Computer Tomography: Computational theory and methods

University of Bergen

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Outline

- Introduction
- Fields of applications
- Principles of X-ray CT
- Reconstruction techniques in 2D
  - Transform based methods (FBP)
  - Algebraic reconstruction techniques
- Con-beam reconstructions
  - Circular
  - Helical
- Conclusions

Definition

Tomography: technique for imaging cross-sections or slices from a set of external measurements of a spatially varying function.

Fields of applications

- X-ray CT
  - medical imaging
  - industrial nondestructive evaluation
  - airport screening
  - microtomography

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- CT principles are also applied in:
  - oceanography
  - astronomy
  - geophysics
  - optics

Some hisrory

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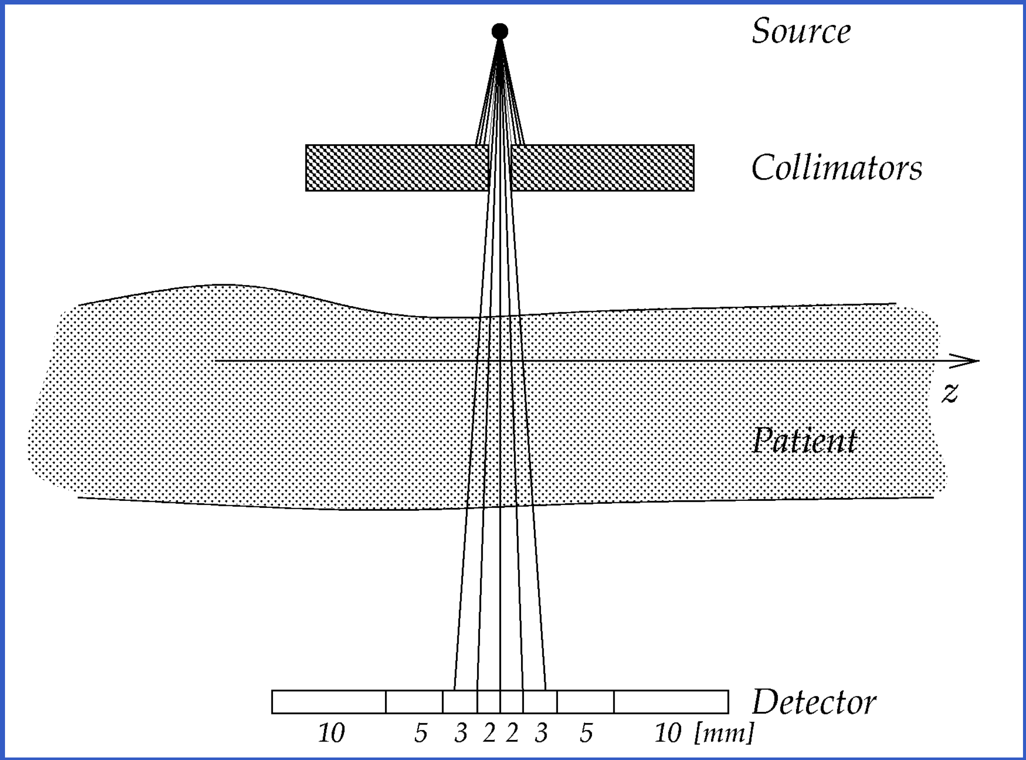
- Hounsfield and Cormack shared the 1979 Nobel Prize for medicine
- since then the field is steadily evolving and givis rise to never ceasing flow of new algorithms

Principles of X-ray CT

Principles of X-ray CT

- X-ray traverses an object along stight line
- attenuated signals from various directions are recorded
- reconstruction algorithms are used to convert the projections



Projections

Let

$f(x, y)$	represents the density of a two-dimensional object
I_{in}	applied intensity form the X-ray source
I_{out}	the attenuated intensity of a ray as it propagates throught the object
	along the line $L = L(\theta, t)$

