Graph Classes with Structured Neighborhoods and Algorithmic Applications*

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Abstract. Boolean-width is a recently introduced graph width parameter. If a boolean decomposition of width w is given, several NP-complete problems, such as Maximum Weight Independent Set, k-Coloring and Minimum Weight Dominating Set are solvable in $O^*(2^{O(w)})$ time [5]. In this paper we study graph classes for which we can compute a decomposition of logarithmic boolean-width in polynomial time. Since $2^{O(\log n)} = n^{O(1)}$, this gives polynomial time algorithms for the above problems on these graph classes. For interval graphs we show how to construct decompositions where neighborhoods of vertex subsets are nested. We generalize this idea to neighborhoods that can be represented by a constant number of vertices. Moreover we show that these decompositions have boolean-width $O(\log n)$. Graph classes having such decompositions include circular arc graphs, circular k-trapezoid graphs, convex graphs, Dilworth k graphs, k-polygon graphs and complements of k-degenerate graphs. Combined with results in [1, 6], this implies that a large class of vertex subset and vertex partitioning problems can be solved in polynomial time on these graph classes.

1 Introduction

Two common ways of coping with NP-hard graph problems are to restrict instances to a certain graph class where the problem is polynomial, or to give FPT algorithms parameterized by a graph width parameter. In this paper we combine these two approaches by exploiting the fact that an FPT algorithm with running-time $2^{O(w)} \cdot poly(n)$ is polynomial whenever w is $O(\log n)$.

A theorem by Courcelle, Makovski and Rotics [10] states that every problem expressible in MSO_1 logic can be solved in linear time on graphs of bounded clique-width. Examples of graph classes with bounded clique-width can be found in Group I of Figure 1. However, many interesting classes of graphs have unbounded clique-width (see [4] and [16]). In order to obtain algorithms for larger classes of graphs, we have to compromise by considering a smaller range of problems or having less efficient running time. An example of such algorithms, related to the results in this paper, was shown by Kratochvíl, Manuel and Miller in [23],

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where a large class of the (σ, ρ) vertex subset problems was shown to be solvable in polynomial time on interval graphs.

Boolean-width is a graph parameter recently introduced by Bui-Xuan, Telle and Vatshelle [5]. They present algorithms for solving Maximum Weight Independent Set and Minimum Weight Dominating Set in $2^{O(w)} \cdot poly(n)$ time, given a decomposition of boolean-width w. In this paper we study classes of graphs with boolean-width $O(\log n)$. We show that a large class of graphs including interval graphs, permutation graphs, convex graphs, circular k-trapezoid graphs, Dilworth k graphs and complements of planar graphs have boolean-width $O(\log n)$ (see Group II of Figure 1). Combining our results with the results in [5] leads to polynomial time algorithms for problems such as Minimum Weight Dominating Set and Maximum Weight Independent Set, for all the graph classes in Group I and II of Figure 1. To our knowledge, this is the first time an FPT algorithm parameterized by a graph width parameter is used to give a polynomial time algorithm on a graph class where the parameter value is not bounded by a constant. Note that our result unifies and generalizes several previous polynomial time algorithms for Minimum Weight Dominating Set. Interestingly, there is no graph class whose boolean-width is known *not* to be $O(\log n)$ for which Minimum Weight Dominating Set can be solved in polynomial time. We are also able to prove that for most of the graph classes discussed in this paper the upper bounds we give on boolean-width are tight up to a constant factor, using the fact that they have clique-width $\Omega(\sqrt{n})$.

In the simple case of interval graphs we show how to construct decompositions such that every cut $(A, \overline{A})$ has nested neighborhoods, i.e. for every pair of vertices of A, the neighborhood of one is a subset of the neighborhood of the other when restricted to $\overline{A}$. We generalize the idea of a cut with nested neighborhoods to the notion of representative-size. We say a cut $(A, \overline{A})$ has representative-size r if every subset of A contains another subset of size at most r having the same neighborhood in $\overline{A}$. We also show that these decompositions have boolean-width $O(\log n)$, since there is only a polynomial number of subsets of constant size. Our proofs depend on having a certain representation of the input graph. For most of the graph classes discussed in this paper the required representation can be computed in polynomial time, meaning we can in polynomial time build a decomposition given a graph belonging to the graph class.

Telle and Proskurowski [30] introduced a framework covering a large class of vertex subset and vertex partitioning problems. This framework includes several well studied problems, among which are Maximum Independent Set and Minimum Dominating Set, but also Perfect Code, k-Coloring, H-Cover, H-Homomorphism and H-Role Assignment. We use the algorithm Bui-Xuan et al. gave in [1, 6] to show that all the problems covered by this framework can be solved in polynomial time on all the graph classes in Group I and II of Figure 1.

In Section 2, we start by introducing standard graph theoretic notions and define boolean-width as well as some of the related terminology. We also define formally the notion of representing a neighborhood by a smaller set of vertices,

