

The order of exponential polar-type splitting for some problems

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We are looking for numerical solution of the value problem

$$\begin{cases} y' = (A + B)y & t > 0 \\ y(0) = y_0 \end{cases}$$

If A and B are constant matrices the solution is

$$y(t) = \exp(t(A + B))y_0.$$

Splitting Methods

- **First order splitting methods:**

$$\begin{aligned} AB &: \exp(\Delta t A) \exp(\Delta t B), \\ BA &: \exp(\Delta t B) \exp(\Delta t A). \end{aligned}$$

- **Second order symmetric splitting method**

$$\begin{aligned} BAB &: \exp(\Delta t B/2) \exp(\Delta t A) \exp(\Delta t B/2) \\ ABA &: \exp(\Delta t A/2) \exp(\Delta t B) \exp(\Delta t A/2) \\ AB + BA &: \frac{1}{2} (\exp(\Delta t A) \exp(\Delta t B) + \exp(\Delta t B) \exp(\Delta t A)) \end{aligned}$$

which approximate $\exp(\Delta t(A + B))$.

$$E = \exp(\Delta t(A + B)) - SM,$$

where $SM = \{AB, BA, ABA, BAB, AB + BA\}$

- **IF \mathbf{A} , \mathbf{B} are nonstiff**

$\|\Delta t A\| \ll 1$ and $\|\Delta t B\| \ll 1$, then $E = e^{\Delta t A} B e^{\Delta t B}$
for all symmetric splittings;

- **IF \mathbf{A} is stiff and \mathbf{B} is nonstiff**

$$A = A_0 + \frac{A_1}{\varepsilon}; \quad \|A_0\| \leq C_0; \quad \|A_1\| \leq C_1;$$

$$C_0, C_1 = \text{const}; \quad 0 < \varepsilon \ll 1 \quad \text{and}$$

$$\|\Delta t A\| \gg 1; \quad \|\Delta t B\| \ll 1; \quad \text{then } E = E_1 + E_2$$

where $E_1 = O(\Delta t, \varepsilon)$ and $E_2 = O(\Delta t^2, \varepsilon)$

Polar-type splittings

$$\begin{aligned} p-t(2) &: \exp(\Delta t A - \frac{1}{2}\Delta t^2[A, B])\exp(\Delta t B), \\ p-t(4) &: \exp(\Delta t A - \frac{1}{2}\Delta t^2[A, B] - \frac{1}{3}\Delta t^3[B, [\\ &\quad \exp(\Delta t B - \frac{1}{12}\Delta t^3[A, [A, B]]) \end{aligned}$$

- **IF \mathbf{A} , \mathbf{B} are nonstiff**

$\|\Delta t A\| \ll 1$ and $\|\Delta t B\| \ll 1$, then

$$E_{p-t(2)} = O(\Delta t^3) \text{ and } E_{p-t(3)} = O(\Delta$$

- **IF \mathbf{A} is stiff and \mathbf{B} is nonstiff**

Example 1:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \quad B = \frac{1}{\varepsilon} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}; \quad \varepsilon = 10$$

Example 2:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \quad B = \frac{1}{\varepsilon} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \varepsilon = 10$$

Example 3:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{pmatrix}; \quad B = \frac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$