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then

$$\int_L f(x,y)ds = -\ln \frac{I_{out}}{I_{in}} = P_\theta(t)$$

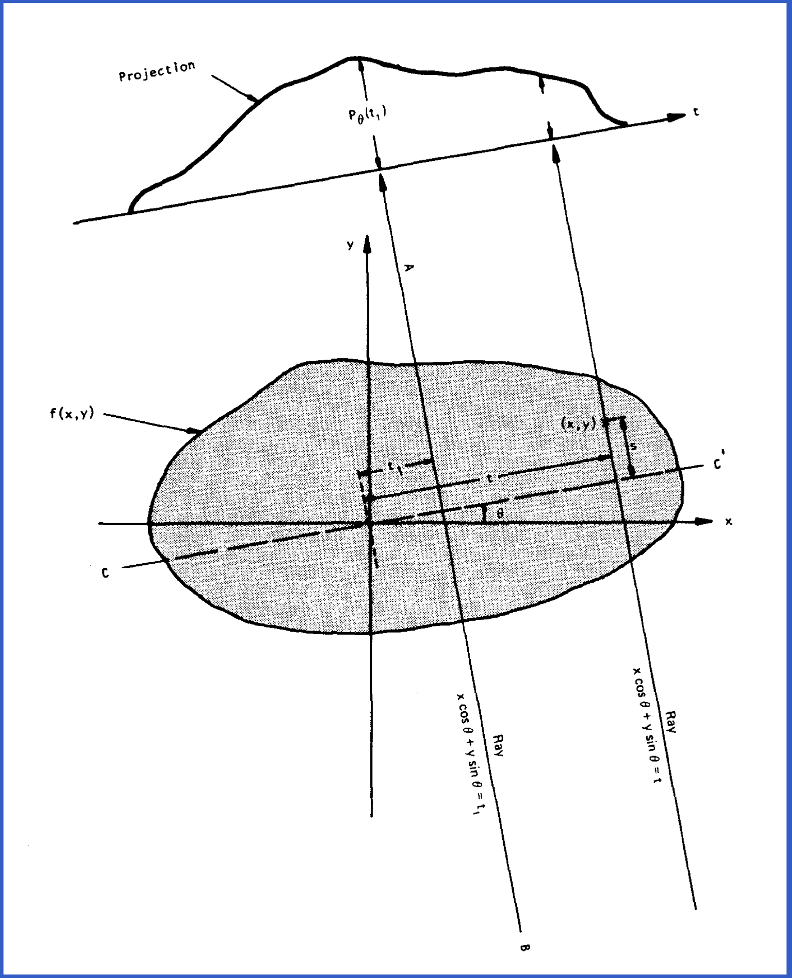
where

θ - projection angle

t - detector position

Projections

$$P_\theta(t) = \int_L f(x,y)ds$$



Projections

$$P_\theta(t) = \int_L f(x,y)ds$$

The line integral can be written as

Radon Transform

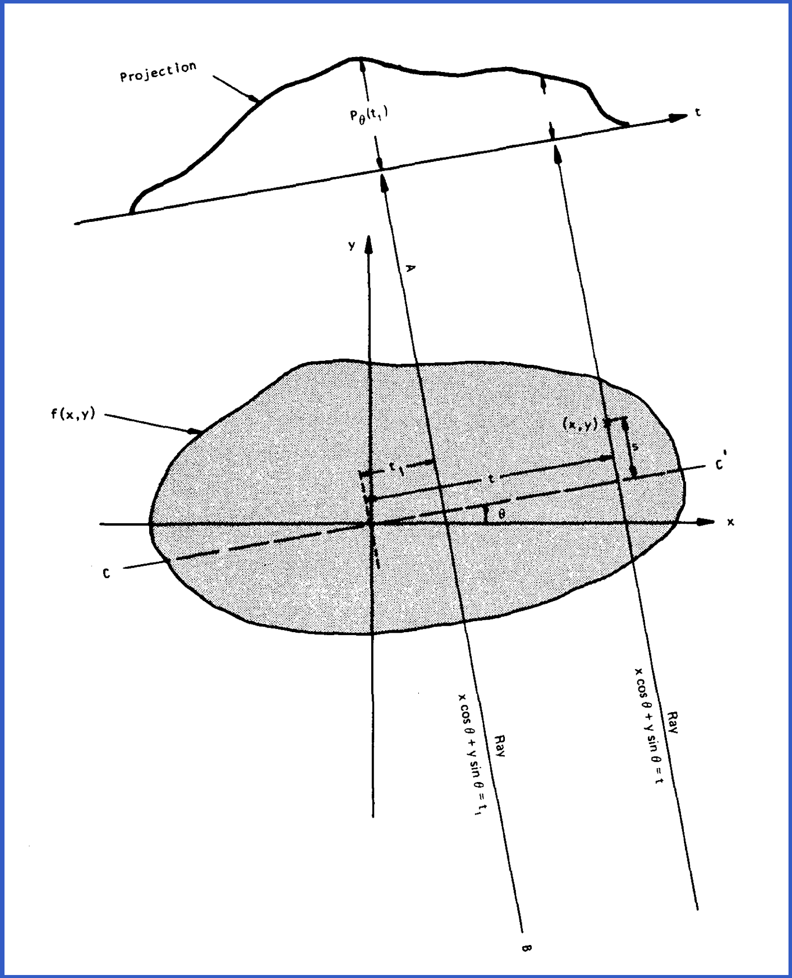
$$P_\theta(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(r(x,y,t))dxdy,$$

where

δ is the Dirac delta function and

$$r(x,y,t) = \cos\theta + y\sin\theta - t$$

(the equation of the X-ray)



Fourier Slice Theorem

Consider the two-dimensional FT of the object function

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

and the one-dimensional FT of the projection $P_\theta(t)$

$$S_\theta(w) = \int_{-\infty}^{\infty} P_\theta(t)e^{-j2\pi wt}dt$$

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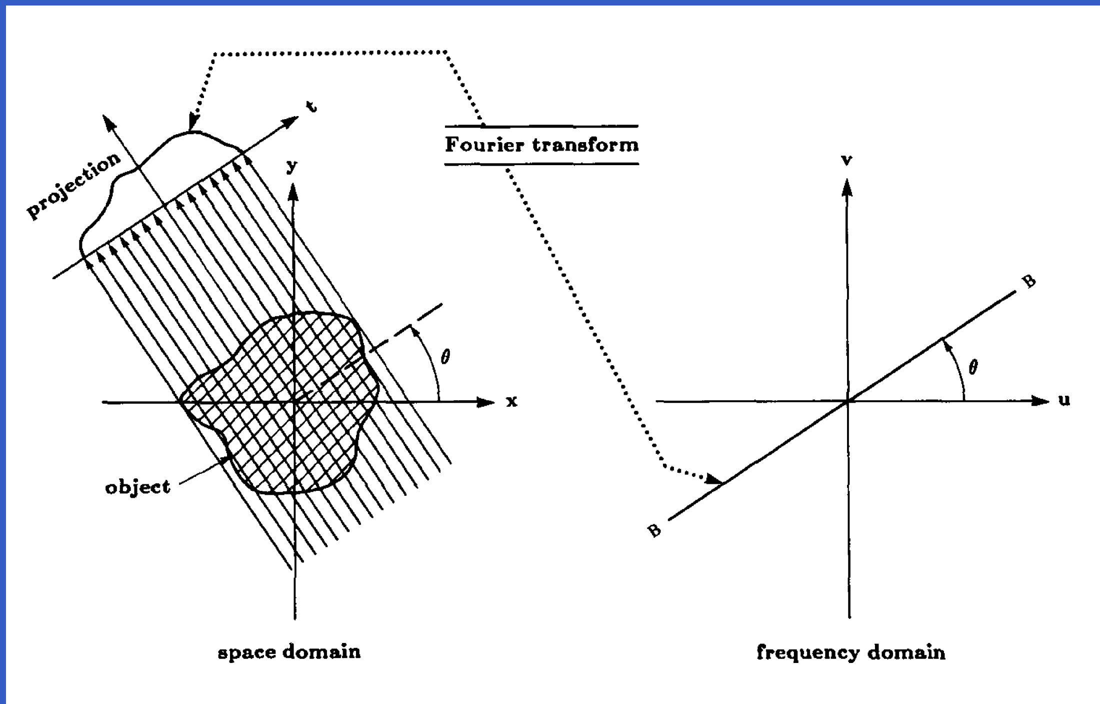
Simplest case, when $v = 0$ $(\theta = 0)$

$$\begin{aligned} F(u,0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dxdy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)dy\right]e^{-j2\pi ux}dx \\ &= \int_{-\infty}^{\infty}\left[\int_L f(x,y)ds\right]e^{-j2\pi ux}dx = \int_{-\infty}^{\infty} P_{\theta=0}(x)e^{-j2\pi ux}dx = S_{\theta=0}(u) \end{aligned}$$

Fourier Slice Theorem

In general for any anglr θ

$$S_\theta(w) = F(w, \theta) = F(w\cos\theta, w\sin\theta).$$



Fourier reconstruction algorithm

- Take projection of an object function at angles $\theta_1, \theta_2, \ldots, \theta_k$
- Fourier transform each projection to obtain the values of $F(u, v)$
- Recover the object function $f(x, y)$ by using the inverse FT

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du dv$$

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Note: infinite number of projections on an interval with lenght π allows exact reconstuction.