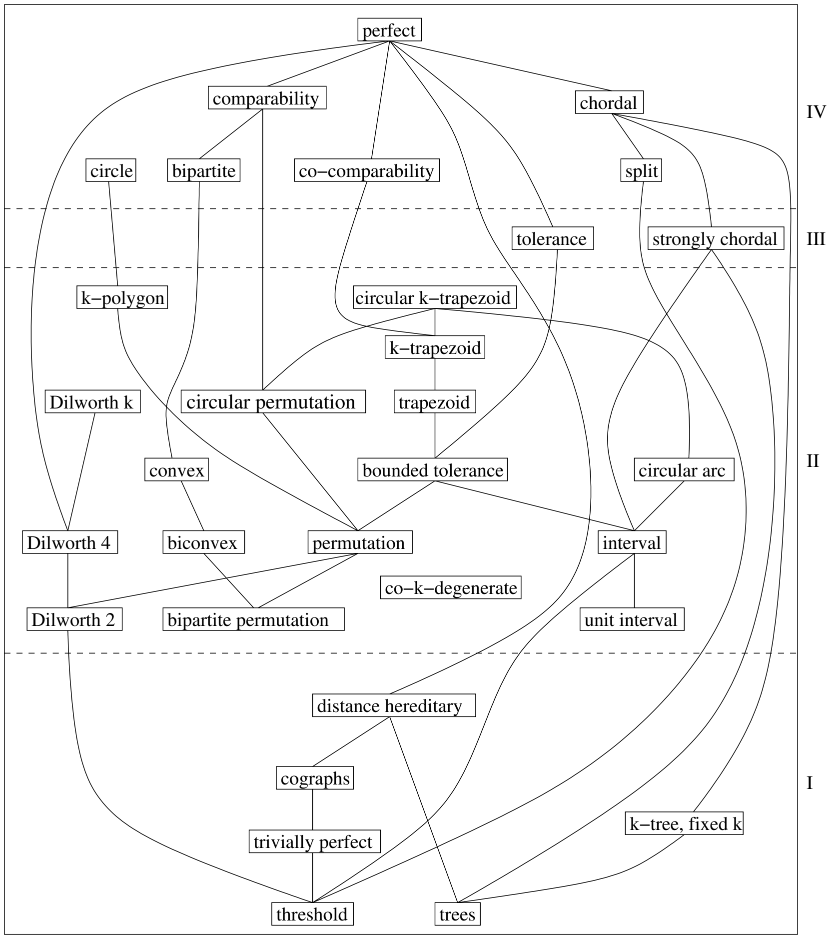

Fig. 1. Inclusion diagram of some well-known graph classes. A link between a higher class A and a lower class B means that B is a subclass of A. (I) Graph classes where boolean-width is bounded by a constant. (II) Graph classes having boolean-width $O(\log n)$. (III) It is unknown whether boolean-width is $O(\log n)$. (IV) There does not always exist a boolean-decomposition of value $O(\log n)$, or it is NP-complete to compute it. Many vertex subset and vertex partitioning problems can be solved in polynomial time on graph classes in Group I and II.

Main result. We show that Dilworth k graphs, convex graphs, trapezoid graphs, circular permutation graphs, circular arc graphs and complements of k-degenerate graphs and circular k-trapezoid graphs have decompositions where neighborhoods can be represented by a constant number of vertices. This implies that a large class of vertex subset and vertex partitioning problems are solvable in polynomial time on these graph classes given their intersection model. This generalizes algorithms for many problems on many graph classes, like [2, 8, 11, 12, 13, 14, 17, 18, 24, 27, 31].

whose size we call "representative-size" and relate this notion to boolean-width. In section 3, we show classes of graphs having representative-size bounded by a constant. In section 4, we show that constant representative-size allows to apply the results in [1, 6] and get polynomial time algorithms for the large class of vertex subset and vertex partitioning problems defined by Telle and Proskurowski [30]. Finally, in section 5 we show that our upper bounds are tight up to a constant factor and give evidence that a large class of graphs cannot have logarithmic boolean-width.

2 Framework

All graphs considered in this paper are undirected, finite and simple. A *graph* G is a pair (V, E) where V is the set of vertices of G and E is the set of edges. The *neighborhood* of a vertex u, denoted by $N(u)$, is the set of vertices u such that the edge $\{u, v\}$ is in E. Given a set $A \subseteq V$, we denote by $\overline{A}$ the complement of A in V, i.e. $V \setminus A$. We call a bipartition $(A, \overline{A})$ of V a *cut* of G. Given a cut $(A, \overline{A})$ of G and $u \in A$, we call the set $N(u) \cap \overline{A}$ the *neighborhood of u across* $(A, \overline{A})$.

When applying divide-and-conquer to solve a graph problem, we first need to divide the input graph. A common way to store the information of how to divide a graph is to use a decomposition tree. The choice of a decomposition tree greatly influences the running time of any algorithm using the decomposition tree. In order to choose the best decomposition tree, we evaluate a decomposition tree by using a cut function. The following formalism is referred to as *branch decomposition* of a cut function and is standard in graph and matroid theory (see, e.g., [15, 26, 29]).

Definition 1. *A* decomposition tree *of a graph $G = (V, E)$ is a pair (T, δ) where T is a tree having internal nodes of degree three and $|V|$ leaves, and δ is a bijection between the vertices of G and the leaves of T. Every edge of T defines a cut $(A, \overline{A})$ of the graph via δ, by the leaves of the two subtrees of T we get by removing the edge. Let $f : 2^V \to \mathbb{R}$ be a symmetric function, i.e. $f(A) = f(\overline{A})$ for all $A \subseteq V$, also called a* cut function. *The f*-width of (T, δ) *is the maximum value of $f(A)$, taken over all cuts $(A, \overline{A})$ of G given by an edge of T. The f*-width of G *is the minimum f-width over all decomposition trees of G.*

The following equivalence relation on subsets of A was introduced in [5] and serves as a basis for the definition of boolean-width:

Definition 2. *Let $G = (V, E)$ be a graph and $A \subseteq V$. Two vertex subsets $X, X' \subseteq A$ are* neighborhood equivalent *with respect to A, denoted by $X \equiv_A X'$, if $N(X) \cap \overline{A} = N(X') \cap \overline{A}$. We denote by $nec(\equiv_A)$ the number of equivalence classes of $\equiv_A$.*

Definition 3. **[5]** *The cut-bool function of a graph G is defined as cut-bool$(A) = \log_2 nec(\equiv_A)$. Using Definition 1 with $f =$ cut-bool we define the boolean-width of a decomposition, denoted $boolw(T, \delta)$, and the boolean-width of a graph, denoted $boolw(G)$.*

It is known from boolean matrix theory that *cut-bool* is symmetric [21]. For more background on boolean-width, see the full version of [5].

