1$ iff both A and B are eigenvectors of M.

Remember:

$$F = C \circ A + D^* \circ B, \quad G = D \circ A - C^* \circ B$$

$$\Updownarrow$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} C & D^* \\ D & -C^* \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Type-IV

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'$*$' : $A^* = MA$

'$\circ$' : $C(y) \circ A(z) := C(y)A(z)$ — y, z disjoint variables

$A(z) \circ B(z) := \langle A, B \rangle$

Therefore,

$$RQ_2((A,B),M) = \frac{A \circ A^* + B \circ B^*}{\langle A, A \rangle + \langle B, B \rangle}.$$

Moreover, as

$$F(x) \circ F^{*}(x) + G(x) \circ G^{*}(x)$$

$$= \Big(C(y) \circ C^{+}(y) + D(y) \circ D^{*}(y) \Big) \Big(A(z) A^{*}(z) + B(z) B^{*}(z) \Big).$$

Moreover, as

$$F(x)\circ F^*(x) + G(x)\circ G^*(x)$$
$$= \left(C(y)\circ C^+(y) + D(y)\circ D^+(y)\right)\left(A(z)A^+(z) + B(z)B^+(z)\right).$$

Then,

$$RQ_2\left((F,G), M_{FG}\right) = \frac{\lambda_{FG}}{\langle F,F\rangle + \langle G,G\rangle},$$

where $\lambda_{FG} = \lambda_{CD}\lambda_{AB}$, and $M_{FG} = M_{CD} \otimes M_{AB}$.

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$\lambda_{FG} = \pm 1$ and F, G eigenvectors of M_{FG}.

Special case if $M_{AB} = M_{CD} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

... construct eigenvectors of Hadamard transform.

Complementary Pairs - Boolean functions

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Type-I: $c_j^*(u_j)=c_j(u_j+1)$, Type-II: $c_j^*=c_j$

Type-III: $c_j^*=c_j+l_j$, $l_j=u_j\cdot 1$.

Note:

for Type-II:

$$f_j = (c_j + d_j)(f_{j-1} + g_{j-1}) + c_j + f_{j-1}$$

$$g_j = (c_j + d_j)(f_{j-1} + g_{j-1}) + d_j + g_{j-1}.$$

Note:

for Type-II:

$$f_j = (c_j + d_j)(f_{j-1} + g_{j-1}) + c_j + f_{j-1}$$
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identical to indirect construction for bent functions (Carlet), and the special case being a construction for self-dual bent functions.

Note:

for Type-II:

eigenvectors of Hadamard transform

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identical to indirect construction for bent functions (Carlet), and the special case being a construction for self-dual bent functions.

Complementary Sets - Boolean functions

closed-form:

$$f_j = \sum_{p=0}^{j} \Bigg(v_p + o_p(w_p) + \sum_{k=0}^{t-1} \begin{pmatrix} \Theta_{p,k} & 0 \\ 0 & \hat{\Theta}_{p,k} \end{pmatrix}_k J_k$$

$$\times \sum_{q=0}^{p-1} \left(\prod_{r=q+1}^{p-1} e_{\Theta,r,k} \right) v_q \Bigg).$$

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vector of length 2^t

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permutations: $\mathbb{F}_2^t \to \mathbb{F}_2^t$

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Complementary Sets - Boolean functions

alternative closed-form:

$$f_j = \left(f_{j,s},\, s \in \mathbb{F}_2^t\right),$$

$$f_{j,s} = \sum_{q=0}^{j} v_q\left(h_{\theta,q}\right) + o_q\left(w_q\right),$$

$$h_q = \bigwedge_{p=q+1}^{j+1} w_p, \qquad w_{j,k} = c_{j,k} + d_{j,k}^*$$

Complementary Sets - Boolean Functions

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Complementary Sets - Boolean Functions

alternative closed-form:

Note: '*' can take any valid definition.

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size 2^t vector

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- Type-I BPSK "primitive" sequence pairs known at lengths 2, 10, 20, 26. Any more at length > 100??

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- Type-II or III BPSK primitive sequence pairs??.... e.g. length-3 Type-III pair known. .. arrays?
- Generalise the above from pairs to sets, to larger alphabets, near-complementary.

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