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Discretising on the square $-A/2 < x < A/2; -A/2 < y < A/2$

$$f(x,y) \approx \frac{1}{A^2}\sum_{m=-N/2}^{N/2}\sum_{n=-N/2}^{N/2} F\left(\frac{m}{A},\frac{n}{A}\right)e^{j2\pi((m/A)x+(n/A)y)},$$

where$N$ is even integer.

The values of F(u,v) on a square grid are obtained from the available values along the radial lines by interpolation.

Reconstruction for Parallel Projection

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} F(w,\theta)e^{j2\pi w \overbrace{(x\cos\theta + y\sin\theta)}^{t}} wdwd\theta$$

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$$= \int_{0}^{\pi}\underbrace{\int_{-\infty}^{\infty} S_\theta(w)|w|e^{j2\pi wt}dw}_{Q_\theta(t)}\,d\theta$$

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$$= \int_{0}^{\pi}\left[\int_{-\infty}^{\infty} P_\theta(\tau)\underbrace{\int_{-\infty}^{\infty}|w|e^{j2\pi w(t-\tau)}dw}_{h(t-\tau)}\,d\tau\right]d\theta = \int_{0}^{\pi}\left[\int_{-\infty}^{\infty} P_\theta(\tau)h(t-\tau)d\tau\right]d\theta$$

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$$= \int_{0}^{\pi}(P_\theta * h)(x\cos\theta + y\sin\theta)d\theta = \int_{0}^{\pi} Q_\theta(x\cos\theta + y\sin\theta)d\theta$$

Reconstruction for Parallel Projection

$$
\begin{aligned}
f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,dudv \\
&= \int_0^{2\pi}\int_0^{\infty} F(w,\theta)e^{j2\pi w\overbrace{(x\cos\theta+y\sin\theta)}^{t}}\,wdwd\theta \\
&= \int_0^{\pi}\underbrace{\int_{-\infty}^{\infty} S_\theta(w)|w|e^{j2\pi wt}dw}_{Q_\theta(t)}\,d\theta = \int_0^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_\theta(\tau)e^{-j2\pi w\tau}d\tau|w|e^{j2\pi wt}dwd\theta \\
&= \int_0^{\pi}\left[\int_{-\infty}^{\infty} P_\theta(\tau)\underbrace{\int_{-\infty}^{\infty}|w|e^{j2\pi w(t-\tau)}dw}_{h(t-\tau)}\,d\tau\right]d\theta = \int_0^{\pi}\left[\int_{-\infty}^{\infty} P_\theta(\tau)h(t-\tau)d\tau\right]d\theta \\
&= \underbrace{\int_0^{\pi}\underbrace{(P_\theta * h)}_{\text{Filtering}}(x\cos\theta+y\sin\theta)d\theta}_{\text{Backprojection}} = \int_0^{\pi} Q_\theta(x\cos\theta+y\sin\theta)d\theta
\end{aligned}
$$

The FBP algorithm

Step 1: Filtering

$$Q_\theta(t) = \int_{-\infty}^{\infty} S_\theta(w)|w|e^{j2\pi wt}dw$$

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Use badlimited FT to compute $S_\theta(w)$ i.e.

$$S_\theta(w) \approx S_\theta\left(m\frac{2W}{N}\right) = \frac{1}{2W}\sum_{k=-N/2}^{N/2-1} P_\theta\left(\frac{k}{2W}\right)e^{-j2\pi(mk/N)}, \quad m = -\frac{N}{2}, \ldots, \frac{N}{2},$$

where W is a frequency band limit and N is some (large) number corresponding to the number of the sampling points. (FFT)

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where W is a frequency band limit and N is some (large) number corresponding to the number of the sampling points. (FFT)
Therefore

$$Q_\theta\left(\frac{k}{2W}\right) \approx \left(\frac{2W}{N}\right)\sum_{m=-N/2}^{N/2} S_\theta\left(m\frac{2W}{N}\right)\left|m\frac{2W}{N}\right|e^{j2\pi(mk/N)}, \quad k = -\frac{N}{2},\ldots,\frac{N}{2}.$$

Inverse DFT of the product $S_\theta(m(2W/N))$ and $|m(2W/N)|$.

The FBP algorithm

Step 1: Filtering

Superior results are obtained by deemphasizing the high frequencies with the Hamming window function H.

$$Q_\theta\left(\frac{k}{2W}\right) \approx \left(\frac{2W}{N}\right) \sum_{m=-N/2}^{N/2} S_\theta\left(m\frac{2W}{N}\right)\left|m\frac{2W}{N}\right| H\left(m\frac{2W}{N}\right) e^{j2\pi(mk/N)}$$

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where $\phi(k/"W)$ is the inverse DFT of $|m(2W/N)|H(m(2W/N))$.

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Step 2: Backprojection

Reconstruc $f(x, y)$ by the formula

$$f(x,y) \approx \frac{\pi}{K} \sum_{i=1}^{K} Q_{\theta_i}(x\cos\theta_i + y\sin\theta_i)$$

where K is the number of the projection angles θ_i.

Note The values of Q_θ in Step 2 can be approxiamted from the values obtained in Step 1 by interpolation (linear).

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Computational cost: Backprojection summation —> $\mathcal{O}(N^3)$