Definition 4 (Representative-size). *Let $G = (V, E)$ be a graph and $(A, \overline{A})$ a cut of G. We say that the cut $(A, \overline{A})$ has representative-size r if r is the smallest integer such that for every subset S of A, there exists a set $S' \subseteq S$ with $|S'| \leq r$ and $S \equiv_A S'$. We denote by rep-size(A) the representative-size of the cut $(A, \overline{A})$. Using Definition 1 with $f =$ rep-size we define the representative-size of a decomposition, denoted rep-size(T, δ), and the representative-size of a graph, denoted rep-size(G).*

The next lemma relates representative-size and boolean-width:

Lemma 1. *Let $G = (V, E)$ be a graph, and (T, δ) a decomposition of G. If the representative-size of (T, δ) is r, then the boolean-width of (T, δ) is at most $r \log_2(|V|)$.*

Proof. For any cut $(A, \overline{A})$ of the decomposition (T, δ), we know that $rep\text{-}size(A)$ is at most r. This means that given any set $S \in A$, there exists a set S' such that $|S'| \leq r$ and $S \equiv_A S'$. Clearly, there are at most $\binom{|V|}{r} \leq |V|^r$ subsets of A of cardinality at most r. Hence we have that boolean-width is at most $\log_2 |V|^r = r \log_2 |V|$. □

Caterpillar decompositions are decompositions where the underlying tree is a path with one leaf added as neighbor of each internal node of the path. Many of our proofs will construct caterpillar decompositions. To describe a caterpillar decomposition of a graph G, we only give an ordering $v_1, \ldots, v_n$ of the vertices of G. To construct the caterpillar decomposition (T, δ) from an ordering, first construct a caterpillar T from a path $u_1, \ldots, u_n$ of length $|V|$. Then let δ be a mapping of v_1 to u_1, v_n to u_n and for all $i \in \{2, \ldots, n-1\}$, of v_i to the leaf attached to u_i.

Many of the graph classes we study in this paper are special cases of intersection graphs. Let $\mathcal{F}$ be a family of nonempty sets. The *intersection graph* of $\mathcal{F}$ is obtained by representing each set in $\mathcal{F}$ by a vertex and connecting two vertices by an edge if and only if their corresponding sets intersect. The intersection model $\mathcal{F}$ usually consists of geometrical objects such as intervals of the real line.

3 Upper Bounds on Boolean-width of Graph Classes

In this section we prove upper bounds on the boolean-width of several classes of graphs. Throughout the paper, when talking about a class of graphs, we denote by n the number of vertices $|V|$. We say that a class of graphs $\mathcal{C}$ has boolean-width $f(n)$ if every graph belonging to $\mathcal{C}$ has boolean-width at most $f(n)$. In particular, we focus on classes of graphs having boolean-width $O(\log n)$. We prove that the graph classes in Group II of Figure 1 have representative-size bounded by a constant. Combining this with Lemma 1 implies that they also have boolean-width $O(\log n)$.

First, we give a sketch of the proof for interval graphs showing that they have representative-size 1. We build the decomposition by ordering the vertices by the left endpoint of their intervals, then across each cut $(A, \overline{A})$ of the decomposition the neighborhood of the vertices are nested in order of right endpoint of their intervals. This means that, for every pair of vertices of A, the neighborhood of one is a subset of the neighborhood of the other when restricted to $\overline{A}$. Now we extend this idea to circular-arc graphs, which are the intersection graphs of arcs on a circle.

Lemma 2. *Given a circular-arc graph G we can, in polynomial time, compute a decomposition of G having representative-size at most 2 and boolean-width at most $2 \log n$.*

Proof. We compute the circular-arc intersection model of G in polynomial time using the algorithm of McConnell [25]. Let p be an arbitrary point on the circle. We define the distance of an arc from p as follows: if the arc contains p, then the distance is 0, otherwise it is the minimum distance between p and any point of the arc. For any vertex u, we denote by arc_u the arc corresponding to u. Note that since p is an arbitrary point then no pair of arcs have the same distance from p unless they intersect.

Build a caterpillar decomposition by adding the vertices in order of increasing distance of their associated arc from p, breaking ties arbitrarily. Note that this decomposition can be computed in polynomial time. We now consider any cut $(A, \overline{A})$ of this decomposition. By construction, for every $x \in A, y \in \overline{A}$, the distance of arc_x from p is less than or equal to the distance of arc_y from p.

Now, we prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2$ and $S \equiv_A S'$. Let d be the smallest distance from p to the arc of any vertex in $\overline{A}$. Let p^+ be the point on the circle which is at distance d going in clockwise direction from p. Likewise, p^- is the point at distance d going in counter-clockwise direction from p. We build S' starting from the empty set. If there exists a vertex in S whose arc contains p^+, then let u be one such vertex with arc_u extending furthest from p^+ in clockwise direction and add u to S'. Likewise, if there exists a vertex in S whose arc contains p^-, then let v be one such vertex with arc_v extending furthest from p^- in counter-clockwise direction and add v to S'. Now we prove that $N(S) \cap \overline{A} = N(S') \cap \overline{A}$.

Let z be some vertex of $N(S) \cap \overline{A}$, if no such z exists $S' = \emptyset$ satisfies the lemma. Assume for contradiction that $z \notin N(S')$. Let w be a vertex of $N(z) \cap S$ and p_i any point contained in both arc_w and arc_z. Since any arc of a vertex in A is at distance at most d from p and p_i is at distance at least d from p, then arc_w contains both p_i and a point of distance at most d from p. We can assume without loss of generality that arc_w contains all points from p^+ to p_i in clockwise direction. Since arc_u is the arc extending furthest in clockwise direction from p^+, arc_u will also contain p_i, contradicting the choice of p_i.