Fast backprojections (Nilsson'96) —> $\mathcal{O}(N^2 \log N)$

Reconstruction for Fan Projections

Equiangular Rays

Let $R_\beta(\gamma)$ be a fan projection

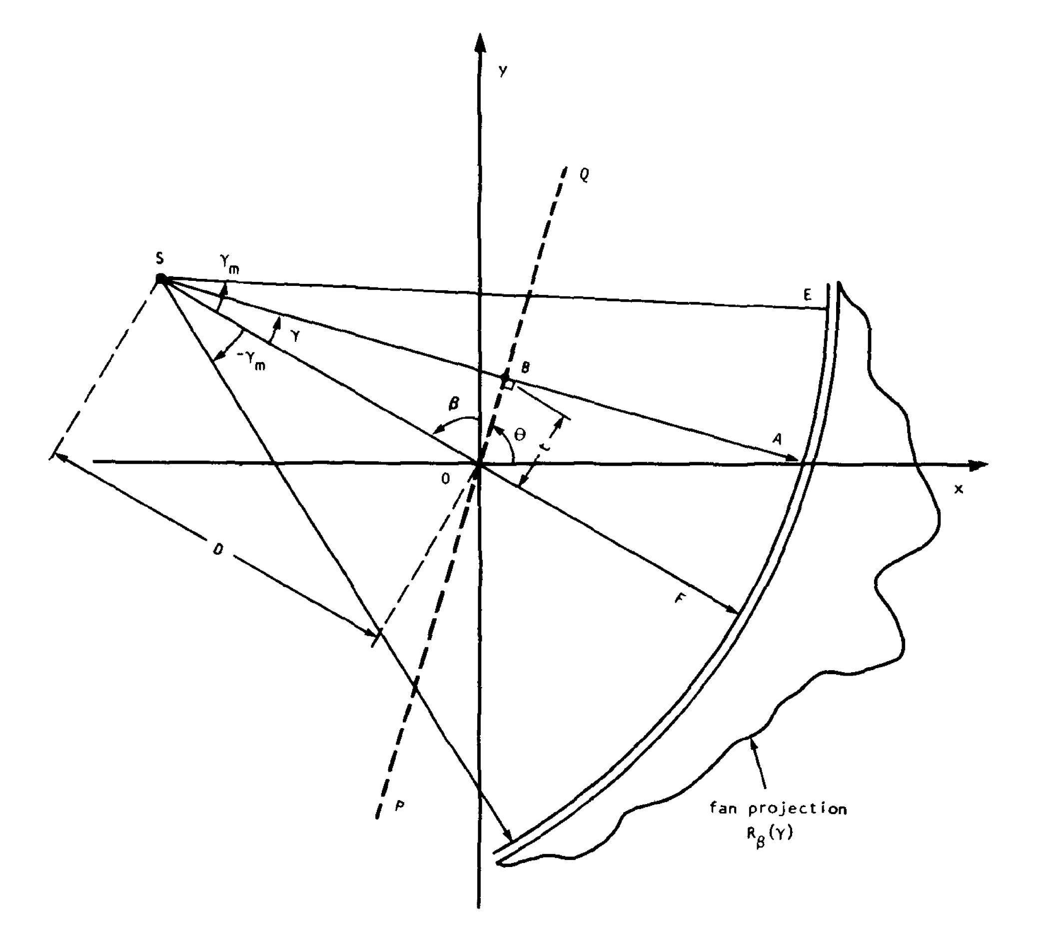
β is the angle between the source and a reference axis.

γ gives the location of a ray

$D =$|SO| source to origin dist.

L - source to point (x, y) dist.
In polar coordinates $[(r, \phi)]$

$L = \sqrt{D + (r\sin\eta)^2 + (r\cos\eta)^2}$,
where $\eta = \beta - \phi$.



Reconstruction for Fan Projections

Equiangular Rays

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} P_\theta(t) h(x\cos\theta + y\sin\theta - t) dt d\theta$$

$$f(r,\phi) = \frac{1}{2} \int_0^{2\pi} \int_{-\infty}^{\infty} P_\theta(t) h(r\cos(\theta - \phi) - t) dt d\theta$$

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&= \dots \\
&= \int_0^{2\pi} \frac{1}{L^2} Q_\beta(\gamma')d\beta,
\end{aligned}
$$

where

$$
\begin{aligned}
Q_\beta(\gamma) &= R'_\beta(\gamma) * g(\gamma) \\
R'_\beta(\gamma) &= R_\beta(\gamma)\cdot D\cdot\cos\gamma \\
g(\gamma) &= \frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 h(\gamma)
\end{aligned}
$$

Weighted Backprojection Algorithm

Equiangular Rays

Let
- α is the sampling interval of the projection
- β_i are the angles at which projections are taken

Step 1 For each $R_{\beta_i}(n\alpha),\ n \in \mathbb{N}$, generate the corresponding modified projection

$$R'_{\beta_i}(n\alpha) = R_{\beta}(n\alpha) \cdot D \cdot \cos n\alpha$$

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Step 2 Convolve $R'_{\beta_i}(n\alpha)$ with $g(n\alpha) = \frac{1}{2}\left(\frac{n\alpha}{\sin n\alpha}\right)^2 h(n\alpha)$ to generate the corresponding filtered projection (FFT)

$$Q_{\beta_i}(n\alpha) = R'_{\beta_i}(n\alpha) * g(n\alpha)$$

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$$Q_{\beta_i}(n\alpha) = R'_{\beta_i}(n\alpha) * g(n\alpha) * k(n\alpha),$$

where $k(n\alpha)$ is a smoothing filter.

Weighted Backprojection Algorithm

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Step 3 Perform a weighted backprojection

$$f(x,y) \approx \Delta\beta \sum_{i=1} K \frac{1}{L^2(x,y,\beta_1)} Q_{\beta_i}(\gamma'),$$

where γ' is the angle of a ray that passes through the point (x,y) and $\Delta\beta = 2\pi/K$.

Reconstruction for Fan Projections

Equally Spaced Detectors

Let $R_\beta(s)$ be a fan projection

β is the angle between the source and a reference axis.

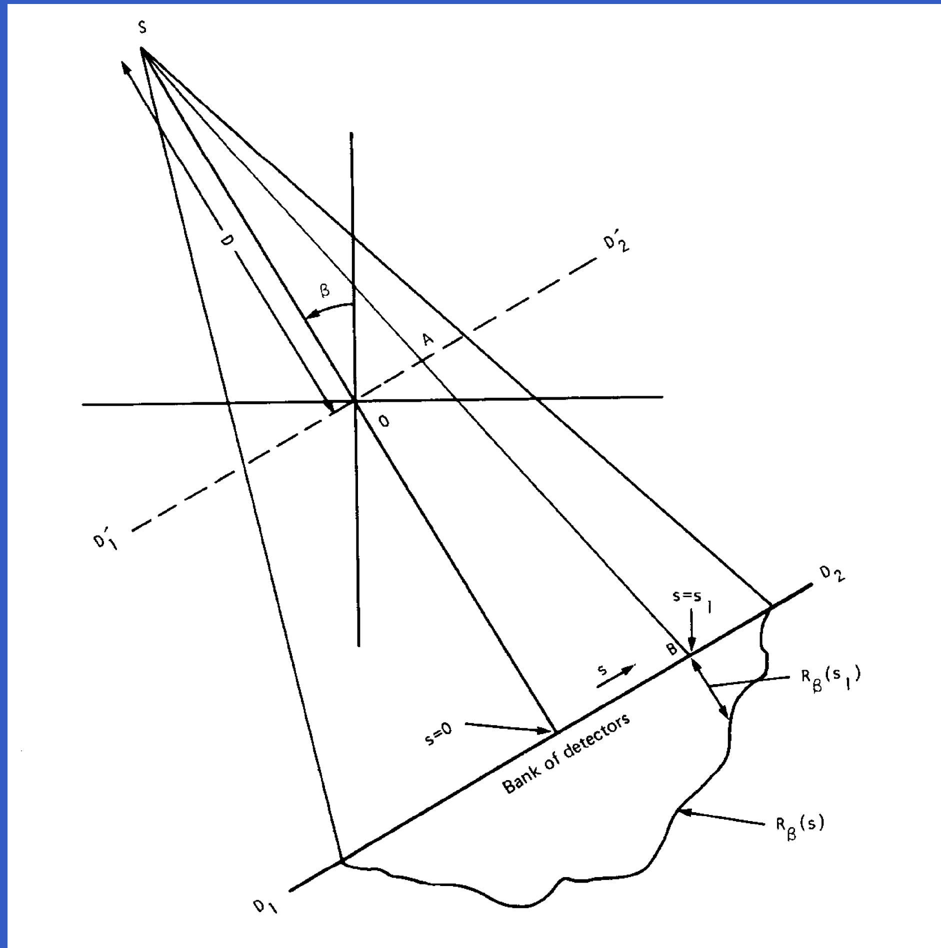
s is the distance along the detector

$D =$|SO| source to origin dist.

U - the dist. source to projection of a pixel on the central ray

In polar coordinates $[(r,\phi)]$

$$U = \frac{D + r\sin(\beta-\phi)}{D}$$



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&= \ldots \\
&= \int_0^{2\pi} \frac{1}{U^2} Q_\beta(s')\,d\beta,
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$$

where

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\begin{aligned}
Q_\beta(s) &= R'_\beta(s) * g(s) \\
R'_\beta(s) &= R_\beta(s)\cdot\frac{D}{\sqrt{D^2+s^2}} \\
g(s) &= \frac{1}{2}h(s)
\end{aligned}
$$

Algebraic reconstruction techniques

Advantages

- easy to handle different rays geometry
- more adaptable to missing data - better image quality
- metal artifacts are reduced
- less radiation dose

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Disadvantage

Higher computational cost!

Image and Projection Representation

The idea: Assume that the cross section consists of array of unknowns

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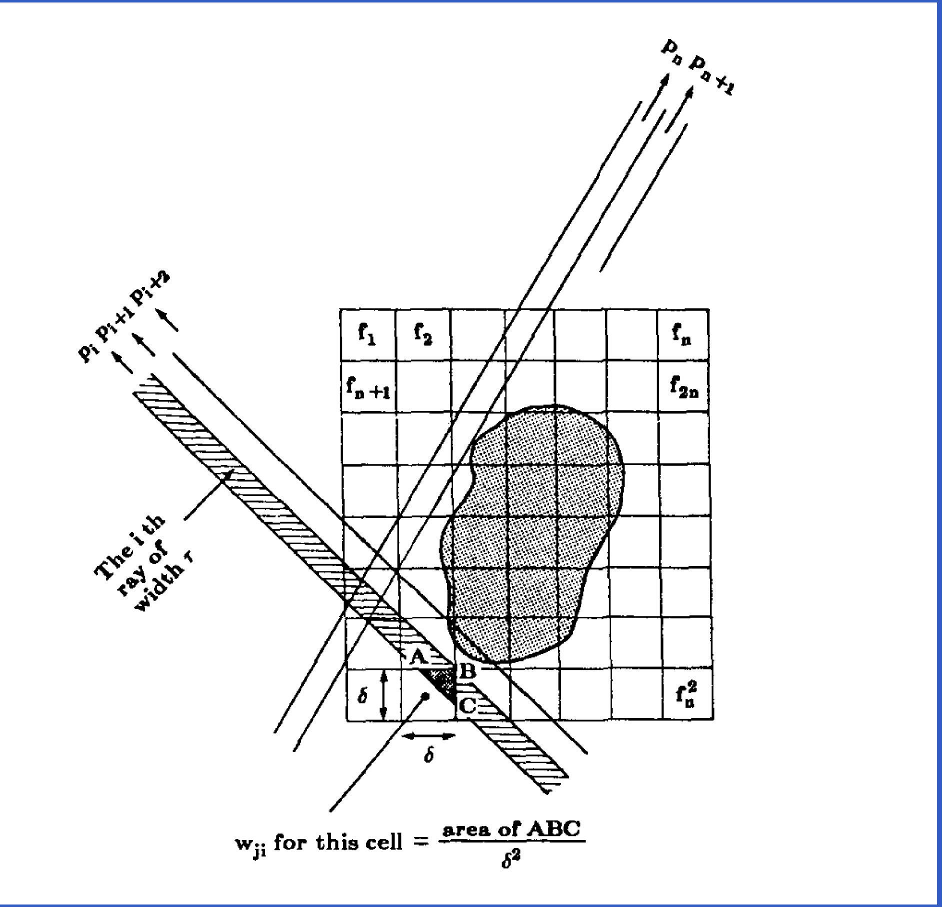
N - total number of cells