Therefore $S \equiv_A S'$, which implies that the decomposition we built has representative-size at most 2. By applying Lemma 1 it follows that circular-arc graphs have boolean-width at most $2 \log n$. □

We show a similar result for several other classes of graphs but their definitions and proofs are in the appendix due to space limitation. The proof for circular-arc graphs contains all the important ideas. The definitions of the graph classes can also be found in [3] or [28].

Theorem 1. *Convex graphs, circular-arc graphs, circular permutation graphs and trapezoid graphs have representative-size $O(1)$ and boolean-width $O(\log n)$.*

The graph classes in Group II of Figure 1 involving a parameter k are dealt with in Theorem 2. As an example, the proof showing that k-trapezoid graphs have representative-size at most k can be sketched as follows. A k-trapezoid is the polygon obtained by choosing an interval on each of k parallel lines in the plane and connecting the left and right endpoints of each neighboring interval. k-trapezoid graphs are intersection graphs of k-trapezoids. First, we build the caterpillar decomposition by ordering the k-trapezoids by their leftmost point. Then, for any cut $(A, \overline{A})$ of the decomposition and any set $S \subseteq A$, there is one k-trapezoid extending further to the right on each of the k lines. We call the set of vertices associated with these k-trapezoids S'. Moreover, for every vertex of S, any of its neighbors in $\overline{A}$ is also adjacent to at least one of the vertices in S'. Hence we have $S' \subseteq S, |S'| \leq k$ and $S' \equiv_A S$.

Theorem 2. *Complements of k-degenerate graphs, Dilworth k graphs, k-polygon graphs and circular k-trapezoid graphs have representative-size $O(k)$ and thus boolean-width $O(k \log n)$.*

Note that Theorem 1 and 2 encompass all graph classes in Group I and II of Figure 1. We find it interesting to note that some of these classes are seemingly unrelated to each other, but they all have decompositions sharing a common neighborhood structure, which allows for efficient dynamic programming approaches on a large class of problems. In particular, we combine these results with the following:

Theorem 3 (Bui-Xuan, Telle, Vatshelle [5]). *For any graph $G = (V, E)$,* Minimum Weight Dominating Set *can be solved in $O(|V|^2 + |V| \cdot w \cdot 2^{3 \cdot w})$ time when given a decomposition of G having boolean-width w.*

Combining Theorem 3 with Theorem 1 and 2, we get:

Corollary 1. Minimum Weight Dominating Set *can be solved in polynomial time on all the graph classes in Group I and II of Figure 1.*

The next section shows how to extend this result to a larger class of problems.

4 Vertex partitioning problems

In [30] Proskurowski and Telle introduced a generalized framework for handling many types of vertex subset and vertex partitioning problems in a unified manner. These problems can be described by a degree constraint matrix.

Definition 5. *A degree constraint matrix D_q is a q by q matrix with entries being finite or co-finite subsets of natural numbers. A D_q-partition in a graph G is a partition $\{V_1, V_2, ..., V_q\}$ of V such that for $1 \leq i, j \leq q$ we have $\forall v \in V_i$: $|N(v) \cap V_j| \in D_q[i, j]$.*

A D_q vertex partitioning problem is the problem of finding a D_q partition satisfying a given D_q matrix and optionally maximizing or minimizing the weight of a given class of the D_q partition. This formalism was introduced by Telle and Proskurowski and encompass several well studied problems, such Maximum Independent Set, Minimum Dominating Set, Perfect Code, k-Coloring, H-Cover, H-Homomorphism and H-Role Assignment. The class of (σ, ρ) vertex subset problems is a subset of D_q vertex partitioning problems. For example, Maximum Independent Dominating Set is encoded by a 2 by 2 matrix with entries $[1, 1] = \{0\}, [1, 2] = \{0, 1, \dots\}, [2, 1] = \{1, 2, \dots\}$ and $[2, 2] = \{0, 1, \dots\}$, and maximizing the size of V_1. H-Homomorphism for a graph H on q vertices simply asks for the existence of a partition satisfying the q by q matrix constructed from the adjacency matrix of H by replacing entry 0 with $\{0\}$ and 1 with $\{0, 1, \dots\}$. Telle and Proskurowski showed that all D_q-problems are solvable in FPT time parameterized by tree-width [30]. Kobler and Rotics showed that D_q-problems are solvable on graphs of bounded clique-width [22], and with a little effort their algorithms can be made into FPT algorithms. Bui-Xuan et al. showed that D_q-problems are FPT when parameterized by boolean-width [1]. Kratochvíl et al. [23] showed that a subset of the D_q-problems are solvable in polynomial time on interval graphs. We generalize the results of [23] by showing that all D_q-problems are solvable in polynomial time on many well known graph classes, including interval graphs.

We will apply the algorithm of Bui-Xuan et al. [1], where the bottleneck for running time is the number of equivalence classes of d-neighborhoods. When solving a D_q-problem, the integer value $d(D_q)$ needed depends on the degree constraint matrix in the following way. Let $d(\{0, 1, \dots\}) = 0$. For every finite or co-finite non-empty set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max x : x \in \mu, \max x : x \notin \mu)$. For a matrix D_q, the value $d(D_q)$ will be $\max_{i,j} d(D_q[i, j])$. When there is no ambiguity, we denote $d(D_q)$ by d. Note that d depends only on the problem and hence can be treated as a constant.

Definition 6 (d-neighbor equivalence). *Let $G = (V, E)$ be a graph, $A \subseteq V$ and d a positive integer. Two vertex subsets $X \subseteq A$ and $X' \subseteq A$ are d-neighbor equivalent with respect to A, denoted $X \equiv_A^d X'$ if:*
$\forall v \in \overline{A}, (|N(v) \cap X| = |N(v) \cap X'|)$ *or* $(|N(v) \cap X| \geq d$ *and* $|N(v) \cap X'| \geq d)$
We denote by $nec(\equiv_A^d)$ the number of equivalence classes of $\equiv_A^d$.

Note that X and X' are 1-neighborhood equivalent with regard to A if and only if $N(X) \cap \overline{A} = N(X') \cap \overline{A}$ and thus $nec(\equiv_A) = nec(\equiv_A^1)$. We show a connection between representative-size and d-neighbor equivalence.

Lemma 3. *Let $G = (V, E)$ be a graph and $(A, \overline{A})$ a cut of G. If rep-size$(A) = r$, then for every positive integer d and every set $X \subseteq A$, there exists $X_d \subseteq X$ such that $|X_d| \leq d \cdot r$ and $X_d \equiv_A^d X$.*

Proof. We prove the statement by induction on d. Let $R \subseteq X$ be an inclusion minimal set such that $N(R) \cap \overline{A} = N(X) \cap \overline{A}$. Since the representative-size of $(A, \overline{A})$ is r, we have that $|R| \leq r$. For $d \leq 1$ the lemma holds since $R \equiv_A^1 X$. Assume the induction hypothesis true up to $i-1$, then we show it true for i. By induction hypothesis there exists $X_{i-1} \subseteq (X \setminus R)$ such that $X_{i-1} \equiv_A^{i-1} (X \setminus R)$ and $|X_{i-1}| \leq r \cdot (i-1)$. Thus it is enough to show $X_i \equiv_A^i X$, for $X_i = X_{i-1} \cup R$.

We partition the nodes of $\overline{A}$ into (P, Q) such that $\forall v \in P$, we have $|N(v) \cap (X \setminus R)| = |N(v) \cap X_{i-1}|$ and $\forall v \in Q$, we have $|N(v) \cap (X \setminus R)| \geq i-1$ and $|N(v) \cap X_{i-1}| \geq i-1$. Since $R \cap X_{i-1} = \emptyset$ and $R \subseteq X$, we know $|N(v) \cap (X \setminus R)| = |N(v) \cap X| - |N(v) \cap R|$ and $|N(v) \cap (X_{i-1} \cup R)| = |N(v) \cap X_{i-1}| + |N(v) \cap R|$. Hence for every vertex $v \in P$, we have $|N(v) \cap X| = |N(v) \cap X_{i-1}| + |N(v) \cap R| = |N(v) \cap (X_{i-1} \cup R)|$. Since $i > 1$, then for every vertex $v \in Q$ we have $N(v) \cap R \neq \emptyset$. Since $X \equiv_A R$, then for every vertex $v \in Q$ we have $|N(v) \cap X| \geq i$ and $|N(v) \cap X_i| \geq d$.

Since (P, Q) is a partition we get $X_i \equiv_A^i X$ and $|X_i| \leq r \cdot i$, thus by induction the lemma holds for all i. □

For a decomposition (T, δ) of a graph G, let $nec_d(T, \delta)$ be the maximum $nec(\equiv_A^d)$ over all cuts $(A, \overline{A})$ of (T, δ).

Lemma 4. *Let $G = (V, E)$ be a graph, (T, δ) a decomposition of G and d a positive integer. If rep-size$(T, \delta) = r$, then $nec_d(T, \delta) \leq |V|^{d \cdot r}$.*

Proof. For any cut $(A, \overline{A})$ of the decomposition (T, δ), we know that $rep\text{-}size(A)$ is at most r. From Lemma 3 we know that for any $S \subseteq A$ there exists a set S' such that $|S'| \leq d \cdot r$ and $S \equiv_A^d S'$. Clearly, there are at most $\binom{|V|}{d \cdot r} \leq |V|^{d \cdot r}$ subsets of A of cardinality at most $d \cdot r$. Hence $nec_d(T, \delta) \leq |V|^{d \cdot r}$. □

By combining Lemma 4 with Theorem 1 and Theorem 2 we get:

Theorem 4. *Let $G = (V, E)$ be a graph in Group I or II of Figure 1, then we can in polynomial time compute a decomposition (T, δ) such that $nec_d(T, \delta)$ is polynomial in $|V|$ assuming an intersection model of G is provided.*

Theorem 5 (Bui-Xuan, Telle, Vatshelle [6][1]). *For any graph $G = (V, E)$ and (T, δ) a decomposition of G, all D_q vertex partitioning problems can be solved in $O(nec_d(T, \delta)^{3 \cdot q} \cdot poly(|V|))$ time.*

Combining Theorem 4 with Theorem 5, we get:

Corollary 2. *All D_q vertex partitioning problems can be solved in polynomial time on all the graph classes in Group I and II of Figure 1 assuming an intersection model of the input graph is provided.*

[1] [6] is an arXiv version of [1] containing a more fitting version of this theorem.

5 Lower bounds

We say that a class of graphs $\mathcal{C}$ has boolean-width $\Omega(f(n))$ if there exists an infinite family of graphs in $\mathcal{C}$ all having boolean-width $\Omega(f(n))$. In this section we show that the upper bounds we gave on the boolean-width are tight in two senses. Firstly, for all graph classes (except Dilworth k graphs) in Group II of Figure 1, we are able to show that they have boolean-width $\Omega(\log n)$. Secondly, we show that for all graph classes in Group IV of Figure 1, it is highly unlikely that they have boolean-width $O(\log n)$. Note the following result on the relation between boolean-width and some other width parameters:

Theorem 6 (Bui-Xuan, Telle, Vatshelle [5]). *For any graph G it holds that $\log rw(G) - 1 \leq \log cw(G) - 1 \leq boolw(G)$, where $boolw(G), rw(G)$ and $cw(G)$ denote respectively the boolean-width, rank-width and clique-width of G.*

Hence if a graph class has rank-width or clique-width $\Omega(n^c)$ for some constant $c > 0$, then this graph class also has boolean-width $\Omega(\log n)$. We use this to show that the bounds we give in this paper are tight up to a constant factor.