$f_j = const$ is the value of $f(x, y)$
in the j-th cell

p_i - the i-th line integral *ray-sum*



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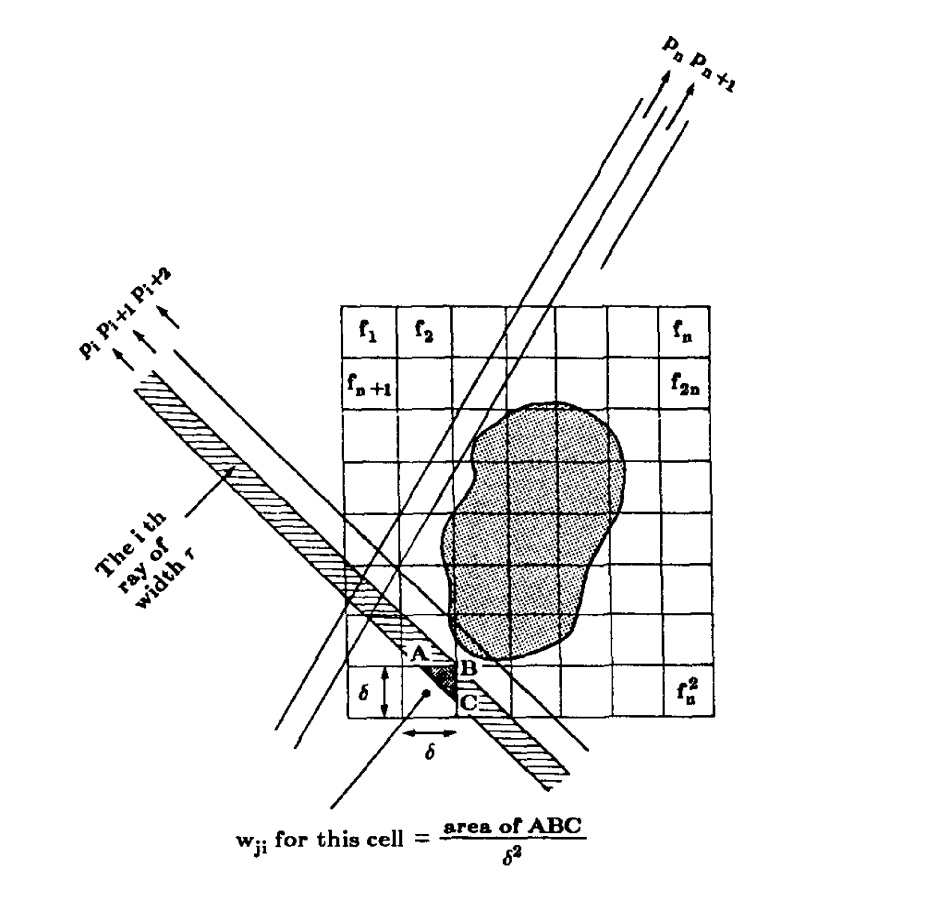

$$\sum_{j=1}^{N} w_{ij} f_j = p_i, \quad i = 1, 2, \ldots, M$$

where

M is the total number of rays

w_{ij} represents the contribution of the j-th cell to the i-th ray integral

Projection methods

Therefore we have to solve the following linear system of equations

$$
\begin{aligned}
w_{11} f_1 + w_{12} f_2 + \cdots + w_{1N} f_N &= p_1 \\
w_{21} f_1 + w_{22} f_2 + \cdots + w_{2N} f_N &= p_2 \\
\vdots \\
w_{M1} f_1 + w_{M2} f_2 + \cdots + w_{MN} f_N &= p_M
\end{aligned}
$$

for large values of M and N.

Projection methods

Therefore we have to solve the following linear system of equations

$$
\begin{aligned}
w_{11} f_1 + w_{12} f_2 + \cdots + w_{1N} f_N &= p_1 \\
w_{21} f_1 + w_{22} f_2 + \cdots + w_{2N} f_N &= p_2 \\
&\vdots \\
w_{M1} f_1 + w_{M2} f_2 + \cdots + w_{MN} f_N &= p_M
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for large values of M and N.

Use iterative methods - projestions (Kaczmarz'37, Tanabe'71)

Let $f^{(0)} = \left(f_1^{(0)}, f_2^{(0)}, \dots f_N^{(0)}\right)$ is the initial guess

$$f^{(i)} = f^{(i-1)} - \frac{\left(f^{(i-1)} \cdot w_i - p_i\right)}{w_i \cdot w_i} w_i \quad i = 1, 2, \dots,$$

where $w_i = (w_{i1}, w_{i,2}, \dots w_{iN})$

Projection methods

Theorem (Tanabe'71)
If there exists a unique solution $f^{(s)}$ to the system of equations, then

$$\lim_{k\to\infty} f^{(kM)} = f^{(s)}$$

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- Full orthogonalization is computationally not feasible
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• When $M > N$ the process oscillate in the neighborhood of the intersections of the hyperplanes

• When $M < N$ the process converges to a solution $f'^{(s)}$, such that $|f^{(0)} - f'^{(s)}|$ is minimized.

Projection methods

Consider again

$$f^{(i)} = f^{(i-1)} - \frac{\left(f^{(i-1)} \cdot w_i - p_i\right)}{w_i \cdot w_i} w_i \quad i = 1, 2, \ldots,$$

It can also be written as

$$f_j^{(i)} = f_j^{(i-1)} + \frac{p_i - q_i}{\sum_{k=1}^N w_{ik}^2} w_{ij} \quad j = 1, 2, \ldots, N, \quad i = 1, 2, \ldots,$$

where

$$q_i = f^{(i-1)} \cdot w_i = \sum_{k=1}^N f_k^{(i-1)} w_{ik}.$$

Therefore

$$f_j^{(i)} = f_j^{(i-1)} + \Delta f_j^{(i)},$$

where

$$\Delta f_j^{(i)} = \frac{p_i - q_i}{\sum_{k=1}^N w_{ik}^2} w_{ij}.$$

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The main computational diffuculty is in the calculation, storage and retrieval of the weight coefficients w_{ij}.

Algebraic Reconstruction Tech.(ART)

- Replace w_{ij} by 0 or 1, depending wether the center of the j-th cell is within the i-th ray. Thus we can use

$$\Delta f_j^{(i)} = \frac{p_i - q_i}{N_i}$$

for all cells whos centers are within the i-th ray.
$N_i = \sum_{k=1}^{N} w_{ik}^2$ is the number of cells whose centers are within the i-th ray.

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- Superior approxiamtion is

$$\Delta f_j^{(i)} = \frac{p_i}{L_i} - \frac{q_i}{N_i},$$

where L_i is the lenght of the i-th ray trought the reconstruction region.

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- It is possible to reduce the noce by

relaxation (i.e. update a pixel by $\lambda_i \Delta f_j^{(i)},\ \lambda < 1$)

SIRT - change the cell values after going through all of the equations

SART (Simultaneous ART)

First reported by Anderson and Kak '84

- Reduce the error in the approximation of the ray integrals

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Insted of solving

$$p_i = \sum_{j=1}^{N} w_{ij} f_j, \quad i = 1, 2, \ldots, M.$$

We use again the Radon Transform

$$p_i = R_i f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\delta(r_i(x,y))dxdy,$$

where $r_i(x,y) = 0$ is the equation of the i-th ray and R_i is the projection operator.

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where $r_i(x,y) = 0$ is the equation of the $i^{\text{-th}}$ ray and R_i is the projection operator.