Lemma 5. *All graph classes in Group II of Figure 1 (except Dilworth k graphs), have boolean-width $\Theta(\log n)$.*

Proof. Brandstädt and Lozin showed in [4] an infinite family of bipartite permutation graphs with clique-width $\Omega(\sqrt{n})$. Likewise, Golumbic and Rotics showed in [16] an infinite family of unit interval graphs with clique-width $\Omega(\sqrt{n})$. Moreover, Jelínek showed in [19] that $q \times q$ grids have rank-width exactly $q - 1$. Note that if a graph G has rank-width w, then its complement $\overline{G}$ has rank-width $w \pm 1$. Since all grids are 2-degenerate, then complements of 2-degenerate graphs have rank-width $\Omega(\sqrt{n})$. From Theorem 6, it follows that these three graph classes have boolean-width $\Theta(\log n)$. Hence the lemma follows since all graph classes in Group II of Figure 1 contain one these graph classes. □

Another interesting question to ask is whether there exist more graph classes having logarithmic boolean-width. For some graph classes it is possible to provide examples of an infinite family of graphs having non-logarithmic boolean-width, for example the grid. However, for some classes of graphs, we do not know any example of infinite family of graphs having non-logarithmic boolean-width. We are nonetheless able to provide some lower bounds:

Lemma 6. *For all the classes in Group IV of Figure 1, either they do not have boolean-width $O(\log n)$, or a decomposition of boolean-width $O(\log n)$ cannot be computed in polynomial time, unless $P = NP$.*

Proof. Note first that for all the classes of graphs in Group IV of Figure 1, Minimum Weight Dominating Set is NP-complete (see [9], [7] and [20]). Moreover, Minimum Weight Dominating Set can be solved in time $O(2^{3 \cdot boolw} \cdot poly(n))$. Assume now that there exists a class $\mathcal{C}$ in Group IV of Figure 1 having boolean-width $O(\log n)$ and where such decompositions can be computed in polynomial time. Then Minimum Weight Dominating Set can be computed in time $O(2^{O(\log n)} \cdot poly(n))$, which is a polynomial of n. Hence if a class of graphs on which Minimum Weight Dominating Set is NP-complete has boolean-width $O(\log n)$, then decompositions of boolean-width $O(\log n)$ cannot be computed in polynomial time, unless $P = NP$. □

Note that this holds not only for Minimum Weight Dominating Set, but as long as there exists a problem which can be solved in $O(2^{O(boolw} \cdot poly(n))$ time. Finally, we get better lower bounds by working under a stronger hypothesis. The Exponential Time Hypothesis (ETH) states that there does not exists an algorithm for solving 3-SAT running in time $2^{o(n)}$. We can reformulate Lemma 6 as follows:

Lemma 7. *For all the classes in Group IV of Figure 1, either they do not have boolean-width $n^{o(1)}$, or a decomposition of boolean-width $n^{o(1)}$ cannot be computed in time $2^{o(n)}$, unless ETH fails.*

Proof. Assume for contradiction that there exists a class of graphs $\mathcal{C}$ in Group IV of Figure 1 for which a decomposition of boolean-width $n^{o(1)}$ can be computed in time $2^{o(n)}$. Recall that Minimum Weight Dominating Set is NP-complete on all the classes in Group IV of Figure 1. Hence, there is a polynomial time reduction from k-SAT to Minimum Weight Dominating Set on $\mathcal{C}$ such that from any instance I of k-SAT, a graph $G = (V, E)$ belonging to $\mathcal{C}$ can be built such that $|V| \leq |I|^c$, for some constant $c > 0$ and solving Minimum Weight Dominating Set on G implies a solution to k-SAT on I. Recall that Minimum Weight Dominating Set can be solved in $2^{3 \cdot boolw(G)} \cdot poly(|V|)$. Finally, since we assumed we could compute a decomposition of boolean-width $n^{o(1)}$ in time $2^{o(n)}$, the instance I can be solved in $2^{3 \cdot |V|^{o(1)}} \cdot poly(|V|)$, which is equivalent to $2^{3 \cdot |I|^{o(c)}} \cdot poly(|I|^c)$. This would imply that we could the instance I in time $2^o(|I|)$. Hence the Lemma follows. □

This means for instance that if split graphs have boolean-width *poly-log*(n), then it is NP-hard to compute a decomposition of split graphs having boolean-width within a factor $\log(n)$ of the optimum.

6 Conclusion

We have shown that all graph classes in Group II of Figure 1 have logarithmic boolean-width and we can compute such decompositions of logarithmic boolean-width, answering an open question from [5]. Applying the algorithm for vertex partitioning problems (as well as their weighted versions) in [1, 6], we show several graph classes for which a large class of vertex partitioning problems can be solved in polynomial time. What is the boolean-width of the graph classes in Group III of Figure 1? Is there any graph class not having boolean-width $O(\log n)$ where Minimum Weight Dominating Set is polynomially solvable?

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Appendix

Dilworth *k* graphs

Definition 7. *Two vertices x and y are said to be comparable if either $N(y) \subseteq N[x]$ or $N(x) \subseteq N[y]$. The Dilworth number of a graph is the largest number of pairwise incomparable vertices of the graph. A graph is a Dilworth k graph if it has Dilworth number k.*

Lemma 8. *Let G be a Dilworth k graph, then we can compute in polynomial time a decomposition of representative-size at most k.*

Proof. Let us consider any cut $(A, \bar{A})$ of G. Let $S' \subseteq S$ be an inclusion minimal set such that $S' \equiv_A S$. We want to prove that $|S'| \leq k$.