Let $\{b_j(x,y)\}_{j=1}^{N}$ be basis functions and let

$$f(x,y) \approx \widetilde{f}(x,y) = \sum_{j=1}^{N} g_j b_j(x,y)$$

then

$$p_i = R_i f(x,y) \approx R_i \widetilde{f}(x,y) = \sum_{j=1}^{N} g_j a_{ij}$$

wher $a_{ij} = R_i b_j(x,y)$.

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We solve

$$p_i = \sum_{j=1}^{N} g_j a_{ij}, \quad i = 1, 2, \ldots, M.$$

- The two systems are the same for the *pixel basis*

$$b_j(x,y) = \begin{cases} 1 & \text{inside the } j\text{-th pixel} \\ 0 & \text{everywhere else} \end{cases}$$

- Superior reconstructions are obtained by using bilinear elements

pyramid shaped basis

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The overall SART iterative scheme is

$$g_j^{(k+1)} = g_j^{(k)} + \frac{\lambda_k}{\sum_{i=1}^{M} a_{ij}} \sum_{i=1}^{M} a_{ij} \frac{p_i - a_i \cdot g^{(k)}}{\sum_{j=1}^{N} a_{ij}}, \quad j = 1, 2, \ldots, N,$$

where a_i is the i-th row of A and $\lambda_k > 0$ are relaxation coefficients.

More iterative algorithms

Censor and Elfving'02

The idea: Use generalized projections and diagonal weighting matrices to reflect the sparsity of the A matrix.

Definition: A family $\{G_i\}_{i=1}^{M}$ of diagonal $N \times N$ matrices with diagonal elements $g_{ij} > 0$ is called *Sparsity Pattern Oriented* (SPO) with respect to $M \times N$ matix A if:

1) $\sum_{i=1}^{M} G_i = I$ 2) $g_{ij} = 0$ iff $a_{ij} = 0$ for every $i = 1, 2, \ldots, M$.

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Diagonal Weighting (DWE)

$$g_j^{(k+1)} = g_j^{(k)} + \lambda_k \sum_{\substack{i=1 \\ g_{ij} \neq 0}}^{M} a_{ij} \frac{p_i - a_i \cdot g^{(k)}}{\sum_{\substack{l=1 \\ g_{il} \neq 0}}^{N} \frac{a_{il}^2}{g_{il}}}, \quad j = 1, 2, \ldots, N.$$

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Special case

$$g_{ij} = \begin{cases} \frac{1}{s_j} & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

where s_j is the number of the nonzero elements in column j of A.

Component Averaging (CAV)

$$g_j^{(k+1)} = g_j^{(k)} + \lambda_k \sum_{i=1}^{M} a_{ij} \frac{p_i - a_i \cdot g^{(k)}}{\sum_{l=1}^{N} s_l a_{il}^2}, \quad j = 1, 2, \ldots, N.$$

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- Jiang and Wang'03 ART, SART, CAV, DWE are special cases of the Landweber method

$$g^{(k+1)} = g^{(k)} + \lambda_k V^{-1} A^* W (p - A g^{(k)})$$

3D Reconstructions

- Slice by Slice

Disadvantages

- low speed
- discrete nature
- higher radiation dose

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  Disadvantages

  - low speed

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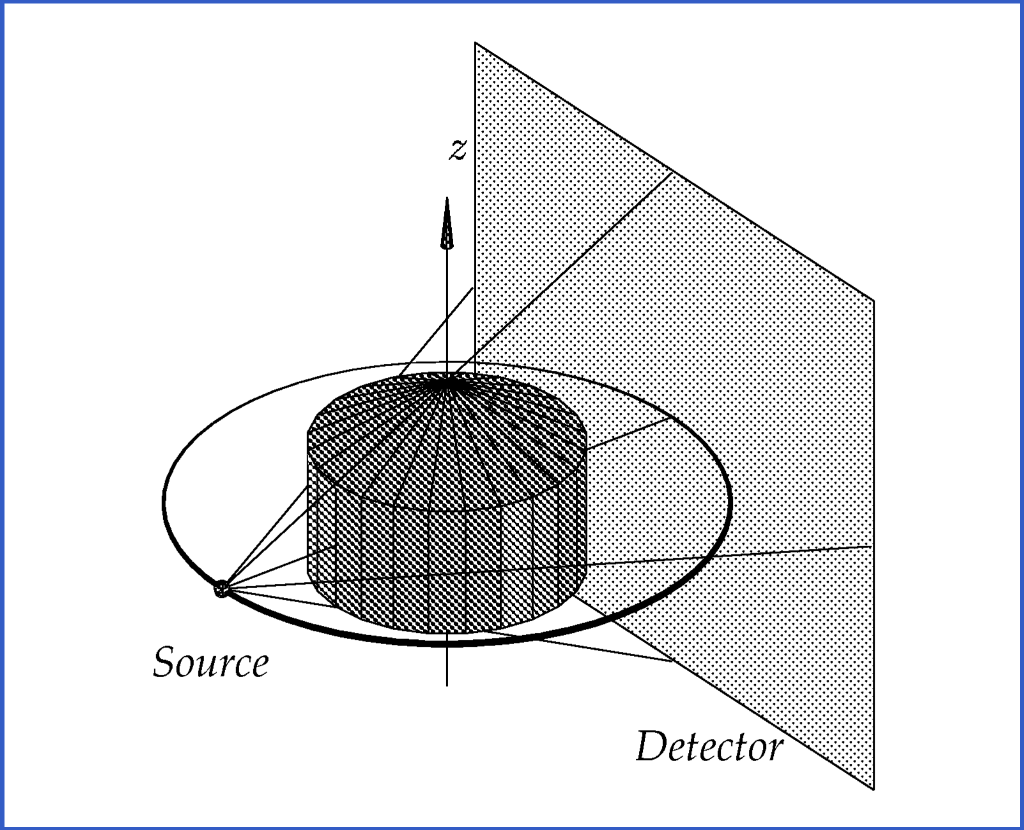
- Cone-Beam Reconstructions