Assume for contradiction that $|S'| > k$. Since V cannot contain more than k pairwise incomparable vertices, S' contains two vertices x and y such that $N(y) \subseteq N[x]$. Hence $S' \equiv_A (S' \setminus \{y\})$, which contradicts the assumption that S' was inclusion minimal. Therefore any decomposition has representative-size at most k and by applying Lemma 1 it follows that Dilworth k graphs have boolean-width at most $k \log n$. □

Convex graphs

Definition 8. *A graph $G = (V, E)$ is convex if G is bipartite with color classes X and Y and an ordering $x_1, \ldots, x_{|X|}$ of X such that for every vertex $u \in Y$ and $x_i, x_j \in N(u)$, we have for every vertex $x_t \in X$ that if $i < t < j$ then $x_t \in N(u)$.*

Lemma 9. *Let G be a convex graph, then we can compute in polynomial time a decomposition of representative-size* 1.

Proof. Since G is convex we can in polynomial time find a bipartition (X, Y) of V and σ_X an ordering of X such that for every vertex $u \in Y$ and $x, y \in N(u)$. Hence we have for every vertex $z \in X$ that if $\sigma_X(x) < \sigma_X(z) < \sigma_X(y)$ then $z \in N(u)$. We construct a total ordering σ of V from σ_X by keeping the ordering of vertices in X and for each vertex $v \in Y$ we insert v immediately after the last element of $N(v)$. We construct a caterpillar decomposition from the order σ.

Let us now consider a cut $(A, \overline{A})$ of the decomposition. We want to prove that for any subset S of A, there exists a set $S' \subseteq S$ such that $S' \equiv_A S$ and $|S'| \leq 1$. Note that by construction of σ, we have $\forall v \in Y \cap A, N(v) \cap \overline{A} = \emptyset$, hence we can assume $S' \subseteq X \cap S$.

Let $v_1, v_2, \ldots, v_t$ be the ordering of the vertices of $X \cap S$ induced by σ. Since all the vertices in $Y \cap \overline{A}$ appear later in σ than v_t, then we have for every vertex $v \in Y \cap \overline{A}$, either $v_t \in N(v)$ or $N(v) \cap S = \emptyset$. Hence we can set $S' = \{v_t\}$ and have $S \equiv_A S'$. This implies that the decomposition we built has representative-size at most 1. By applying Lemma 1 it follows that convex graphs have boolean-width at most $\log n$. □

k-polygon graphs

Definition 9. *A k-polygon graph is the intersection graph of chords (straight lines between two points on distinct sides) of a convex k sided polygon.*

Lemma 10. *Let G be a k-polygon graph, then we can compute in polynomial time a decomposition of representative-size at most $2k$.*

Proof. Let p be an arbitrary corner of the k-polygon. We measure the distance of a point from p as the distance around the edge of the k-polygon in clockwise direction. We define the distance of a chord from p as the minimum distance of any endpoint of the chord from p. We build a caterpillar decomposition of G by ordering the vertices of G by increasing distance of their corresponding chord from p.

We now consider any cut $(A, \overline{A})$ of the decomposition. Now, we prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2k$ and $S' \equiv_A S$. We denote by d the maximum distance from p to a chord of any vertex in A. We can observe that, by construction of the decomposition, for every vertex u in A, if both endpoints of the chord corresponding to u are at distance at most d from p, then $N(u) \cap \overline{A} = \emptyset$. Now, we identify each side of the k-polygon with an index $i \in \{1, ..., k\}$ ordered in clockwise direction starting from p. For each side we define $S_i \subseteq S$ as the set of vertices of which line has an endpoint on side i. Each vertex of G belongs to exactly 2 such sets. We also define for each side i such that $S_i \neq \emptyset$, the vertex l_i as the vertex in S_i of which endpoint on side i is closest from p. Likewise r_i is the vertex in S_i of which endpoint on side i is furthest from p.

Let $S' = \bigcup_i \{l_i, r_i\}$. We now claim that $S' \equiv_A S$. More precisely, we claim that for every $S_i \neq \emptyset$, $S_i \equiv_A \{r_i, l_i\}$. Let us assume for contradiction that there exists a vertex $x \in (N(S) \setminus N(S')) \cap \overline{A}$. Let $a < b$ be the indexes of the two sides containing the endpoint of the chord associated to x. Let p_a and p_b be the endpoints of the chord corresponding to x on side a and b respectively.

Let $y \in S$ be a neighbor of x. Then there exists an integer j such that $y \in S_j$ and $a \leq j \leq b$. If $a < j < b$, then any vertex in S_j is adjacent to x. Let us assume now, without loss of generality, that $a = j$. Let q_r be the endpoint on side j of the chord corresponding to r_j. The chord corresponding to r_j partitions the k-polygon in two parts. Since the distance of p_a from p is at most the distance of q_r from p and the distance of p_b from p is at least the distance of q_r from p, then we have $x \in N(S')$.

Therefore $S \equiv_A S'$, which implies that the decomposition we built has representative-size at most $2k$. By applying Lemma 1 it follows that k-polygon graphs have boolean-width at most $2k \log n$. □

Circular *k*-trapezoid graphs

Definition 10. *Let $C_1, \ldots, C_k$ be k parallel circles on the surface of a cylinder, all orthogonal to its axis. In order to build a circular k-trapezoid, first choose two points s_i and e_i on each line. Then, make two non-intersecting paths s and e by joining s_i to s_{i+1} and e_i to e_{i+1} respectively by straight lines for each $i \in \{1, \ldots, k-1\}$. A circular k-trapezoid is the polygon defined by s, e and the arcs going from s_1 to e_1 and s_k to e_k in clockwise direction. A circular k-trapezoid graph is the intersection graph of circular k-trapezoids.*

Lemma 11. *Let G be a circular k-trapezoid graph, then we can compute in polynomial time a decomposition of representative-size at most $2k$, given a circular k-trapezoid model.*

Proof. Let p be an arbitrary point on C_1. We define the distance of a k-polygon from p as the minimum distance between p and any point of the arc of the k-polygon on C_1. For any vertex u, we denote by gon_u the k-polygon corresponding to u and $arc_{u,i}$ the arc of C_i contained in gon_u.