  - Circular

  - Helical

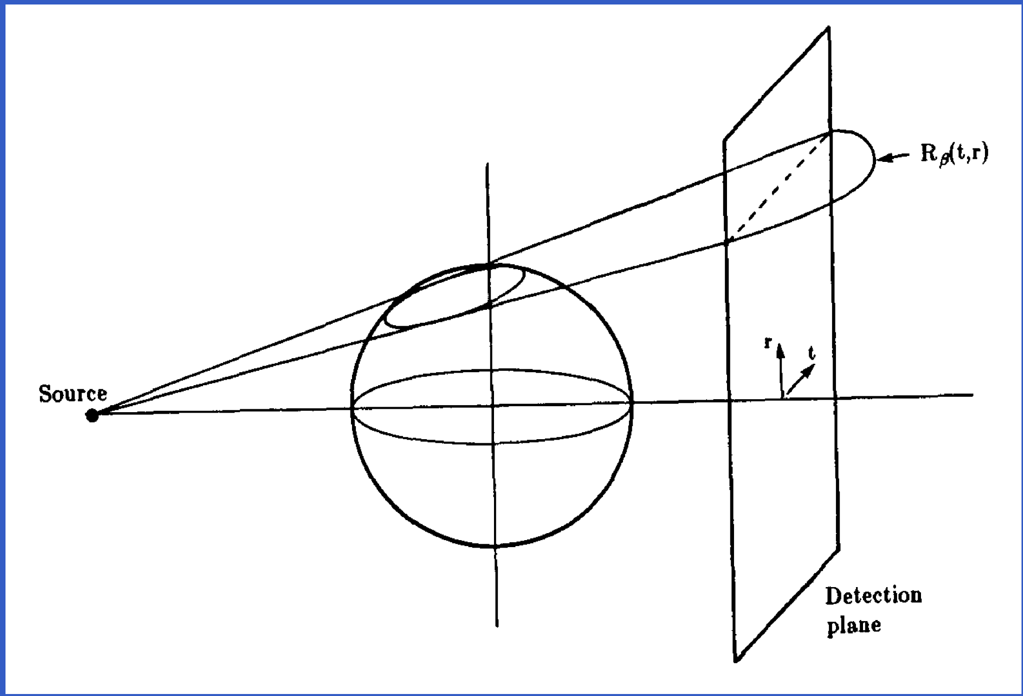
Circular Cone-Beam Reconstructions



Circular Cone-Beam Reconstructions

The Idea: Apply fan-beam FBP to each plane in the cone

Feldkamp, Davis, Kress ’84 (FDK)



Circular Cone-Beam Reconstructions

The Idea: Apply fan-beam FBP to
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Feldkamp, Davis, Kress '84 (FDK)

A ray from the cone is described by

$$t = x\cos\theta + y\sin\theta$$
$$r = -(-x\sin\theta + y\cos\theta)\sin\gamma + z\cos\gamma$$

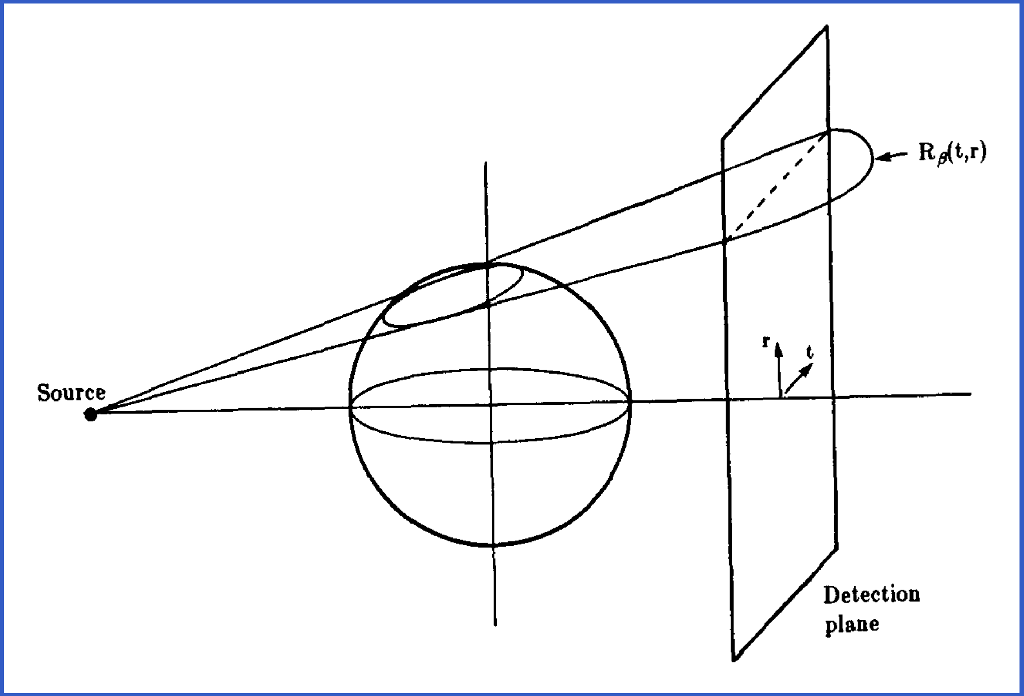
where

(t, θ) locate a ray in a fan

(r, γ) specify the location ot the fan

θ - projection angle

γ - fan angle



(x, y, z) —> (t, s, z) (rotation by θ degrees around the z-axes)

The FDK Algorithm

Based on the FBP Algorithm for equispatial rays

$$f(t,s,z) = \frac{1}{2}\int_0^{2\pi} \frac{D^2}{(D-s)^2} \int_{-\infty}^{\infty} R_\beta(p,\xi) \frac{D}{\sqrt{D^2+\xi^2+p^2}} h\left(\frac{D\,t}{D-s} - p\right) dp d\beta,$$

where $D = |SO|$ and $\xi = \frac{D}{D-s} z$.

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where $D=|SO|$ and $\xi=\frac{D}{D-s}z$.

The Algorithm

Step 1 Compute the modified projection

$$R'_\beta(p,\xi)=\frac{D}{\sqrt{D^2+\xi^2+p^2}}R_\beta(p,\xi)$$

Step 2 Convolve $R'_\beta(p,\xi)$ with $h(p)/2$

$$Q_\beta(p,\xi)=R'_\beta(p,\xi)*\frac{1}{2}h(p)$$

Step 3 Backproject over the 3D reconstruction grid

$$f(t,s,z)=\int_0^{2\pi}\frac{D^2}{(D-s)^2}Q_\beta(\frac{D\,t}{D-s},\frac{D\,z}{D-s})d\beta$$

The Tuy–Smith sufficiency condition

Exact reconstruction is possible if all planes intersecting the object also intersect the source trajectory at least once.

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Circular trajectory does not satisfay this condition since a plane parallel to the trajectory may also intersect the object.

Exact reconstruction form helical cone-beam data is possible. Tam'95 Kudo, Noo, Defrise '98

Conclusions

The challenges:

- Three dimentional images
- Better quality
- Faster algorithms
- Restricted in time

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