Build a caterpillar decomposition by adding the vertices in order of increasing distance of their associated k-polygon from p, breaking ties arbitrarily. We now consider any cut $(A, \overline{A})$ of this decomposition. By construction, for every $x \in A, y \in \overline{A}$, the distance of gon_x from p is less than or equal to the distance of gon_y from p.

Now, we prove that for any set $S \subseteq A$, there exists a subset $S' \subseteq S$ such that $|S'| \leq 2 \cdot k$ and $S \equiv_A S'$. Let d be the smallest distance from p to $arc_{u,i}$ for any vertex $u \in \overline{A}$. Let p^+ be the point on C_1 which is at distance d going in clockwise direction from p. Likewise, p^- is the point on C_1 at distance d going in counter-clockwise direction from p. If there exists a vertex in S whose arc contains p^+, then let u be one such vertex with $arc_{u,1}$ extending furthest from p^+ in clockwise direction and add u to S'. Otherwise, u is the vertex whose arc is closest from p^+ in counter-clockwise direction. Likewise, if there exists a vertex in S whose arc contains p^-, then let v be one such vertex with $arc_{v,1}$ extending furthest from p^- in counter-clockwise direction and add v to S'. Otherwise, v is the vertex whose arc is closest from p^- in clockwise direction.

If $\overline{A} \subseteq N(u) \cup N(v)$ then $S' = \{u, v\}$ satisfies the lemma. Hence we assume there exists a vertex $w \in \overline{A} \setminus N(S)$. We define the k-polygon $span(S)$ as the circular k-trapezoid of minimum area which contains every point of $gon_u, \forall u \in S$ and the arc of $span(S)$ on C_1 is the minimum arc which contains every point of $arc_{x,1}, \forall x \in S$ and no point of $arc_{w,1}$. Note that $span(S)$ can be described by an arc $span_i(S)$ on each circle C_i and We build S' starting from the empty set. For each $i \in \{1, k\}$ and endpoints q_i^+ and q_i^- of $span_i(S)$, add to S' vertices $u, v \in S$ such that $arc_{u,i}$ contains q_i^+ and $arc_{u,i}$ contains q_i^-. Note that u and v are not necessarily distinct. We denote by $N(span(S))$ the set $\{x \in V : gon_x$ intersects $span(S)\}$. Note also that $N(S') \cap \overline{A} \subseteq N(S) \cap \overline{A} \subseteq N(span(S)) \cap \overline{A}$. Hence it is enough to show that $N(S') \cap \overline{A} = span(S) \cap \overline{A}$.

Let z be some vertex of $N(S) \cap \overline{A}$, if no such z exists $S' = \emptyset$ satisfies the lemma. Assume for contradiction that $z \notin N(S')$. First, note that if $\exists i : arc_{z,i}$

contains one of the endpoints of $span_i(S)$, then by construction $z \in N(S')$. Note also that if $\exists i$: $arc_{z,i}$ do not intersect $span_i(S)$ and $arc_{z,i+1}$ do not intersect $span_{i+1}(S)$ but there is a point between C_i and C_{i+1} in which gon_z intersects $span(S)$, then $A \subseteq N(z)$. Moreover, if $arc_{z,1}$ is properly contained in $span_1(S)$, then it intersects either $arc_{u,1}$ or $arc_{v,1}$. Hence we know that $arc_{z,1}$ does not intersect $span_1(S)$ and let i be the smallest index such that $arc_{z,i}$ is properly contained in $span_i(S)$. Then, let Q^+ be the shortest path from q^+_{i-1} to q^+_i containing only points of $\bigcup_{x\in S'} gon_x$. Likewise, let Q^- be the shortest path from q^-_{i-1} to q^-_i containing only points of $\bigcup_{x\in S'} gon_x$. Q^+ and Q^- together with the arcs $span_{i-1}(S)$ and $span_i(S)$ form a closed curve which defines an area B. Since gon_z contain a point inside B and another outside B and does not intersect both the arcs $span_{i-1}(S)$ and $span_i(S)$, then gon_z intersects $\bigcup_{x\in S'} gon_x$. This contradicts the assumption that $z \notin N(S')$.

Therefore $S \equiv_A S'$, which implies that the decomposition we built has representative-size at most $2k$. By applying Lemma 1 it follows that circular k-trapezoid graphs have boolean-width at most $2k \log n$. □

Complements of k degenerate graphs

Definition 11. *A graph $G = (V, E)$ is k-degenerate if there exists an elimination ordering $v_1, \ldots, v_n$ of the vertices of G such that $\forall i \in \{1, \ldots, n\}, |\{v_j : j > i$ and $v_j \in N(v_i)\}| \leq k$.*

Lemma 12. *Let G be the complement of a k-degenerate graph, then we can compute in polynomial time a decomposition of representative-size at most $k+1$.*

Proof. We build a caterpillar decomposition of G using the elimination ordering induced by the k-degeneracy of $\overline{G}$. We consider a cut $(A, \overline{A})$ of the decomposition and a set $S \subseteq A$.

Note first that since $\overline{G}$ is k-degenerate, every vertex of A has at most k neighbors in $\overline{A}$. Therefore, in G every vertex of A has at least $|\overline{A}| - k$ neighbors in $\overline{A}$. Let x be a vertex of S and $U = \overline{A} \setminus N(x)$. Clearly $|U| \leq k$. We build S' in the following way: let S' be equal to $\{x\}$, for each vertex $u \in U \cap N(S)$, add one vertex from $N(u) \cap S$ to S'. Clearly $|S'| \leq k + 1$ and $N(S') \cap \overline{A} = (N(x) \cap \overline{A}) \cup U = N(S) \cap \overline{A}$.

Therefore $S \equiv_A S'$ and $|S'| \leq k + 1$, which implies that the decomposition we built has representative-size at most $k + 1$, and by applying Lemma 1 it follows that complement of k-degenerate graphs have boolean-width at most $(k + 1) \log n$. □